1. (10 points) Evaluate each of the following limits.

(a) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sqrt{x}}\right)$

(b) $\lim_{x\to -\infty}\frac{\sqrt{x^2-3}}{4x+1}$

(c) $\lim_{x\to 0}\frac{\frac{1}{x+2}-\frac{2}{x+4}}{\sin(x)}$

(d) $\lim_{x\to 3}\frac{2-\sqrt{7-x}}{x^2-5x+6}$

(e) $\lim_{h\to 0}\frac{\ln(e+h)-1}{h}$, (by interpreting it as a derivative)

2. (3 points) Find the horizontal and vertical asymptotes of $f(x)=\frac{3e^x+1}{5e^x-2}$.

3. (3 points) Let $f(x) = \begin{cases} x^2+6x+12 & \text{if } x<-2, \\ ax+b & \text{if } x\geq -2. \end{cases}$ Find all pairs of values a and b so that f is differentiable everywhere.

4. (3 points) Give the rule of a function of the form $f(x)=\frac{(Ax-B)(Cx-D)}{(Ex-F)(Gx-H)}$ that has **all** of the following properties:

- $\lim_{x\to\infty} f(x)=1$.
- $\lim_{x\to 3} f(x)$ exists, but $f(3)$ does not.
- $\lim_{x\to 2^-} f(x)=\infty$.

5. (16 points) Find $\frac{dy}{dx}$ for each of the following. Do not simplify your answers.

(a) $y=\frac{x^9}{9}-\frac{9}{x^9}+9^x+\log_9(x)+\sqrt[9]{x}+9^9$

(b) $y=\frac{\ln(x)\sec(2x-1)}{7}$

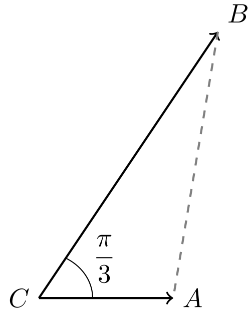
(c) $y=\sqrt{(e^x-3)^2+\frac{x}{\sin(x)}}$

(d) $y=\tan(x^9+x^x)$ (Hint: what is $\frac{d}{dx}(x^x)$?)

6. (3 points) Let $f(x)=xg(x)$. If $g'(3)=2$, and the slope of the line tangent to $f(x)$ at $x=3$ is 14, determine the value of $g(3)$.

7. (5 points) Allan and Balla both start walking from the same point. Allan walks east at a rate of 1.5 km/h, while Balla jogs $\frac{\pi}{3}$ radians north of east at a rate of 4 km/h. How fast is the distance between them increasing after 2 hours?

[Hint: Recall the law of cosines: $\boldsymbol{c^2=a^2+b^2-2ab\cos C}$ for a triangle of with side lengths a, b and c, and where C is the angle opposite the side c.]



8. (4 points) Find the equation of the line tangent to the curve defined by $x^2+y^2=(2x^2+2y^2-1)^2$ at the point $(0,\frac{1}{2})$.

9. (4 points) (a) State (or explain) the Mean Value Theorem.

(b) Suppose $f(x)$ is a twice-differentiable function, and $f(x)$ has at least two critical values. Show that $f''(x)$ has at least one root.

10. (4 points) Find the intervals of increase/decrease of $f(x)=9x^{2/3}(x-20)$, and coordinates of all local extrema.

11. (4 points) Given

$$f(x)=(1+\sin(x))^{2/3}, \qquad f''(x)=\frac{-2(1+2\sin(x))}{9\sqrt[3]{1+\sin(x)}}$$

find the intervals of concavity and points of inflection of $f(x)$ **over the interval** $\boldsymbol{[0,2\pi]}$.

12. (5 points) Given the following information about a function $f(x)$:

Domain: $\mathbb{R}\setminus\{0\}$
***x*-intercepts:** $x=-1$, $x=2$, $x=9$,
$f(-5)=7,\ f(-3)=5,\ f(6)=-3,\ f(10)=4$

$\lim_{x\to 0^-} f(x)=-\infty, \qquad \lim_{x\to\infty} f(x)=6, \qquad \lim_{x\to-\infty} f(x)=1$

$\boldsymbol{f'(x)>0}$ **:** $(-\infty,-5)\cup(6,\infty)$

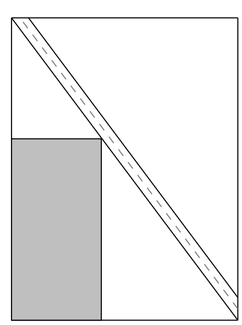
$\boldsymbol{f'(x)<0}$ **:** $(-5,0)\cup(0,6)$

$\boldsymbol{f''(x)>0}$ **:** $(-\infty,-5)\cup(-5,-3)\cup(0,10)$

$\boldsymbol{f''(x)<0}$ **:** $(-3,0)\cup(10,\infty)$

Sketch a graph of f which takes all of this information into account. Label all intercepts, asymptotes, extrema and inflection points.

13. (6 points) A 600m by 800m rectangular park has a bicycle path cutting though it diagonally (see below). A rectangular play area is to be fenced off in the space below the bike path. What dimensions of the play area will allow it to have the largest possible area?



14. (4 points) Use **differentiation** to verify that the following formula is correct.

$$\int \frac{1}{(x^2+4)^{3/2}}\,dx = \frac{x}{4\sqrt{x^2+4}}+C.$$

15. (3 points) Find $\int_0^3 \sqrt{9-x^2}\,dx$, by interpreting in terms of area.

