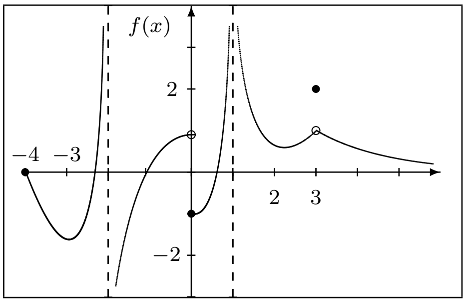
1. Given the graph of $f(x)$, answer the following.
Use $+\infty, -\infty$ or "does not exist" where appropriate.



- Evaluate

a) $\lim_{x\to -2^-} f(x) =$

b) $\lim_{x\to 0} f(x) =$

c) $\lim_{x\to 0^+} f(x) =$

d) $\lim_{x\to 3} f(x) =$

- Estimate **e)** $f'(-1) =$ **f)** $f'(3) =$

2. Evaluate the limit or explain why it does not exist. Use $+\infty, -\infty$ or "does not exist" where appropriate.

a) $\lim_{x\to -1} \frac{3x^3+2x^2+1}{x^2+1}$ **b)** $\lim_{x\to 5} \frac{1}{x^2-25}\left(\frac{1}{x-8}+\frac{1}{3}\right)$ **c)** $\lim_{x\to -3^-} \frac{|x+3|}{x^2-9}$ **d)** $\lim_{x\to 0} \frac{\sin^2(3x)}{9x^2}$

e) $\lim_{x\to\infty} \frac{\sqrt{9x^4+2x+17}}{x^2-3x+1}$

3. **a)** State the definition of **"function f is continuous at $x=a$"**.

b) Find the domain and the range of f for $f(x) = \begin{cases} x+1 & \text{if } x<-2 \\ \frac{x-1}{x^2-1} & \text{if } -2\le x\le 2 \\ x+\frac{1}{3} & \text{if } 2<x \end{cases}$

c) Find **all** the discontinuities of f. **Justify** your answers by using your definition in **a)**.

4. Find **all** values of c which make $f(x) = \begin{cases} \cos(x)+c^2 & \text{if } x<0 \\ e^x+2+c & \text{if } x\ge 0 \end{cases}$ a continuous function.

5. State a **limit definition of the derivative** and use it to differentiate $f(x) = \frac{1}{2x+3}$ at $x=a$.

6. Differentiate and do **not** simplify.

a) $f(x) = 3^x + \frac{x^3}{3} - \frac{3}{\sqrt[3]{x}} + e^{3x} - \tan(3x) + \ln(3)$ **b)** $f(x) = (2x - e^x + 1)(1-2\cos(x))^2$

c) $f(x) = \ln(\ln(\ln(x)))$ **d)** $f(x) = a^x x^a$ where a is a constant. **e)** $f(x) = \frac{\sin(x)\cos(x)\tan^3(x)}{\sqrt{x}}$

7. Find $f'(0)$ given that $f(x) = \sqrt{(x+1)(x^2+1)(x^3+1)(x^4+1)}$

8. Consider the function $f(x) = e^x$.

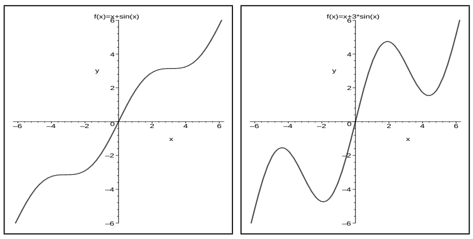
a) Find an equation of the tangent line to the graph of f at the point $(a, f(a))$.

b) Find an equation of the tangent line to the graph of f that passes through the origin $(0,0)$.

9. Given a constant $b>0$, consider the function

$$f(x) = x + b\sin(x)$$

over the interval $[-2\pi, 2\pi]$.
Two such functions, with $b=1$ and $b=3$ respectively, are shown on the graphs.

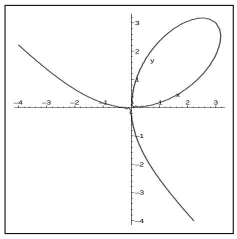


a) Find all x-coordinates in $[-2\pi, 2\pi]$ where the tangent line to the graph of f is the line $y = x+b$.

b) Are the points described in part **a)** local maximum points of f? **Justify!**

c) Show that all inflection points of the graph of f lie on the line $y=x$.

