1. Evaluate $\dfrac{d}{dx}\Big\{\sin\big(\arccos\sqrt{1-x^2}\,\big)\Big\}$, and simplify your answer.

2. Evaluate the following integrals.

a. $\displaystyle\int_1^{\sqrt{2}} \frac{4+2\sqrt{x^2-1}}{x\sqrt{x^2-1}}\,dx$ b. $\displaystyle\int_1^5 \frac{x+2}{\sqrt{2x-1}}\,dx$

c. $\displaystyle\int e^{-2x}\cos 6x\,dx$ d. $\displaystyle\int \sqrt{t+1}\log\sqrt{t+1}\,dt$

e. $\displaystyle\int_0^{\frac{1}{2}\pi} \sin^3 x\cos^3 x\,dx$ f. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-36}}$ g. $\displaystyle\int \frac{x+4}{x(x^2+2)}\,dx$

3. Evaluate the following improper integrals.

a. $\displaystyle\int_2^{\infty} \frac{1}{1-x^2}\,dx$ b. $\displaystyle\int_0^2 \frac{x}{x^2-4}\,dx$

4. Evaluate the following limits.

a. $\displaystyle\lim_{x\to 0^+} \frac{(\log x)^2}{1+x^{-1}}$ b. $\displaystyle\lim_{x\to 0}(\sec x)^{\cot^2 x}$ c. $\displaystyle\lim_{x\to\infty}\left\{\frac{x^2+2}{x-3}-\frac{(x-2)^3}{x^2+1}\right\}$

5. Find the area of the region (in quadrant I) bounded by the graphs of

$$y=\frac{2}{x},\quad y=\frac{3x}{x^2+2}\quad\text{and}\quad x=1.$$

Give the exact answer in simplified form only: no decimals.

6. Let $\mathscr{R}$ be the region bounded by the graphs of

$$y=\frac{x^2}{4},\quad y=x^3-3x+3,\quad x=-2\quad\text{and}\quad x=2.$$

a. Set up, *but do not evaluate*, an integral that represents the volume of the solid generated by revolving $\mathscr{R}$ about the x-axis.

b. Find the volume of the solid generated by revolving $\mathscr{R}$ about the line $x=3$. *Give the exact answer in simplified form only: no decimals.*

7. Find a solution to the differential equation

$$y'=\frac{\sqrt{1-y^2}}{1+x^2};\quad y(1)=0.$$

8. Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series whose n^{th} partial sum is given by $s_n=\dfrac{2n+1}{n+2}$.

a. Evaluate $\displaystyle\sum_{n=1}^{\infty} a_n$. b. Find a_5.

9. What can you say about the convergence of each series based only on the limit of its general term?

a. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n}$ b. $\frac{1}{2}+1+\frac{1}{4}+1+\frac{1}{8}+1+\frac{1}{16}+1+\cdots$

10. Determine whether each of the following series converges or diverges; if it converges, find the sum. Justify your answers.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{5(-4)^{n+2}}{3^{2n+1}}$ b. $\displaystyle\sum_{n=1}^{\infty}\log\frac{2n-1}{2n+1}$

11. Determine whether each of the following series converges or diverges. State the tests you use and verify that the conditions for using them are satisfied.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$ b. $\displaystyle\sum_{k=1}^{\infty}\frac{\cos^2 k}{k\sqrt{k}}$ c. $\displaystyle\sum_{n=1}^{\infty}\frac{e^{\sqrt{n}}}{\sqrt{n}}$ d. $\displaystyle\sum_{n=2}^{\infty}\sin\left(\frac{2}{n}\right)$

12. Determine whether each of the following series converges absolutely, conditionally or diverges. Justify your answers.

a. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-n}{2n+1}\right)^{3n}$ b. $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{\log n}{\sqrt{n}}$

13. Find the radius and interval of convergence for the power series

$$\sum_{n=1}^{\infty}\frac{3^n}{2n+1}(x-2)^{n+1}.$$

14. Find the Taylor series of $f(x)=\cos 2x$ centred at $\frac{1}{2}\pi$. State the first four *non-zero* terms and give the formula for the n^{th} term.

Solution outlines

1. $\sin\big(\arccos\sqrt{1-x^2}\,\big)=|x|$, and hence $\dfrac{d}{dx}\Big\{\sin\big(\arccos\sqrt{1-x^2}\,\big)\Big\}=\dfrac{x}{|x|}$ (1 if $x>0$ and -1 if $x<0$).

