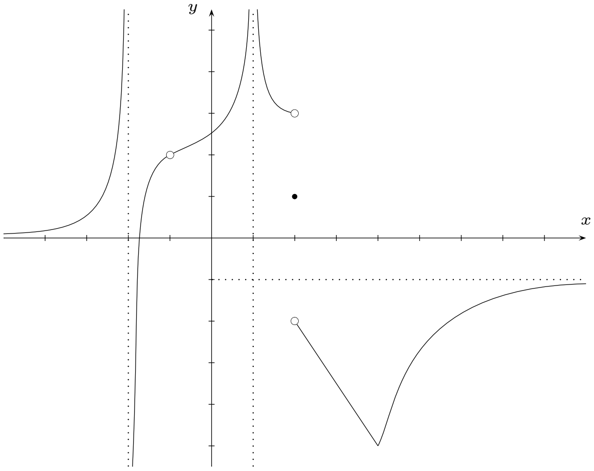
201-NYA-05 May 2010 Final Exam

(1) Refer to the following sketch (with unit lengths marked along the coordinate axes) to answer the questions below.



(a) Evaluate the following. Use ∞, $-\infty$ or "does not exist" as appropriate.

(i) $\lim_{x\to -2^-} f(x)$
(ii) $\lim_{x\to -1} f(x)$
(iii) $\lim_{x\to -\infty} f(x)$
(iv) $f(2)$
(v) $\lim_{x\to 1} f(x)$
(vi) $\lim_{x\to 2} f(x)$
(vii) $\lim_{x\to 4} f(x)$
(viii) $\lim_{x\to \infty} f(x)$

(b) List the values of x at which f is discontinuous.
(c) List the values of x at which f is continuous but not differentiable.

(2) Evaluate the following limits. Use ∞, $-\infty$ or "does not exist" as appropriate.

(a) $\lim_{x\to -2} \dfrac{x^3-4x}{3x^2+7x+2}$
(b) $\lim_{x\to \infty} \dfrac{2x^3-2x^2+4x-1}{8-5x+3x^2-3x^3}$
(c) $\lim_{\vartheta\to 0} \dfrac{\sqrt{2\vartheta+3}-\sqrt{3}}{\sin\vartheta}$
(d) $\lim_{x\to 3} \dfrac{\dfrac{3}{7}-\dfrac{x}{3x-2}}{x-3}$

(3) (a) State the definition of the derivative of a function f.
(b) Use the definition to find the derivative of $f(x)=x/(x-2)$.
(c) Check your answer to Part b using the laws of differentiation.

(4) Find all values of c such that

$$f(x)=\begin{cases}\dfrac{3c}{x^2-10} & \text{if } x<-4, \text{ and}\\ \sqrt{c-2x} & \text{if } -4\le x\le -2.\end{cases}$$

is continuous at -4.

(5) Find an equation of each line which is tangent to the graph of $y=x^2+3x+4$ and passes through the point $(2,5)$.

(6) Sketch the graph of $f(x)=x^{2/3}(x-3)$, given that $f'(x)=\dfrac{5x-6}{3x^{1/3}}$ and $f''(x)=\dfrac{10x+6}{9x^{4/3}}$. Make sure your solution includes all intercepts, asymptotes, intervals of monotonicity and concavity, extrema, and points of inflection.

(7) Find all absolute extrema of:

(a) $f(x)=x^{1/3}e^{3x}$ on $[-1,1]$;
(b) $g(x)=\dfrac{2\cos x}{\sin x-2}$ on $[-\frac{1}{2}\pi,\frac{1}{2}\pi]$.

(8) Find all critical numbers of the function $y=(4x+3)^3(3x+3)^4$.

(9) Find the dimensions of the rectangle of largest area which has two vertices on the x-axis and two vertices above the x-axis, bounded by the curve $y=16-2x^2$.

(10) For each of the following, find $\dfrac{dy}{dx}$.

(a) $y=x^7(x^3-2x\sin 3x)^{2/3}$
(b) $y=5x^\pi-5/x+\sqrt[5]{x^2}-\log_5 x+5\cdot 2^x$
(c) $y=\ln\sqrt[3]{\tan 2x^4}$
(d) $y=\sec\big(\cos(\tan\pi x)\big)$
(e) $xy^2+y\ln x=x$

(11) Use logarithmic differentiation to find $\dfrac{dy}{dx}$.