10. The graph of $x^3 + y^3 = 6xy$ is called a *folium of Descartes*.



a) Find $\dfrac{dy}{dx}$.

b) Find the slope of the tangent line to the curve at $(\frac{4}{3}, \frac{8}{3})$.

c) Find the point $(x_0, y_0) \neq (0, 0)$, at which the tangent line is horizontal.

11. A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 2 m/s. How rapidly is the area enclosed by the ripple increasing when it is 400 m^2.

12. Find **all** local and absolute extrema of $f(x) = x^4 - 2x^2 - 3$ on the interval $[-2, 2]$. Sketch the graph of f over this interval. **Justify!**

13. A cylindrical storage tank is to hold 6000 cubic metres of highly-toxic industrial by-product. Per square metre, concrete for the circular base costs \$100, plastic for the roof costs \$50, and steel for the cylindrical wall costs \$200. Find the dimensions of the least expensive such storage tank. **Note**: Volume of right cylinder = (area of base)×(altitude)

14. Given that $g''(x) = 6x - \dfrac{1}{x^2} + \pi^2 \sin(\pi x)$ with $g'(1) = 4$ and $g(1) = 1$, find $g(x)$.

15. Given $f(x) = \dfrac{x}{(x^2-1)^{1/3}}$, then $f'(x) = \dfrac{x^2-3}{3\,(x^2-1)^{4/3}}$ and $f''(x) = \dfrac{2\,x\,(9-x^2)}{9\,(x^2-1)^{7/3}}$.

a) Find all vertical and horizontal asymptotes.

b) Find the increase/decrease intervals of f.

c) Find the local and absolute extrema.

d) Find the upward/downward concavity intervals.

e) Find the points of inflection.

f) Sketch the graph of f. Indicate clearly the asymptotes, extrema, concavity and points of inflection.

16. Consider the definite integral $\displaystyle\int_3^{11} \frac{4+x}{x}\, dx$.

a) Find an approximate value by calculating the Riemann sum $\displaystyle\sum_{i=1}^{4} f(x_i)\Delta x_i$ letting $\Delta x_i = 2$ and taking x_i as the midpoint, for all i.

b) Find the exact value using the **Fundamental Theorem of Calculus**.

17. **a)** Sketch the graph of f for $f(x) = \begin{cases} 1 & \text{if } -2 \le x < -1 \\ 1 + \sqrt{1-x^2} & \text{if } -1 \le x \le 1 \\ x & \text{if } \ 1 < x \le 2 \end{cases}$

b) Evaluate $\displaystyle\int_{-2}^{2} f(x)\, dx$ by interpreting it in terms of areas.

18. Evaluate the following integrals.

a) $\displaystyle\int \left(\pi + e^x - \frac{1}{x} + \sin(x)\right) dx$

b) $\displaystyle\int \frac{2 + 2x - x^2}{\sqrt{x}}\, dx$

c) $\displaystyle\int \frac{1 + \sin(x)}{\cos^2(x)}\, dx$

d) $\displaystyle\int_{-3}^{3} \sin^3(x)\, dx$

19. Consider the function $g(x) = \displaystyle\int_{x^2}^{\pi} e^t\, dt$.

a) Evaluate $g(\sqrt{\pi}\,)$.

b) Using the **Fundamental Theorem of Calculus**, find $g'(x)$.

c) Evaluate $g'(1)$.

Answers

1. **a)** $+\infty$ **b)** dne **c)** -1 **d)** 1 **e)** 2 **f)** dne

2. **a)** 0 **b)** $-\frac{1}{90}$ **c)** $\frac{1}{6}$ **d)** 1 **e)** 3

3. **a)** $\lim_{x\to a} f(x) = f(a)$ **b)** dom$(f) = \mathcal{R} \setminus \{-1, 1\}$ range$(f) = (-\infty, -1] \cup [\frac{1}{3}, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$

c) $c = \pm 1:\ c \notin$dom(f), $c = 2:\ \lim_{x\to c^-} f(x) = \frac{1}{3} \neq \frac{7}{3} = \lim_{x\to c^+} f(x)$

4. $c = -1$, $c = 2$ **5.** $\lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = -\dfrac{2}{(2a+3)^2}$