2. a. Separating terms and simplifying reveals two basic integrals:

$$\int_1^{\sqrt{2}}\frac{4+2\sqrt{x^2-1}}{x\sqrt{x^2-1}}\,dx=(4\operatorname{arcsec}x+2\log x)\Big|_1^{\sqrt{2}}=\pi+\log 2.$$

b. Partial integration gives

$$\int_1^5\frac{x+2}{\sqrt{2x-1}}\,dx=\Big\{(x+2)\sqrt{2x-1}-\tfrac{1}{3}(2x-1)\Big\}\Big|_1^5$$
$$=\tfrac{1}{3}(x+7)\sqrt{2x-1}\Big|_1^5=\tfrac{28}{3}.$$

c. By the product rule for differentiation,

$$\frac{d}{dx}\big\{e^{-2x}\cos 6x\big\}=-2e^{-2x}\cos 6x-6e^{-2x}\sin x,\quad\text{and}$$
$$\frac{d}{dx}\big\{e^{-2x}\sin 6x\big\}=\quad 6e^{-2x}\cos 6x-2e^{-2x}\sin x.$$

Subtracting the first equation from three times the second, and then rearranging the corresponding integral equation, one obtains

$$\int e^{-2x}\cos 6x\,dx=\tfrac{1}{20}e^{-2x}(3\sin 6x-\cos 6x)+C.$$

d. Integrate by parts after revising the logarithmic factor, to obtain

$$\tfrac{1}{2}\int\sqrt{t+1}\log(t+1)\,dt=\tfrac{1}{3}(t+1)^{3/2}\log(t+1)-\tfrac{2}{9}(t+1)^{3/2}+C$$
$$=\tfrac{1}{9}\big(3\log(t+1)-2\big)\sqrt{(t+1)^3}+C.$$

e. Letting $t=\sin x$ gives

$$\int_0^{\frac{1}{2}\pi}\sin^3 x\cos^3 x\,dx=\int_0^1 t^3(1-t^2)\,dt=\tfrac{1}{12}t^4(3-2t^2)\Big|_0^1=\tfrac{1}{12}.$$

f. Let $t=\sqrt{x^2-36}/x$, or $t^2=1-36/x^2$, so that $\frac{1}{36}t\,dt=dx/x^3$, and hence

$$\int\frac{dx}{x^2\sqrt{x^2-36}}=\int\frac{x}{\sqrt{x^2-36}}\cdot\frac{dx}{x^3}=\tfrac{1}{36}\int dt=\frac{\sqrt{x^2-36}}{36x}+C.$$

g. Resolve the integrand into partial fractions and split the last fraction to integrate.

$$\int\frac{x+4}{x(x^2+2)}\,dx=\int\left\{\frac{2}{x}-\frac{2x-1}{x^2+2}\right\}dx$$
$$=\int\left\{\frac{2}{x}-\frac{2x}{x^2+2}+\frac{1}{x^2+2}\right\}dx$$
$$=\log\frac{x^2}{x^2+2}+\tfrac{1}{2}\sqrt{2}\arctan\tfrac{1}{2}x\sqrt{2}+C.$$

3. a.

$$\int_2^{\infty}\frac{dx}{1-x^2}=\lim_{t\to\infty}\int_2^t\frac{dx}{1-x^2}=\lim_{t\to\infty}\tfrac{1}{2}\log\frac{t+1}{t-1}-\tfrac{1}{2}\log 3=-\tfrac{1}{2}\log 3$$

b. The integral diverges, because

$$\int_0^2\frac{x\,dx}{x^2-4}=\lim_{\alpha\to 2^-}\int_0^{\alpha}\frac{x\,dx}{x^2-4}=\lim_{\alpha\to 2^-}\tfrac{1}{2}\log(4-\alpha^2)-\log 2=-\infty.$$

4. a. If $y=1/x$, then

$$\lim_{x\to 0^+}\frac{(\log x)^2}{1+x^{-1}}=\lim_{y\to\infty}\frac{(\log y)^2}{1+y}=\lim_{y\to\infty}\frac{2}{y}=0,$$

with two applications of l'Hôpital's rule.

b. We have $\lim\limits_{x\to0}(\sec x)^{\cot^2 x} = \lim\limits_{x\to0} e^{-\cos^2 x(\log\cos x)/\sin^2 x} = \sqrt{e}$, since

$$\lim_{x\to0}\frac{\log\cos x}{\sin^2 x} = \lim_{x\to0}\frac{-1}{2\cos^2 x} = -\tfrac12, \quad \text{by l'Hôpital's rule.}$$

c. Combining terms, and extracting the dominant powers of x from the numerator and denominator, yields

$$\lim_{x\to\infty}\left\{\frac{x^2+2}{x-3} - \frac{(x-2)^3}{x^2+1}\right\} = \lim_{x\to\infty}\frac{9-17/x+44/x^2-24/x^3}{(1-3/x)(1+1/x^2)} = 9.$$