16. (2 points) Express $\int_0^{\pi} x\cos(2x)\,dx$ as the limit of a Riemann Sum. Do not evaluate the limit.

17. (3 points) Express $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left(5+\frac{4}{n}i\right)\sin\left(2+\frac{4}{n}i\right)\frac{4}{n}$ as a definite integral starting at $a=3$.

18. (11 points) Evaluate each of the following integrals

(a) $\int(\sqrt{x}-3)^2\,dx$

(b) $\int \dfrac{\tan(\theta)-\cos^2(\theta)+\sec(\theta)}{\cos(\theta)}\,d\theta$

(c) $\int_1^2\left(e^y-\dfrac{3}{y^2}\right)dy$

(d) $\int_5^5 \cot(3x^2-9)\,dx$

19. (3 points) Given the relation

$$\int_0^y e^{-t}\,dt = 4+\int_2^{x^2}\sin^2(t)\,dt,$$

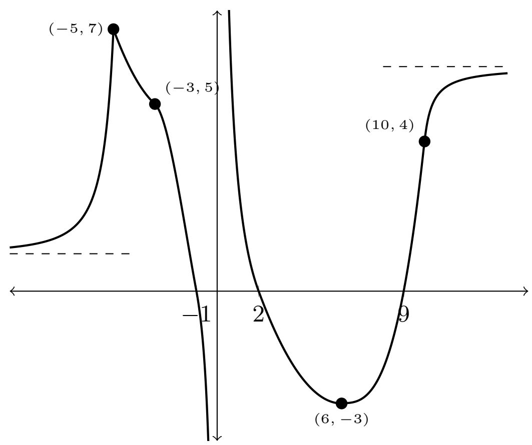
use implicit differentiation to find $\dfrac{dy}{dx}$.

20. (4 points) Answer true or false, justifying your answer with an explanation or counterexample:

- If $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=1$.
- If $f(x)$ is differentiable at $x=a$, $\lim\limits_{x\to a} f(x)$ must exist.
- If f is increasing on $[a,b]$, then $f'(x)>0$ for every x in (a,b).
- If f and g are both continuous on $[a,b]$, then $\int_a^b f(x)\cdot g(x)\,dx = \int_a^b f(x)\,dx\cdot\int_a^b g(x)\,dx$.

Answers:

1. (a) ∞
   (b) $-\frac{1}{4}$
   (c) $-\frac{1}{8}$
   (d) $\frac{1}{4}$
   (e) $\frac{1}{e}$
2. HA: $y=-\frac{1}{2}, \frac{3}{5}$, VA: $x=\ln(2/5)$
3. $a=2$, $b=8$
4. Answers will vary.
5. (a) $x^8-81x^{-10}+9^x\ln 9+\frac{1}{x\ln 9}+\frac{1}{9}x^{-8/9}$
   (b) $\frac{1}{7}(\frac{1}{x}\sec(2x-1)+\sec(2x-1)\tan(2x-1)(2)\ln(x))$
   (c) $\frac{1}{2}((e^x-3)^2+\frac{x}{\sin x})^{\frac{-1}{2}}(2(e^x-3)e^x+\frac{\sin x - x\cos x}{\sin^2 x}))$
   (d) $\sec^2(x^9+x^x)(9x^8+x^x(\ln x+1))$
6. $g(3)=8$
7. $\frac{7}{2}$ km/h
8. $y=\frac{1}{2}$
9. (a) If f is a function continuous on $[a,b]$ and differentiable on (a,b) then there exists a number $c\in(a,b)$ with $f(b)-f(a)=f'(c)(b-a)$.
   (b) Since f has two critical values, and f'' exists everywhere, we know that f' has at least two roots, say $x_1<x_2$. Then f' is continuous and differentiable on $[x_1,x_2]$, and $f'(x_1)=f'(x_2)$. So by Rolle's theorem, there exists $c\in(x_1,x_2)$ with $f''(c)=0$.
10. Increasing: $(-\infty,0),(8,\infty)$, Decreasing: $(0,8)$
    Local Max: $(0,0)$, Local Min: $(8,-432)$
11. CU: $[\frac{7\pi}{6},\frac{11\pi}{6}]$, CD: $[0,\frac{7\pi}{6}],[\frac{11\pi}{6},2\pi]$

    POI: $(\frac{7\pi}{6},\frac{1}{\sqrt[3]{4}});(\frac{11\pi}{6},\frac{1}{\sqrt[3]{4}})$
12. 

13. The play area should be 300m × 400m.

15. $\frac{9\pi}{4}$
16. $\lim\limits_{x\to\infty}\sum_{i=1}^{n}\left(\frac{i\pi}{n}\right)\cos\left(\frac{2i\pi}{n}\right)\left(\frac{\pi}{n}\right)$
17. $\int_3^7 (2+x)\sin(-1+x)\,dx$
18. (a) $\frac{1}{2}x^2-4x^{3/2}+9x+C$
    (b) $\sec\theta-\sin\theta+\tan\theta+C$
    (c) $e^2-e-\frac{3}{2}$
    (d) 0
19. $\frac{dy}{dx}=2x\sin^2(x^2)e^y$
20. - False
    - True
    - False
    - False