[10] 1. Evaluate the following limits:

(a) $\displaystyle\lim_{x\to -2} \frac{x^2+2x}{x^2+6x+8}$

(b) $\displaystyle\lim_{x\to -2^-} \frac{x+1}{4-x^2}$

(c) $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{2x^2+1}}{3x-5}$

(d) $\displaystyle\lim_{x\to 4} \frac{\frac{1}{x}-\frac{1}{4}}{2-\sqrt{x}}$

(e) $\displaystyle\lim_{x\to 0} \frac{\tan x - \sin(2x)}{x}$

[4] 2. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x+1}{x^2+x} & \text{if } \quad x < -1 \\ ax+b & \text{if } \ -1 \le x < 2 \\ x^2-2 & \text{if } \quad x \ge 2 \end{cases}$$

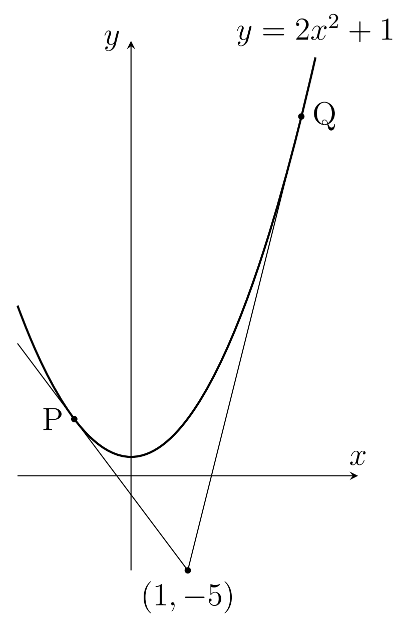
[3] 3. Sketch the graph of a function f such that all the following conditions are satisfied:

- $f(-5) = 0$, $f(-\frac{1}{2}) = 0$ and $f(3)$ is undefined;
- $\displaystyle\lim_{x\to -4} f(x) = \infty$, $\displaystyle\lim_{x\to 1^-} f(x) = \infty$ and $\displaystyle\lim_{x\to 1^+} f(x) = -\infty$;
- $\displaystyle\lim_{x\to \infty} f(x) = -3$.

[4] 4. Find the derivative of $f(x) = \sqrt{x^2+1}$, using the limit definition of the derivative. Verify your answer using the derivative rules.

[1] 5. Evaluate $\displaystyle\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{2}+h\right)-1}{h}$. (Hint : Interpret this as a derivative.)

[3] 6. Find (both coordinates of) each point on the parabola defined by $y = 2x^2+1$ at which the tangent line passes through the point $(1, -5)$.



[4] 7. Find an equation of the tangent line to the curve $x^2 + 2xy + 4y^2 = 13$ at the point $(-1, 2)$.

[15] 8. Find $\dfrac{dy}{dx}$ for each of the following. Do not simplify your answers.

(a) $y = \cos^2(x)\sec(x^2) + \log_3 x + \pi^e$

(b) $y = \dfrac{\tan^2(e^x - 3)}{\ln(3x^2 + 5)}$

(c) $y = (\ln(\cos(e^{3x+7})))^6$

(d) $y = (\cot x)^{\sin x}$

(e) $y = \sqrt[4]{\dfrac{x^5 \sin^2 x}{(x-5)^6}}$ (Use logarithmic differentiation.)

[3] 9. Prove that the equation $x^3 + 33x - 8 = 0$ has exactly one root. Use the intermediate value theorem and Rolle's theorem in your proof.

[5] 10. Given the function $f(x) = \dfrac{2}{x^2} - \dfrac{9}{x^4}$

(a) state the equations of all horizontal and vertical asymptotes of f

(b) find the intervals on which f is increasing or decreasing

(c) find all local maximum or minimum values of f

[9] 11. Given

$$f(x) = x(x-5)^{2/3} \qquad f'(x) = \frac{5(x-3)}{3(x-5)^{1/3}} \qquad f''(x) = \frac{10(x-6)}{9(x-5)^{4/3}}$$

with $3(2^{2/3}) \approx 5$, find:

(a) the domain of f,

(b) x- and y- intercepts,

(c) vertical and horizontal asymptotes, if any,

(d) intervals on which f is increasing or decreasing,

(e) local extrema,

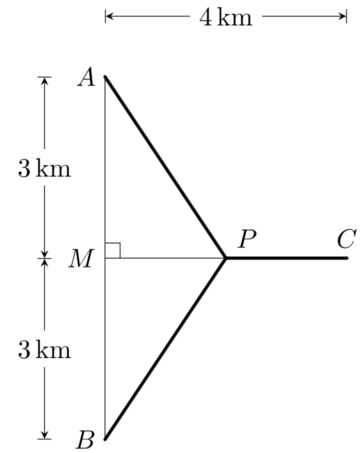
(f) intervals on which f is concave upward or downward,

(g) inflection point(s).

Sketch the graph of f. Label all intercepts, asymptotes, extrema and inflection point(s).

[4] 12. Find the absolute maximum and minimum of $f(t) = 4t^3 - 5t^2 - 8t + 3$ on $[-1, 1]$.

[6] 13. Factory A is 6 kilometres north of factory B, while power plant C is 4 kilometres east of the midpoint M of AB. Power is to be delivered to these two factories via a cable that will run from C to some point P (as in the diagram), where it will split into two branches going to A and B. How far away from the midpoint M should the branch point P be located in order to minimize the total length of the cable between A, B and C?



[3] 14. A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$ cm/s^2. Its initial velocity is $v(0) = -6$ cm/s. and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

[4] 15. Compute $\int_0^2 (2x^3 - 1)dx$ as a limit of Riemann sums. Note that $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

[12] 16. Evaluate each of the following integrals.

(a) $\int \left(e^x + x^3 + 3^x + e^3\right) dx$

(b) $\int \frac{(2x + \sqrt{x})^2}{x^3} dx$

(c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec\theta \tan\theta \csc\theta \, d\theta$

(d) $\int_{-3}^{2} |2x - 1| \, dx$

[3] 17. Evaluate the following limit by expressing it as a definite integral.

$$\lim_{n\to\infty} \frac{1}{n}\left(\sqrt[3]{\frac{1}{n}} + \sqrt[3]{\frac{2}{n}} + \sqrt[3]{\frac{3}{n}} + ... + \sqrt[3]{\frac{n}{n}}\right)$$

[3] 18. Use the Fundamental Theorem of Calculus to find the **second derivative** ($g''(x)$) of $g(x) = \int_{\ln(x)}^{x} te^t dt$.

[4] 19. **True** or **False**? Justify your answers!

(a) If $f(x) = \frac{x^3 - 4x}{x - 2}$, then f has a vertical asymptote at $x = 2$.

(b) If f is continuous at $x = a$ then it must be differentiable at $x = a$.

(c) If $\int f(x)dx = x^2 \ln x + C$, then $f(x) = x + 2x \ln x$.

(d) $\int_{\pi}^{\pi} \sqrt{\tan x} \, dx = 0$

1. a. The numerator and denominator each vanish as $x \to -2$, and factorising by inspection gives $x^2 + 2x = x(x+2)$ and $x^2 + 6x + 8 = (x+4)(x+2)$, so

$$\lim_{x \to -2} \frac{x^2 + 2x}{x^2 + 6x + 8} = \lim_{x \to -2} \frac{x}{x+4} = -1.$$

b. If $x \to -2$ and $x < -2$, then $x + 1 \to -1$, $4 - x^2 \to 0$ and $4 - x^2 < 0$, so $\displaystyle\lim_{\substack{x \to -2 \\ x < -2}} \frac{x+1}{4 - x^2} = \infty$.

c. Extracting dominant powers gives $\displaystyle\lim_{x \to -\infty} \frac{\sqrt{2x^2+1}}{3x - 5} = \lim_{x \to -\infty} \frac{-\sqrt{2 + x^{-2}}}{3 - 5x^{-1}} = -\tfrac{1}{3}\sqrt{2}$.