(a) $y=\dfrac{\sec^2(5x-2)}{x^{12}e^{-x}}$
(b) $y=(\cos x)^{5x^2+1}$

(12) Find an equation of the normal line to the curve $x^3y-2y^3-x+3y=-11$ at the point $(-1,2)$.

(13) Let $f(x)=e^{2x}$. Find a simplified formula for $f^{(n)}(x)$.

(14) A ladder 12 metres long is leaning against a wall. The top does not reach high enough so the bottom of the ladder gets pushed towards the wall at a rate of 10 cm/s. What is the rate of change of the acute angle between the ladder and the floor when the ladder reaches 8 metres up the wall?

(15) Find $f(x)$ given that $f''(x)=4-6x-4x^3$, $f(1)=2$ and $f'(-1)=1$.

(16) For the integral $\displaystyle\int_1^4 \frac{1}{x}\,dx$, approximate its value with a Riemann sum with $n=6$ rectangles using midpoints as sample points.

(17) Evaluate the following integrals.

(a) $\displaystyle\int_1^2 (3x+1)^2\,dx$
(b) $\displaystyle\int (2^x+\sin x+\pi)\,dx$
(c) $\displaystyle\int_{\frac{1}{9}\pi^2}^{\frac{1}{9}\pi^2} \sqrt{\cos\sqrt{x}}\,dx$
(d) $\displaystyle\int \frac{x^3+x^2+x+1}{x}\,dx$
(e) $\displaystyle\int_0^{\frac{1}{3}\pi} \frac{\sin x}{\cos^2 x}\,dx$

(18) Find the derivative of the function $f(x)=\displaystyle\int_{x^2}^0 \sqrt{1+t^2}$.

(19) Sketch the graph of

$$f(x)=\begin{cases}2x-2 & \text{if } x\le 3, \text{ and}\\ 4 & \text{if } x>3.\end{cases}$$

and evaluate $\displaystyle\int_0^6 f(x)\,dx$ by interpreting it in terms of area.

Answers and/or Solutions:

(1) (a) By inspecting the given sketch:

(i) $\lim_{x\to-2^-} f(x) = \infty$;

(ii) $\lim_{x\to-1} f(x) = 2$;

(iii) $\lim_{x\to-\infty} f(x) = 0$;

(iv) $f(2) = 1$;

(v) $\lim_{x\to1} f(x) = \infty$;

(vi) $\lim_{x\to2} f(x)$ does not exist, because $\lim_{x\to2^-} f(x) = 3$ and $\lim_{x\to2^+} f(x) = -2$;

(vii) $\lim_{x\to4} f(x) = -5$;

(viii) $\lim_{x\to\infty} f(x) = -1$.

(b) f has infinite discontinuities at -2 and 1, a removable discontinuity at -1, and a jump discontinuity at 2.

(c) f is continuous but not differentiable at 4, since $\lim_{t\to4^-}\dfrac{f(t)-f(4)}{t-4} = -\frac{3}{2}$ and $\lim_{t\to4^+}\dfrac{f(t)-f(4)}{t-4} > 0$.

(2) (a) Factoring the numerator and denominator and simplifying, gives

$$\lim_{x\to-2}\frac{x^3-4x}{3x^2+7x+2} = \lim_{x\to-2}\frac{x(x-2)(x+2)}{(3x+1)(x+2)} = \lim_{x\to-2}\frac{x(x-2)}{3x+1} = -\tfrac{8}{5}.$$

(b) Extracting the dominant powers of x from the numerator and denominator gives

$$\lim_{x\to\infty}\frac{2x^3-2x^2+4x-1}{8-5x+3x^2-3x^3} = \lim_{x\to\infty}\frac{2-2/x+4/x^2-1/x^3}{8/x^3-5/x^2+3/x-3} = -\tfrac{2}{3}.$$

(c) Rationalizing the numerator, and using the basic limit $\lim_{\vartheta\to0}\dfrac{\sin\vartheta}{\vartheta} = 1$, gives

$$\lim_{\vartheta\to0}\frac{\sqrt{2\vartheta+3}-\sqrt{3}}{\sin\vartheta} = \left(\lim_{\vartheta\to0}\frac{\sin\vartheta}{\vartheta}\right)^{-1}\cdot\lim_{\vartheta\to0}\frac{2}{\sqrt{2\vartheta+3}+\sqrt{3}} = \tfrac{1}{3}\sqrt{3}.$$

(d) Simplifying the complex rational expression in the limit gives

$$\lim_{x\to3}\frac{\dfrac{3}{7}-\dfrac{x}{3x-2}}{x-3} = \lim_{x\to3}\frac{2(x-3)}{7(3x-2)(x-3)} = \lim_{x\to3}\frac{2}{7(3x-2)} = \tfrac{2}{49}.$$

(3) (a) The derivative f' of a function f is defined by

$$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

or, equivalently,

$$f'(x) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x},$$

and the domain of f' is the set of all real numbers x such that this limit exists.