5. Since $2/x - 3x(x^2+2) = (4-x^2)/\big(x(x^2+2)\big)$ is equal to zero if $x=\pm2$ and positive if $1\leqslant x<2$, the area in question is equal to

$$\int_1^2\left\{\frac{2}{x} - \frac{3x}{x^2+2}\right\}dx = \tfrac12\log\frac{x^4}{(x^2+2)^3}\bigg|_1^2 = \tfrac12\log 2.$$

6. a. The solid in question can be decomposed into annuli of inner radius $\frac14x^2$ and outer radius x^3-3x+3, for $-2\leqslant x\leqslant 2$, so its volume is equal to

$$\pi\int_{-2}^{2}\left\{(x^3-3x+3)^2 - (\tfrac14x^2)^2\right\}dx = \pi\int_0^2\left\{2x^6 - \tfrac{97}{8}x^4+18x^2+18\right\}dx$$
$$= \pi\left\{\tfrac27x^7 - \tfrac{97}{40}x^5+6x^3+18x\right\}\bigg|_0^2 = \tfrac{1504}{35}\pi.$$

b. The solid in question can be decomposed into cylindrical shells of radius $3-x$ and height $x^3-\frac14x^2-3x+3$, for $-2\leqslant x\leqslant2$, so its volume is equal to

$$2\pi\int_{-2}^2(3-x)(x^3-\tfrac14x^2-3x+3)\,dx = 2\pi\int_0^2\left\{-2x^4+\tfrac92x^2+18\right\}dx$$
$$= 2\pi\left\{\tfrac25x^5+\tfrac32x^3+18x\right\}\bigg|_0^2 = \tfrac{352}{5}\pi.$$

7. Separating variables gives

$$\int\frac{dy}{\sqrt{1-y^2}} = \int\frac{dx}{1+x^2}, \quad\text{or}\quad \arcsin y = \arctan x + C.$$

Since $y(1)=0$ implies that $C=\frac14\pi$, we must have $x\geqslant-1$, in which case

$$y = \sin(\arctan x - \tfrac14\pi) = \frac{x-1}{\sqrt{2(x^2+1)}}.$$

8. a. The sum of the series is $\displaystyle\sum_{n=1}^\infty a_n = \lim_{n\to\infty}s_n = \lim_{n\to\infty}\frac{2n+1}{n+2} = 2.$

b. The series begins with $n=1$, so $s_n = a_1+\cdots+a_n$ (n terms), and therefore $a_5 = s_5 - s_4 = \frac{11}{7}-\frac32 = \frac1{14}$.

9. a. Since $-1/n < (\cos n)/n < 1/n$ for $n\geqslant1$, and $\lim\limits_{n\to\infty}(\pm1/n)=0$, $\lim\limits_{n\to\infty}(\cos n)/n = 0$ by the Squeeze Theorem. In this case, no conclusion can be drawn about the series only from the limit of its general term.

b. Among infinitely many possibilities:

- if $a_n = 2^{-\frac12(n+1)}\sin^2\frac12\pi n + \cos^2\frac12\pi n$ for $n\geqslant1$, then $a_{2n}=1$ for $n\geqslant1$ and $\lim\limits_{n\to\infty}a_n\neq0$, so $\sum a_n$ diverges by the vanishing criterion;
- if $a_n = \dfrac{(|\pi e-n|+\pi e-n)(\sqrt{2^{n+1}}\cos^2\frac12\pi n+\sin^2\frac12\pi n)}{|\pi e-n|\sqrt{2^{n+3}}}$ for $n\geqslant1$, then $\lim\limits_{n\to\infty}a_n=0$ (since $a_n=0$ for $n\geqslant9$) and no conclusion can be drawn about the series only from the limit of its general term;
- if $a_n = \displaystyle\sum_{i=1}^4\left\{\prod_{\substack{1\leqslant j\leqslant8\\ j\neq2i-1}}\frac{2^{-i}(n-j)}{2i-j-1} + \prod_{\substack{1\leqslant j\leqslant8\\ j\neq2i}}\frac{n-j}{2i-j}\right\}$ for $n\geqslant1$, then $\lim\limits_{n\to\infty}a_n=\infty$ and $\sum a_n$ diverges by the vanishing criterion.