d. If $x > 0$, then

$$\frac{1}{x} - \frac{1}{4} = \frac{4-x}{4x} = \frac{(2-\sqrt{x})(2+\sqrt{x})}{4x}, \quad \text{and so} \quad \lim_{x \to 4} \frac{1/x - 1/4}{2 - \sqrt{x}} = \lim_{x \to 4} \frac{2 + \sqrt{x}}{4x} = \tfrac{1}{4}.$$

e. Since $\tan x - \sin(2x) = \sin x \sec x - 2 \sin x \cos x = (\sin x)(\sec x - 2\cos x)$, it follows that

$$\lim_{x \to 0} \frac{\tan x - \sin(2x)}{x} \lim_{x \to 0} \left\{ \frac{\sin x}{x} \cdot (\sec x - 2\cos x) \right\} = 1(1-2) = -1.$$

2. As $f(x) = 1/x$ if $x < -1$ and elsewhere f is a piecewise polynomial function, f is continuous on $(-\infty, -1)$, $[-1, 2)$ and $[2, \infty)$, so f is continuous on $\mathbb{R}$ if f is continuous at -1 and at 2. Now

$$\lim_{\substack{x \to -1 \\ x < -1}} f(x) = -1, \qquad \lim_{\substack{x \to -1 \\ x > -1}} f(x) = f(-1) = -a + b,$$

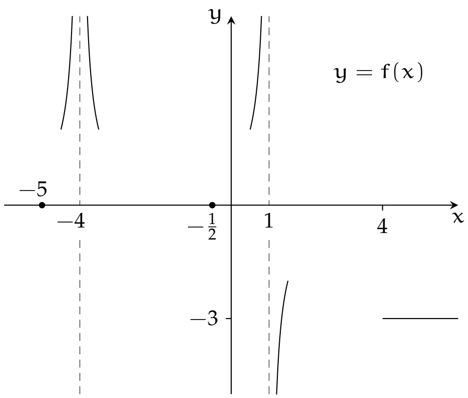
$$\lim_{\substack{x \to 2 \\ x < 2}} f(x) = 2a + b \qquad \text{and} \qquad \lim_{\substack{x \to 2 \\ x < 2}} f(x) = f(2) = 2.$$

Hence, f is continuous at -1 and at 2 if, and only if, $a - b = 1$ and $2a + b = 2$, *i.e.*, $3a = 3$, or $a = 1$, and $b = 0$. Therefore, f is continuous on $\mathbb{R}$ if, and only if, $a = 1$ and $b = 0$.

3. A portion of the graph of such a function, with domain

$$\{-5\} \cup \left[-\tfrac{9}{2}, -4\right) \cup \left(-4, -\tfrac{7}{2}\right] \cup \left\{-\tfrac{1}{2}\right\} \cup \left[\tfrac{1}{2}, 1\right) \cup \left(1, \tfrac{3}{2}\right] \cup [4, \infty),$$

is sketched below.



4. If $y = \sqrt{x^2+1}$ and $y' = \sqrt{x'^2 + 1}$, then $y' - y = \dfrac{y'^2 - y^2}{y' + y} = \dfrac{(x' - x)(x' + x)}{y' + y}$. Hence,

$$\frac{dy}{dx} = \lim_{x' \to x} \frac{y' - y}{x' - x} = \lim_{x' \to x} \frac{x' + x}{y' + y} = \frac{2x}{2y} = \frac{x}{\sqrt{x^2+1}}.$$

This justifies, in this case, the tricks used in $\frac{d}{dx}(x^2+1)^{1/2} = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2+1}}$.

5. By inspection, $\displaystyle\lim_{h \to 0} \frac{\sin\left(\frac{1}{2}\pi + h\right) - 1}{h} = \frac{d}{dx}\{\sin x\}\Big|_{x = \frac{1}{2}\pi} = \cos\left(\tfrac{1}{2}\pi\right) = 0$.

6. The tangent line at (x, y) to the curve defined by $y = 2x^2 + 1$ contains $(1, -5)$ if, and only if,

$$\frac{y+5}{x-1} = \frac{dy}{dx}, \quad \textit{i.e.,} \quad \frac{2x^2+6}{x-1} = 4x, \quad \text{or} \quad x^2 + 3 = 2x^2 - 2x;$$

equivalently, $0 = x^2 - 2x - 3 = (x+1)(x-3)$. Therefore, the tangent lines to the parabola at the points $(-1, 3)$ and $(3, 19)$—and no other points—contain $(1, -5)$.

