(Marks)

(5) 1. Let $f(x) = \begin{cases} \dfrac{x - \sin x}{x^3} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ and let $g(x) = \displaystyle\int_0^x f(t)\,dt$

(a) Find the Maclaurin series of g.

(b) How many terms of the series are required to estimate $g(x)$ to within 10^{-7}, if $-1 \leq x \leq 1$? Justify your answer.

(4) 2. Find the Maclaurin series of $f(x) = \ln\left(x + \sqrt{1+x^2}\,\right)$. (*Hint:* $f'(x) = (1+x^2)^{-1/2}$.)

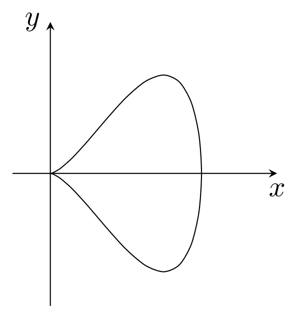
(6) 3. Let $g(x) = \sqrt[4]{x}$. Find the third degree Taylor polynomial $T_3(x)$ for g centered at 16. Use the Lagrange form of the remainder (or Taylor's Inequality) to find an upper bound on the error if $T_3(x)$ is used to estimate $\sqrt[4]{15}$.

(6) 4. (a) On the same set of axes, sketch the graphs of $r = 1 + 2\cos\theta$ and $r = 4\cos\theta$ and find all points of intersection.

(b) Set up appropriate integral(s) needed to compute the area of the region inside $r = 1 + 2\cos\theta$ and outside $r = 4\cos\theta$. (Do not evaluate the integral(s).)

(8) 5. Given the curve $\mathcal{C}$ with parametric equations $x = 1 - \cos t$, $y = (1 - \cos t)\sin t$:

(a) Find $\dfrac{dy}{dx}$. On the graph (right), indicate the direction of increasing t (the orientation).



(b) Find where $\mathcal{C}$ has horizontal and vertical tangents.

(c) Find the area of the region enclosed by $\mathcal{C}$.

(d) Set up, but do not evaluate, an integral which represents the length of $\mathcal{C}$.

(9) 6. Sketch and name each of the following surfaces in $\mathbb{R}^3$. Show all relevant work.

(a) $z = \sqrt{x^2 - 4y^2 + 1}$ (b) $r = 2\sec\theta$ (c) $\rho = \cos\varphi$

(15) 7. A particle P moves along a curve $\boldsymbol{r}(t) = t\,\boldsymbol{i} + 2\cos t\,\boldsymbol{j} + 2\sin t\,\boldsymbol{k}$.

(a) Draw a rough sketch of the curve.

(b) Calculate the length of the curve from $t = 0$ to $t = \pi$.

(c) Find the unit tangent vector $\boldsymbol{T}(t)$, the unit normal vector $\boldsymbol{N}(t)$, the curvature $\kappa(t)$, and the tangential and normal components a_T, a_N of acceleration.

(d) Find the equation of the osculating plane (*i.e.* the plane spanned by $\boldsymbol{T}, \boldsymbol{N}$) at the point where $t = 0$.

(3) 8. Let $z = f(x, y)$ be a surface, and $f(x, y) = c$ one of its level curves in the xy-plane. Assuming this curve is represented by the vector equation $\boldsymbol{r}(t) = \langle x(t), y(t)\rangle$, use the chain rule to show that the gradient of f is always perpendicular to the level curve.

(Marks)

(3) 9. Is the following function continuous at the origin? $f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+2y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(3) 10. If f, g are differentiable functions, and $z = f(x) + g(y)$, $x = s - at$, $y = s + at$, then show that $\dfrac{\partial^2 z}{\partial t^2} = a^2 \dfrac{\partial^2 z}{\partial s^2}$.

(5) 11. Given the (level) surface $g(x,y,z) = x^3 + y^3 + z^3 - xyz = 0$,

(a) Find the directional derivative of g at the point $P(0,-1,1)$ in the direction of $\boldsymbol{v} = \langle 2,-1,2 \rangle$.

(b) Find the equation of the tangent plane to the surface at P.

(c) Show that the space curve $\boldsymbol{r}(t) = \left\langle \frac{1}{4}t^3 - 2, \frac{4}{t} - 3, \cos(t-2) \right\rangle$ is tangent to the surface at P.

(5) 12. Find and classify the critical points of $f(x,y) = 4xy - x^2y - y^3$.

(6) 13. The plane $x + y + z = 12$ intersects the paraboloid $z = x^2 + y^2$ to form an ellipse. Use the method of Lagrange Multipliers to find the point on the intersection that is closest to, and the point on the intersection furthest from, the origin.

(8) 14. Evaluate

(a) $\displaystyle\int_0^1 \int_{\sqrt{3}y}^{\sqrt{4-y^2}} e^{x^2+y^2}\, dx\, dy$ (b) $\displaystyle\int_0^1 \int_{\arcsin(y)}^{\pi/2} e^{\cos x}\, dx\, dy$

(4) 15. Convert $\displaystyle\int_0^{\pi/4} \int_0^{3\sec\theta} r^3 \sin^2\theta\, dr\, d\theta$ to Cartesian coordinates. Evaluate the integral.

(6) 16. Let $\mathcal{S}$ be the solid bounded above by the hemisphere $z = \sqrt{25 - x^2 - y^2}$, below by the xy-plane, and laterally by the cylinder $x^2 + y^2 = 9$. Set up (but do not evaluate) triple integrals to find the volume of $\mathcal{S}$ in:

(a) Cartesian coordinates; (b) cylindrical coordinates; (c) spherical coordinates.