(b)

$$f'(x) = \lim_{t\to x}\frac{\dfrac{t}{t-2}-\dfrac{x}{x-2}}{t-x}.$$

Simplifying this expression, and applying the independence and direct substitution properties of limits then gives

$$\begin{aligned} f'(x) = &= \lim_{t\to x}\frac{t(x-2)-x(t-2)}{(t-2)(x-2)(t-x)} = \lim_{t\to x}\frac{-2(t-x)}{(t-2)(x-2)(t-x)} \\ &= \lim_{t\to x}\frac{-2}{(t-2)(x-2)} \\ &= -\frac{2}{(x-2)^2}, \end{aligned}$$

and the domain of f' is equal to the domain of f (namely $\mathbb{R}\setminus\{2\}$).

(4) Observe that (since we must have $c \geq -4$ for f to be defined on $[-4,-2]$)

$$\lim_{x\to-4^-} f(x) = \tfrac{1}{2}c \quad\text{and}\quad f(-4) = \lim_{x\to-4^+} f(x) = \sqrt{c+8}.$$

So f is continuous at -4 if, and only if, $c = 2\sqrt{c+8}$. This equation implies that $c^2 = 4(c+8)$, or $0 = c^2-4c-32 = (c-8)(c+4)$, *i.e.*, $c = 8$ or $c = -4$. However, only 8 is a solution of the original equation ($c = -4$ turns the original equation into the contradiction $-4 = 4$). Therefore, f is continuous at -4 if, and only if, $c = 8$.

(5) The equation of the line tangent to the graph of $y = x^2+3x+4$ at the point where $x = \xi$ has slope $2\xi+3$, equation $y = \xi^2+3\xi+4+(2\xi+3)(x-\xi)$, and passes through the point $(2,5)$ if, and only if $5 = \xi^2+3\xi+4+(2\xi+3)(2-\xi)$, or $0 = \xi^2-4\xi-5 = (\xi-5)(\xi+1)$, *i.e.*, $\xi = 5$ or $\xi = -1$, where the slope of tangent is, respectively $2(5)+3 = 13$ and $2(-1)+3 = 1$. Therefore, the line $y = 5+13(x-2) = 13x-21$, which is tangent to the parabola at $(5,44)$, and the line $y = 5+(x-2) = x+3$, which is tangent to the parabola at $(-1,2)$, each pass through $(2,5)$, as does no other line tangent to the parabola.

(6) The domain of $f(x) = x^{2/3}(x-3)$ is $\mathbb{R}$, on which f is continuous, so its graph has no vertical asymptotes. Since $f(x) = x^{5/3}(1-3/x)$ for $x \neq 0$, $f(x) \to \pm\infty$ as $x \to \pm\infty$, and the graph of f has no horizontal or oblique asymptotes, or global extrema. $f(x) = 0$ if $x = 0$ or $x = 3$, so the origin and $(3,0)$ are the intercepts of the graph of f. Since $f(x) = x^{2/3}(x-3) = x^{5/3}-3x^{2/3}$, one has

$$f'(x) = \tfrac{5}{3}x^{2/3}-2x^{-1/3} = \tfrac{1}{3}x^{-1/3}(5x-6) \quad\text{for}\quad x\neq0,$$

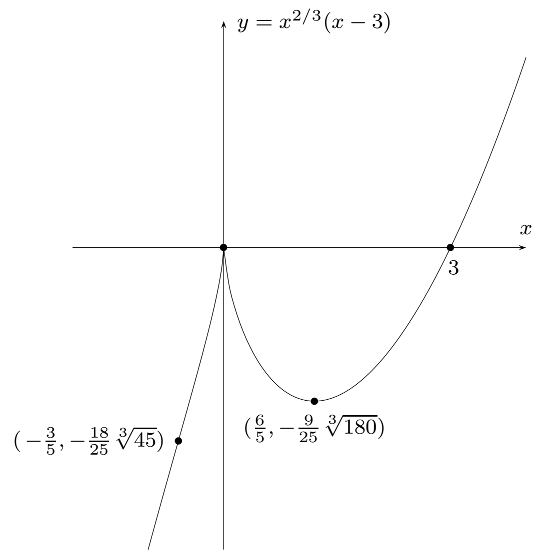
so the critical numbers of f are 0 and $\frac{6}{5}$, $f'(x) > 0$ if $x < 0$ or $x > \frac{6}{5}$ and $f'(x) < 0$ if $0 < x < \frac{6}{5}$. Therefore, f is increasing on $(-\infty,0)$ and on $(\frac{6}{5},\infty)$, and decreasing on $(0,\frac{6}{5})$, with a local maximum at the origin and a local minimum at $(\frac{6}{5},-\frac{9}{25}\sqrt[3]{180})$. Next,