7. If $x^2 + 2xy + 4y^2 = 13$, then implicit differentiation gives

$$\frac{dy}{dx}\bigg|_{\substack{x = -1 \\ y = 2}} = -\frac{x+y}{x+4y}\bigg|_{\substack{x = -1 \\ y = 2}} = -\frac{-1+2}{-1+8} = -\tfrac{1}{7},$$

so the tangent line to the curve at the point $(-1, 2)$ is defined by $x + 7y = 13$.

8. a. If $y = \cos^2(x)\sec(x^2) + \log_3(x) + \pi^e = \dfrac{\cos^2(x)}{\cos(x^2)} + \dfrac{\log x}{\log 3} + \pi^e$, then

$$\frac{dy}{dx} = -\frac{\sin(2x)}{\cos(x^2)} + \frac{2x\cos^2(x)\sin(x^2)}{\cos^2(x^2)} + \frac{1}{x \log 3}.$$

b. If $y = \dfrac{\tan^2(e^x - 3)}{\log(3x^2+5)}$, then $\dfrac{dy}{dx} = \dfrac{2e^x \tan(e^x - 3)\sec^2(e^x - 3)}{\log(3x^2+5)} - \dfrac{6x \tan^2(e^x - 3)}{(3x^2+5)\left(\log(3x^2+5)\right)^2}$.

c. If $y = \left(\log(\cos(e^{3x+7}))\right)^6$, then $\dfrac{dy}{dx} = -18e^{3x+7}\left(\log(\cos(e^{3x+7}))\right)^5 \tan(e^{3x+7})$.

d. If $y = (\cot x)^{\sin x}$, then logarithmic differentiation gives

$$\frac{dy}{dx} = y\frac{d}{dx}\{\log y\} = -(\cot x)^{\sin x}\{(\cos x)\log(\tan x) + \sec x\}.$$

e. If $y = \sqrt[4]{\dfrac{x^5 \sin^2(x)}{(x-5)^6}}$, then logarithmic differentiation gives

$$\frac{dy}{dx} = y\frac{d}{dx}\{\log|y|\} = \frac{1}{4}\sqrt[4]{\frac{x^5\sin^2(x)}{(x-5)^6}}\left\{\frac{5}{x} + 2\cot(x) + \frac{6}{5-x}\right\}$$

9. Since $f(x) = x^3 + 33x - 8$ is a polynomial in x, the Intermediate Value Theorem and the Mean Value Theorem apply to f on any closed interval of positive length. Now

$$f(0) = -8 < 0 \qquad \text{and} \qquad f(1) = 26 > 0,$$

so the Intermediate Value Theorem implies that there is a real number ξ such that $0 < \xi < 1$ and $f(\xi) = 0$. If $\xi' \neq \xi$, then the Mean Value Theorem implies that there is a real number η between ξ and ξ' such that

$$f(\xi') - f(\xi) = f'(\eta)(\xi' - \xi), \qquad \text{or} \qquad f(\xi') = (3\eta^2 + 33)(\xi' - \xi),$$

and hence $|f(\xi')| \geqslant 33|\xi' - \xi| > 0$. Therefore, ξ is the unique real zero of f.

10. If

$$f(x) = \frac{2}{x^2} - \frac{9}{x^4} = \frac{2x^2 - 9}{x^4}, \qquad \text{then} \qquad f'(x) = -\frac{4}{x^3} + \frac{36}{x^5} = \frac{4(9 - x^2)}{x^5}.$$

a. Since f is continuous at every real number besides zero, $\displaystyle\lim_{x \to 0} f(x) = -\infty$ and $\displaystyle\lim_{x \to \pm\infty} f(x) = 0$, the asymptotes of the graph of f are defined by $x = 0$ and $y = 0$.

b. Since $f'(x) > 0$ if $x < -3$ or $0 < x < 3$, and $f'(x), 0$ if $-3 < x < 0$ or $3 < x$, f is increasing on the intervals $(-\infty, -3]$ and $(0, 3]$ (**NOT on the union** $(-\infty, -3] \cup (0, 3]$; for example, $-4 < 1$ but $f(-4) > 0 > f(1)$), and decreasing on the intervals $[-3, 0)$ and $[3, \infty)$.

c. From Parts a and b, it follows that $f(\pm 3) = \frac{1}{9}$ is the (local and global) maximum value of f, and that f has no (local or global) minimum values.