6. **a)** $\ln(3)3^x + x^2 + x^{-4/3} + 3e^{3x} - 3\sec^2(3x) + 0$ **b)** $(2 - e^x)(1 - 2\cos(x))^2 + (2x - e^x + 1)2(1 - 2\cos(x))2\sin(x)$

c) $\dfrac{1}{\ln(\ln(x))\,\ln(x)\,x}$ **d)** $\ln(a)a^x x^a + a^x a x^{a-1}$ **e)** $\dfrac{\sin(x)\cos(x)\tan^3(x)}{\sqrt{x}}\left(\dfrac{\cos(x)}{\sin(x)} - \dfrac{\sin(x)}{\cos(x)} + 3\dfrac{\sec^2(x)}{\tan(x)} - \dfrac{1}{2x}\right)$

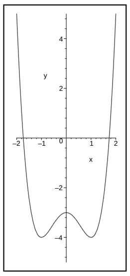
7. $f'(x) = \sqrt{(x+1)(x^2+1)(x^3+1)(x^4+1)}\,\dfrac{1}{2}\left(\dfrac{1}{x+1} + \dfrac{2x}{x^2+1} + \dfrac{3x^2}{x^3+1} + \dfrac{4x^3}{x^4+1}\right) \ \rightarrow\ f'(0) = \dfrac{1}{2}$

8. **a)** $y = e^a(x - a) + e^a$ **b)** $y = e\,x$

9. **a)** $x = -\frac{3\pi}{2}$, $x = \frac{\pi}{2}$ **b)** No! $f'(-\frac{3\pi}{2}) = f'(\frac{\pi}{2}) = 1 \neq 0$

c) $f''(a) = 0$ for $a = 0, \pm\pi, \pm 2\pi$ $\rightarrow$ inflection points: $(a, f(a)) = (a, a)$ for $a = 0, \pm\pi, \pm 2\pi$

10. **a)** $\dfrac{dy}{dx} = -\dfrac{x^2 - 2y}{y^2 - 2x}$ **b)** $\frac{4}{5}$ **c)** $(x_0, y_0) = (2{\cdot}2^{1/3}, 2{\cdot}2^{2/3})$



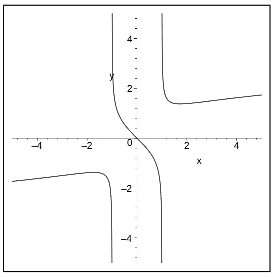
11. $80\sqrt{\pi}$ m^2/s **12.** $f(0) = -3$ local min, $f(\pm 2) = 5$ abs. max, $f(\pm 1) = -4$ abs. (and local) min

13. $r = 20/\sqrt[3]{\pi}, h = 15/\sqrt[3]{\pi}$ **14.** $g(x) = x^3 + \ln(x) - \sin(\pi x) - \pi x + \pi$

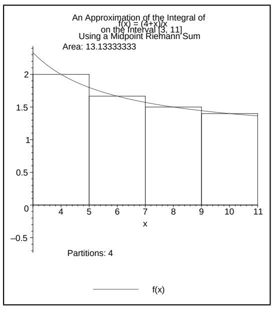
15. **a)** vertical: $x = \pm 1$, horizontal: none **b)** $(-\infty, -\sqrt{3}) \nearrow$, $(-\sqrt{3}, -1) \searrow$, $(-1, 1) \searrow$, $(1, \sqrt{3}) \searrow$, $(\sqrt{3}, \infty) \nearrow$

c) local min $\sqrt{3}$, local max $-\sqrt{3}$, abs. extrema: none **d)** $(-\infty, -3) \uparrow$, $(-3, -1) \downarrow$, $(-1, 0) \uparrow$, $(0, 1) \downarrow$, $(1, 3) \uparrow$, $(3, \infty) \downarrow$

e) $(-3, -\frac{3}{2}), (0, 0), (3, \frac{3}{2})$ **f)**

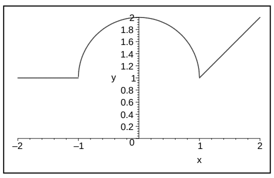


16. **a)** area$= \frac{197}{15} \approx 13.1333$



b) $4\ln(\frac{11}{3}) + 8 \approx 13.1971$

17. **a)**



b) $\frac{9+\pi}{2}$.

18. **a)** $\pi x + e^x - \ln|x| - \cos(x) + C$ **b)** $\left(4 + \frac{4}{3}x - \frac{2}{5}x^2\right)\sqrt{x} + C$ **c)** $\tan(x) + \sec(x) + C$ **d)** 0

19. **a)** $g(\sqrt{\pi}) = 0$ **b)** $g'(x) = -2xe^{x^2}$ **c)** $g'(1) = -2e$