$$f''(x) = \tfrac{10}{9}x^{-1/3}+\tfrac{2}{3}x^{-4/3} = \tfrac{2}{9}x^{-4/3}(5x+3) \quad\text{for}\quad x\neq0,$$

so $f''(x) < 0$ if $x < -\frac{3}{5}$ and $f''(x) > 0$ if $-\frac{3}{5} < x < 0$ or $x > 0$. Therefore, f is concave down on $(-\infty,-\frac{5}{3})$, and concave up on $(-\frac{5}{3},0)$ and on $(0,\infty)$, with a point of inflection at $(-\frac{3}{5},-\frac{18}{25}\sqrt[3]{45})$. Since

$$\lim_{t\to0^\pm}\frac{f(t)-f(0)}{t-0} = \lim_{t\to0^\pm}\frac{t^{2/3}(t-3)}{t} = \lim_{t\to0^\pm} t^{-1/3}(t-3) = \mp\infty,$$

it follows that the graph of f has a vertical cusp at the origin. Below is a sketch of the graph of f, with the points of interest emphasized.



(7) (a) If $f(x) = x^{1/3}e^{3x}$, then

$$f'(x) = \tfrac{1}{3}x^{-2/3}e^{3x}+x^{1/3}e^{3x}3 = \tfrac{1}{3}x^{-2/3}e^{3x}(1+9x) \quad\text{for}\quad x\neq0.$$

Therefore, the critical numbers of f in $[-1,1]$ are $-\frac{1}{9}$ and 0. Evaluating f at these critical numbers and at the endpoints of $[-1,1]$ gives $f(-1) = -e^{-3}$, $f(-\frac{1}{9}) = -\sqrt[3]{(9e)^{-1}}$, $f(0) = 0$, and $f(1) = e^3$. Clearly e^3 is the largest value of f on $[-1,1]$. Next, since 9 is (much) less than e^8 it follows that $e^{-9} < (9e)^{-1}$ and therefore $-\sqrt[3]{(9e)^{-1}} < -e^{-3}$, so that $-\sqrt[3]{(9e)^{-1}}$ is the smallest value of f on $[-1,1]$.

(b) Since

$$g'(x) = \frac{d}{dx}\left\{\frac{2\cos x}{\sin x - 2}\right\} = 2\cdot\frac{(-\sin x)(\sin x-2)-(\cos x)(\cos x)}{(\sin x-2)^2}$$
$$= \frac{2(2\sin x - 1)}{(\sin x - 2)^2},$$

the only critical number of g on $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ occurs where $\sin x = \frac{1}{2}$, *i.e.*, at $\frac{1}{6}\pi$. Since $g(\pm\frac{1}{2}\pi) = 0$ and $g(\frac{1}{6}\pi) = -\frac{2}{3}\sqrt{3}$, it follows that the absolute maximum and minimum values of g on $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ are, respectively, 0 and $-\frac{2}{3}\sqrt{3}$.

(8) Given $y = (4x+3)^3(3x+3)^4 = 81(4x+3)^3(x+1)^4$, one has

$$\frac{dy}{dx} = 81\{12(4x+3)^2(x+1)^4 + 4(4x+3)^3(x+1)^3\}$$
$$= 324(4x+3)^2(x+1)^3(7x+6),$$

so that the critical numbers of y are -1, $-\frac{6}{7}$ and $-\frac{4}{3}$.