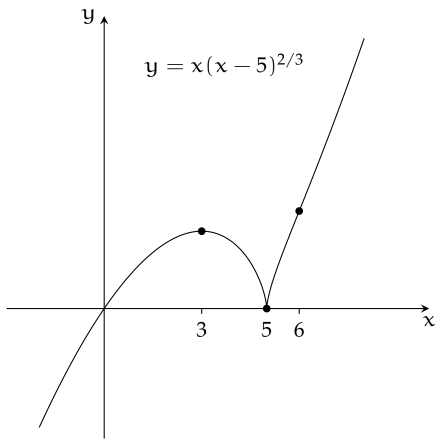
11. Since $y = x(x-5)^{2/3}$ is a continuous function of x on $\mathbb{R}$, and $y = x^{5/3}(1-5x^{-1})^{2/3}$ if $x \neq 0$, the curve has no vertical, horizontal or oblique asymptotes, nor any global extrema. The axis intercepts of the curve are $(0,0)$ and $(5,0)$. Now

$$\frac{dy}{dx} = \frac{5(x-3)}{3(x-5)^{1/3}}, \quad \text{so} \quad \frac{dy}{dx} > 0 \text{ if } x < 3 \text{ or } 5 < x, \quad \text{and} \quad \frac{dy}{dx} < 0 \text{ if } 3 < x < 5.$$

Hence, y is increasing on $(-\infty, 3]$ and on $[5, \infty)$, decreasing on $[3,5]$, and has a local maximum at $(3, 3\sqrt[3]{4})$ and a local minimum at $(5,0)$. Next,

$$\frac{d^2y}{dx^2} = \frac{10(x-6)}{9(x-5)^{4/3}}, \quad \text{so} \quad \frac{d^2y}{dx^2} > 0 \text{ if } 6 < x, \quad \text{and} \quad \frac{d^2y}{dx^2} < 0 \text{ if } x < 5 \text{ or } 5 < x < 6.$$

So the curve is concave up on $[6, \infty)$, concave down on $(-\infty, 5]$ and on $[5, \infty)$, and has a point of inflection at $(6,6)$. In the sketch (which is not to scale—the x-axis is dilated by a factor of 2), the points of interest are emphasised.



12. If $f(t) = 4t^3 - 5t^2 - 8t + 3$, then $f'(t) = 12t^2 - 10t - 8 = 2(2t+1)(3t-4)$, so the critical number of f in $(-1,1)$ is $-\frac{1}{2}$. Since $f(-1) = 2$, $f\left(-\frac{1}{2}\right) = \frac{21}{4}$ and $f(1) = -6$, the largest and smallest values of f on $[-1,1]$ are, respectively, $\frac{21}{4}$ and -6.

13. If x is the distance between M and P, and y is the distance between P and C (each measured in kilometres), then $0 \leqslant x \leqslant 4$ and $x + y = 4$, so $\frac{dy}{dx} = -1$. The total length of the cable is (by Pythagoras' formula) $\ell = 2\sqrt{x^2 + 3^2} + y$. By First Derivative Test (or Snellius' principle), the minimum value of ℓ occurs where

$$\frac{2x}{\sqrt{x^2+3^2}} = 1, \qquad \textit{i.e.,} \qquad 4x^2 = x^2 + 3^2, \qquad \text{or} \qquad x = \sqrt{3} \quad (\text{since } x \geqslant 0).$$

Therefore, P should be $\sqrt{3}$ kilometres east of M to minimise the total length of the cable.

14. If $a = \frac{dv}{dt} = 6t + 4$ and $v_0 = -6$, then by inspection, $v = 3t^2 + 4t - 6$. Likewise, since $v = \frac{ds}{dt}$, if the initial position of the particle is $s_0 = 9$ then $s = t^3 + 2t^2 - 6t + 9$.