(4) 17. Evaluate $\displaystyle\iint_{\mathcal{R}} \cos(x-y)\, dA$, where $\mathcal{R}$ is the region bounded by $x - y = 0$, $x - y = \frac{1}{2}\pi$, $x + y = 2$, $x + y = 4$.

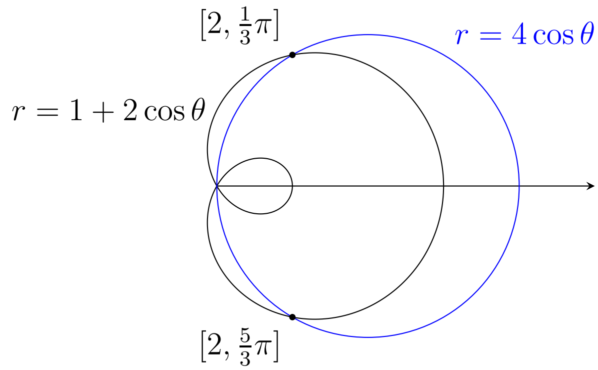
Answers

1. (a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)(2n+3)!}$ (b) 4 terms

2. $\displaystyle x + \sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)!!x^{2n+1}}{n!2^n(2n+1)}$

3. $T_3(x) = 2 + \frac{1}{2\cdot 16}(x-16) - \frac{3}{2!2^7 4^2}(x-16)^2 + \frac{3\cdot 7}{3!2^{11}4^3}(x-16)^3$ $\quad |R_3| \leq 1.4 \times 10^{-6}$

4.

(a) 

Intersection also at the origin.

(b) $\displaystyle A = \int_{\pi/3}^{2\pi/3} (1 + 2\cos\theta)^2 \, d\theta - \int_{\pi/3}^{\pi/2} (4\cos\theta)^2 \, d\theta$

5. (a) $\displaystyle\frac{dy}{dx} = \frac{\cos t + \sin^2 t - \cos^2 t}{\sin t}$. The orientation is clockwise.

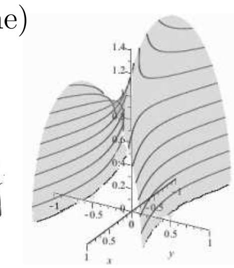
(b) HT: $t = 0$ (the origin); $t = \pm 2\pi/3$, *i.e.* $(3/2, \pm 3\sqrt{3}/4)$
VT: $t = \pi$, *i.e.* $(2, 0)$

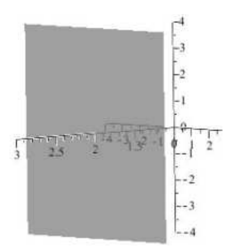
(c) π (d) $2\int_0^{\pi} \sqrt{\sin^2 t + (\cos t + \sin^2 t - \cos^2 t)^2} \, dt$

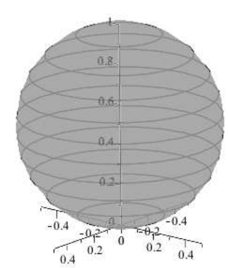
6. (a) Top-half of a hyperboloid of 1 sheet

(b) Plane (parallel to the yz plane)

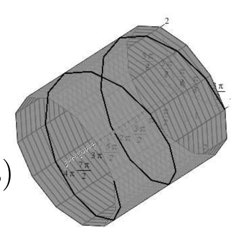
(c) Sphere







7. (a) Helix
(along x axis)



(b) $\pi\sqrt{5}$

(c) $\boldsymbol{T}(t) = \frac{1}{\sqrt{5}}\langle 1, -2\sin t, 2\cos t\rangle$, $\boldsymbol{N}(t) = \langle 0, -\cos t, -\sin t\rangle$, $\kappa(t) = 2/5$, $a_T = 0$, $a_N = 2$

(d) $2x - z = 0$

8. (Assuming the functions are differentiable:) $0 = \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \boldsymbol{\nabla} f \cdot \boldsymbol{r}'$ so $\boldsymbol{\nabla} f$ is perpendicular to (the tangent $\boldsymbol{r}'$ to) the curve.

9. Use the "Squeeze Theorem" to show $\frac{3x^2y}{x^2+2y^2} \to 0$ as $(x,y) \to (0,0)$. So f is continuous.

10. $\frac{\partial z}{\partial s} = f' + g'$, $\frac{\partial z}{\partial t} = -af' + ag'$, so $\frac{\partial^2 z}{\partial s^2} = f'' + g''$, $\frac{\partial^2 z}{\partial t^2} = a^2 f'' + a^2 g''$, so QED.

11. (a) $g_u = 5/3$ (b) $x + 3y + 3z = 0$ (c) $\boldsymbol{r}' \cdot \boldsymbol{n} = \langle 3, -1, 0\rangle \cdot \langle 1, 3, 3\rangle = 0$

12. $(2, 2/\sqrt{3})$ local max; $(2, -2/\sqrt{3})$ local min; $(0,0), (4,0)$ saddle pts.

13. Closest at $(2,2,8)$; furthest at $(-3,-3,18)$.

14. (a) $\frac{\pi}{12}(\mathrm{e}^4 - 1)$ (b) $\mathrm{e} - 1$

15. $\displaystyle\int_0^3 \int_0^x y^2 \, dy \, dx = 27/4$

16. (a) $\displaystyle\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{\sqrt{25-x^2-y^2}} dz \, dy \, dx$

    (b) $\displaystyle\int_0^{2\pi} \int_0^3 \int_0^{\sqrt{25-r^2}} r \, dz \, dr \, d\theta$

    (c) $\displaystyle\int_0^{2\pi} \int_0^{\arctan(3/4)} \int_0^5 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta + \int_0^{2\pi} \int_{\arctan(3/4)}^{\pi/2} \int_0^{3/\sin\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

17. 1