(9) If x denotes the x-coordinate of the top right corner of a rectangle as described, then that rectangle has width $2x$ and height $y = 16 - 2x^2 = 2(8-x^2)$, so its area is given by $A = 4x(8-x^2) = 4(8x - x^3)$, for $0 \le x \le 2\sqrt{2}$. Since A is never negative and is zero at the endpoints of its domain (a closed interval throughout which A is continuous), and since

$$\frac{dA}{dx} = 4(8-3x^2),$$

it follows that the largest value of A occurs at the lone critical number, $\frac{2}{3}\sqrt{6}$, in $[0, 2\sqrt{2}]$. Therefore, the rectangle as described with the largest possible area has height $y = 2(8-\frac{8}{3}) = \frac{32}{3}$ and width $2x = \frac{4}{3}\sqrt{6}$.

(10) (a) Given $y = x^7(x^3 - 2x\sin 3x)^{2/3} = x^{23/3}(x^2 - 2\sin 3x)^{2/3}$, one has

$$\frac{dy}{dx} = \tfrac{23}{3}x^{20/3}(x^2 - 2\sin 3x) + \frac{2x^{23/3}(2x - 6\cos 3x)}{3(x^2-2\sin 3x)^{1/3}}$$
$$= \frac{x^{20/3}(27x^2 - 12x\cos 3x - 46\sin 3x)}{3(x^2 - 2\sin 3x)^{1/3}}$$

(b) $\dfrac{dy}{dx} = 5\pi x^{\pi-1} + 5/x^2 + \frac{2}{5}\sqrt[5]{x^{-3}} - 1/(x\log 5) + +5(\log 2)2^x.$

(c) Given $y = \ln\sqrt[3]{\tan 2x^4} = \frac{1}{3}\ln(\tan 2x^4)$, one has

$$\frac{dy}{dx} = \frac{8x^3\sec^2 2x^4}{3\tan 2x^4} = \frac{8x^3}{3\sin 2x^4\cos 2x^4} = \tfrac{16}{3}x^3\csc 4x^4.$$

(d) $\dfrac{dy}{dx} = -\pi\sec\bigl(\cos(\tan\pi x)\bigr)\tan\bigl(\cos(\tan\pi x)\bigr)\sin(\tan\pi x)\sec^2\pi x.$

(e) Differentiating $xy^2 + y\ln x = x$ implicitly with respect to x gives

$$y^2 + 2xy\frac{dy}{dx} + \frac{dy}{dx}\ln x + \frac{y}{x} = 1, \quad\text{or}\quad x(2xy + \ln x)\frac{dy}{dx} = x(1-y^2) - y;$$

therefore,

$$\frac{dy}{dx} = \frac{x(1-y^2)-y}{x(2xy + \ln x)}.$$

(11) (a) Since $\ln|y| = 2\ln|\sec(5x-2)| - 12\ln|x| + x$, it follows that

$$\frac{dy}{dx} = y\frac{d}{dx}\{2\ln|\sec(5x-2)| - 12\ln|x| + x\}$$
$$= y\{10\tan(5x-2) - 12/x + 1\}$$
$$= x^{-13}e^x\sec^2(5x-2)\bigl(10x\tan(5x-2) + x - 12\bigr).$$

(b) Since $\ln y = (5x^2+1)\ln(\cos x)$, it follows that

$$\frac{dy}{dx} = y\frac{d}{dx}\{(5x^2+1)\ln(\cos x)\}$$
$$= (\cos x)^{5x^2+1}\bigl(10x\ln(\cos x) - (5x^2+1)\tan x\bigr).$$

(12) Differentiating the given equation implicitly with respect to y gives

$$(3x^2y - 1)\frac{dx}{dy} + x^3 - 6y^2 + 3 = 0,$$

and so slope of the line normal to the given curve at $(-1,2)$ is equal to

$$-\frac{dx}{dy}\bigg|_{(x,y)=(-1,2)} = \frac{(-1)^3 - 6(2)^2 + 3}{3(-1)^2(2) - 1} = -\tfrac{22}{5}.$$

Therefore, an equation of the normal line in question is $22x + 5y = -12$.

(13) Computing the first few derivatives of $f(x) = e^{2x}$ reveals a pattern.

$$f'(x) = e^{2x}\cdot 2$$
$$f''(x) = e^{2x}\cdot 2\cdot 2$$
$$\text{etc.}$$

so $f^{(n)}(x) = 2^n e^x$

(14) If ϑ denotes the acute angle between the ladder and the floor, x denotes the distance (in metres) from the bottom of the ladder to the wall and y denotes the distance (in metres) from the top of the ladder to the floor, then $x = 12\cos\vartheta$ and $y = 12\sin\vartheta$. Differentiating the first relation with respect to time gives

$$-\frac{1}{10} = \frac{dx}{dt} = -12\sin\vartheta\,\frac{d\vartheta}{dt} = -y\frac{d\vartheta}{dt}, \qquad\text{or}\qquad \frac{d\vartheta}{dt} = \frac{1}{10y}.$$

Therefore, when the top of the ladder reaches 8 metres up the wall, the acute angle it makes with the floor is increasing at a rate of 1/80 radians per second.