15. If $[0,2]$ is divided into k subintervals of equal length, then the corresponding right endpoint Riemann sum of $2x^3 - 1$ is

$$\begin{aligned}\mathcal{R}_k &= \frac{2}{k}\sum_{j=1}^{k}\left\{2\left(\frac{2}{k}j\right)^3 - 1\right\} = \frac{2}{k}\left\{\frac{16}{k^3}\sum_{j=1}^{k} j^3 - k\right\} = 2\left\{\frac{16}{k^4}\cdot\frac{1}{4}k^2(k+1)^2 - 1\right\} \\ &= 2\left\{4\left(1+\frac{1}{k}\right)^2 - 1\right\}.\end{aligned}$$

Therefore, $\displaystyle\int_0^2 (2x^3 - 1)\, dx = \lim_{k\to\infty} \mathcal{R}_k = 2(4-1) = 6.$

16. a. Integrating by inspection (and noting that $3^x = e^{(\log 3)x}$) gives

$$\int (e^x + x^3 + 3^x + e^3)\, dx = e^x + \tfrac{1}{4}x^4 + 3^x(\log 3)^{-1} + e^3 x + a.$$

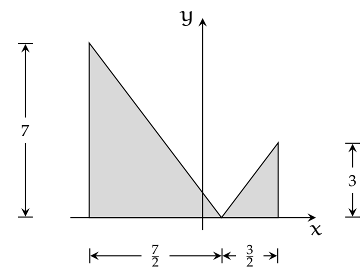
b. Since $(2x + \sqrt{x})^2 = 4x^2 + 4x\sqrt{x} + x$, it follows that

$$\int \frac{(2x+\sqrt{x})^2}{x^3}\, dx = \int \left(4x^{-1} + 4x^{-3/2} + x^{-2}\right) dx = 4\log x - 8x^{-1/2} - x^{-1} + b.$$

c. Since $\sec\vartheta \tan\vartheta \csc\vartheta = \sec^2\vartheta$, it follows that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sec\vartheta \tan\vartheta \csc\vartheta\, d\vartheta = \tan\vartheta \Big|_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} = \sqrt{3} - \tfrac{1}{3}\sqrt{3} = \tfrac{2}{3}\sqrt{3}.$$

d. Below is a sketch of the graph of $y = |2x - 1|$ on $[-3, 2]$ (not to scale).



The definite integral is the sum of the areas of the shaded triangles, *i.e.*,

$$\int_{-3}^{2} |2x - 1|\, dx = \tfrac{49}{4} + \tfrac{9}{4} = \tfrac{29}{2}.$$

17. The expression in the limit is a right endpoint Riemann sum of $\sqrt[3]{x}$, with $[0,1]$ divided into n subintervals of equal length, *i.e.*,

$$\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{\nu=1}^{n}\sqrt[3]{\frac{\nu}{n}}\right\} = \int_0^1 \sqrt[3]{x}\, dx = \tfrac{3}{4}x^{4/3}\Big|_0^1 = \tfrac{3}{4}.$$

18. By the interval additivity of the definite integral and the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_{\log x}^{x} te^t\, dt = \frac{d}{dx}\int_0^x te^t\, dt - \frac{d}{dx}\int_0^{\log x} te^t\, dt = xe^x - \frac{(\log x)e^{\log x}}{x} = xe^x - \log x.$$

Therefore,

$$\frac{d^2}{dx^2}\int_{\log x}^{x} te^t\, dt = \frac{d}{dx}\{xe^x - \log x\} = e^x(x+1) - x^{-1},$$

where the last expression is interpreted only for positive values of x.

19. a. If $x \neq 2$, then

$$y = \frac{x^3 - 4x}{x-2} = \frac{x(x-2)(x+2)}{x-2} = x(x+2), \qquad \text{and hence} \qquad \lim_{x\to 2} y = 8.$$

So the curve has a hole, not a vertical asymptote, where $x = 2$, and the statement is false.

b. The absolute value function is continuous but not differentiable at 0, so the statement is false.

c. Since $\dfrac{d}{dx}\{x^2 \log x\} = 2x\log x + x^2 \cdot x^{-1} = 2x\log x + x$, the statement is true.

d. Since $\sqrt{\tan x}$ is defined if $x = \pi$, the definite integral of $\sqrt{\tan x}$ on $[\pi, \pi]$ is zero, so the statement is true.