No content in sight—just a matter of "guess what teacher wants to hear."

10. a. This is a geometric series with first term $-\frac{320}{27}$ and common ratio $-\frac49$, so

$$\sum_{n=1}^\infty\frac{5(-4)^{n+2}}{3^{2n+1}} = \frac{-320/27}{1-(-4/9)} = -\frac{320}{39}.$$

b. This is a divergent telescoping series, since

$$\sum_{n=1}^\infty\log\frac{2n-1}{2n+1} = \lim_{n\to\infty}\sum_{k=1}^n\log\frac{2k-1}{2k+1} = -\lim_{n\to\infty}\log(2n+1) = -\infty.$$

11. a. Let $\alpha_n = (n!)^2/(2n!)$; then

$$\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \lim_{n\to\infty}\frac{n+1}{2(2n+1)} = \tfrac14,$$

which is smaller than one. Therefore, $\sum\alpha_n$ converges by the ratio test.

b. Since

$$0 < \frac{\cos^2 k}{k\sqrt{k}} < \frac{1}{k\sqrt{k}}$$

for $k\geqslant1$,

$$\sum_{k=1}^\infty\frac{\cos^2 k}{k\sqrt{k}}$$

converges with the p-series $\sum k^{-3/2}$ by the comparison test.

c. Since $e^x > x$ for all real numbers x, the series

$$\sum_{n=1}^\infty\frac{e^{\sqrt{n}}}{\sqrt{n}}$$

diverges by the vanishing criterion.

d. Since $\frac12\vartheta < \vartheta\cos\vartheta < \sin\vartheta$ if $0<\vartheta<\frac13\pi$, $1/n < \sin(2/n)$ if $n\geqslant2$, and so

$$\sum_{n=2}^\infty\sin\left(\frac2n\right)$$

diverges with the harmonic series by the comparison test.

12. a. Let $\alpha_n = \big(-n/(2n+1)\big)^{3n}$; then

$$\lim_{n\to\infty}\sqrt[n]{|\alpha_n|} = \lim_{n\to\infty}\left(\frac{1}{2+1/n}\right)^3 = \tfrac18,$$

which is less than one. Therefore, $\sum\alpha_n$ is absolutely convergent by the root test.

b. Let $f(x) = (\log x)/\sqrt{x}$; then f is decreasing on $(\,e^2,\infty\,)$, since

$$f'(x) = (2-\log x)/(2x\sqrt{x}) < 0$$

if $x>e^2$, $f(x)>0$ if $x>1$, and $f(x)\to0$ as $x\to\infty$ by (one application of) l'Hôpital's rule. Therefore, $\sum(-1)^nf(n)$ converges by the alternating series test. On the other hand, $f(x) > 1/\sqrt{x}$ if $x>e$, so $\sum f(n)$ diverges with the p-series $\sum n^{-1/2}$ by the comparison test. Hence, the series

$$\sum_{n=2}^\infty(-1)^n\frac{\log n}{\sqrt{n}}$$

is conditionally convergent.

13. Let $\alpha_n = 3^n(x-2)^{n+1}/(2n+1)$; then

$$\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \lim_{n\to\infty}\frac{3(2n+1)}{2n+3}|x-2| = 3|x-2|,$$

so by the ratio test $\sum\alpha_n$ converges absolutely if $\frac53 < x < \frac73$ and diverges if $x<\frac53$ or $x>\frac73$. If $x=\frac53$ then $\alpha_n = (-1)^{n+1}/(2n+1)$, so $\sum\alpha_n$ converges by the alternating series test (since $\{1/(2n+1)\}$ is positive and decreasing, and its limit is zero). If $x=\frac73$ then $\alpha_n = 1/(2n+1)\geqslant 1/(3n)$ if $n\geqslant1$, so $\sum\alpha_n$ diverges with the harmonic series by the comparison test. Therefore, the interval of convergence of $\sum\alpha_n$ is $[\,\frac53,\frac73\,)$; its radius of convergence is $\frac13$.

14. Using the Maclaurin series of $\cos t$, where $t=2x-\pi$, gives

$$\cos 2x = -\cos(2x-\pi) = -\cos2\big(x-\tfrac12\pi\big)$$
$$= \sum_{k=0}^\infty\frac{(-1)^{k+1}2^{2k}}{(2k)!}\big(x-\tfrac12\pi\big)^{2k}$$
$$= -1+2\big(x-\tfrac12\pi\big)^2 - \tfrac23\big(x-\tfrac12\pi\big)^4+\tfrac4{45}\big(x-\tfrac12\pi\big)^6-\ldots$$