(15) Antidifferentiating f'' gives $f'(x) = 4x - 3x^2 - x^4 + C$. Use the fact that $f'(-1) = 1$ to find that $C = 9$. Then antidifferentiating f' gives $f(x) = 2x^2 - x^3 - \frac{1}{5}x^5 + 9x + D$, and the fact that $f(1) = 2$ gives us $D = -\frac{39}{5}$, so finally $f(x) = -\frac{39}{5} + 9x + 2x^2 - x^3 - \frac{1}{5}x^5$.

Alternatively, one could solve the problem as follows: We have

$$f'(x) = f'(-1) + \int_{-1}^{x} f''(t)\,dt = 1 + \int_{-1}^{x}(4 - 6t - 4t^3)\,dt$$
$$= 1 + \{4t - 3t^2 - t^4\}\Big|_{-1}^{x} = 9 + 4x - 3x^2 - x^4, \quad\text{and}$$
$$f(x) = f(1) + \int_1^x f'(t)\,dt = 2 + \int_1^x (9 + 4t - 3t^2 - t^4)\,dt$$
$$= 2 + \{9t + 2t^2 - t^3 - \tfrac{1}{5}t^5\}\Big|_1^x = -\tfrac{39}{5} + 9x + 2x^2 - x^3 - \tfrac{1}{5}x^5.$$

(16) Dividing $[1,4]$ into six subintervals of equal length gives $\Delta x = \dfrac{1}{2}$, so we approximate $\displaystyle\int_1^4 \frac{1}{x}\,dx$ with

$$\frac{1}{2}\left(f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + \ldots + f\left(\frac{15}{4}\right)\right)$$
$$= \frac{1}{2}\left(\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11} + \frac{4}{13} + \frac{4}{15}\right)$$

(17) (a) Expanding and integrating term by term gives

$$\int_1^2 (3x+1)^2\,dx = \int_1^2 (9x^2 + 6x + 1)\,dx = (3x^3 + 3x^2 + x)\Big|_1^2 = 31.$$

(b) Integrating term by term gives

$$\int (2^x + \sin x + \pi)\,dx = 2^x/(\ln 2) - \cos x + \pi x + C.$$

(c) Since the integral over $[a,a]$ a function defined at a is zero,

$$\int_{\frac{1}{9}\pi^2}^{\frac{1}{9}\pi^2} \sqrt{\cos\sqrt{x}}\,dx = 0.$$

(d) Dividing and integrating term by term gives

$$\int \frac{x^3 + x^2 + x + 1}{x}\,dx = \int (x^2 + x + 1 + 1/x)\,dx$$
$$= \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + x + \log|x| + C.$$

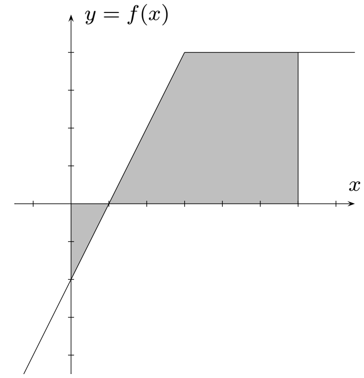
(e) Revising the integrand and using a standard integral formula gives

$$\int_0^{\frac{1}{3}\pi} \frac{\sin x}{\cos^2 x}\,dx = \int_0^{\frac{1}{3}\pi} \sec x \tan x\,dx = \sec x \Big|_0^{\frac{1}{3}\pi} = 1.$$

(18) First, note that $f(x) = -\int_0^{x^2} \sqrt{1+t^2}\,dt$.

Then, by the Chain Rule and the (First) Fundamental Theorem of Calculus,
$f'(x) = -2x\sqrt{1+x^4}$

(19) Here is a sketch of the graph of f, with the region representing the integral shaded.



The region below the x-axis is a triangle with base 1, height 2, and area is 1. The region above the x-axis decomposes naturally into a triangle with base 2, height 4, and area 4, and a rectangle with base 3, height 4, and area 12. Therefore,

$$\int_0^6 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx + \int_3^6 f(x)\,dx = -1+4+12 = 15.$$