Cal III - DDB - Final Exam May 2008

Marks

(10)

1. Let curve C have parametric equations

$$x = t^2 - 4 \;\; ; \;\; y = 2t - t^2$$

$i)$ Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$ii)$ Find the point on C where the tangent line is horizontal.

$iii)$ **Set up** the integral needed to find:

$a)$ the area of the region bounded by C and the $x - axsis$

$b)$ the length of C on the interval $0 \le t \le 2$

2. $a)$ Sketch the polar curves $r_1 = 2 + 2\cos\theta$ and $r_2 = 6\cos\theta$ on

(8)

the same set of axes.

$b)$ Find the points of intersection.

$c)$ **Set up** the integrals needed to find:

$i)$ the area common to both

$ii)$ the length of $r_1 = 2 + 2\cos\theta$

3. A particle moves along the space curve C defined by

(9)

$$\vec{r}\,(t) = \left\langle t^3,\ \sqrt{\frac{3}{2}}\ t^2\ ,\ t \right\rangle$$

Find: $i)$ the length of C from $t = 0$ to $t = 1$

$ii)$ the tangential and normal components of acceleration

$iii)$ the curvature at $t = 1$

4. Sketch the following:

(6)

$a)$ the space currve defined by $\vec{r}\,(t) = \langle t\cos t,\ t\sin t,\ t \rangle$

$b)$ the graph of the function $z = \sqrt{x^2 + y^2 - 9}$

$c)$ the 3 level curves of $f(x,y) = x - y^2$ corresponding to $c = 0,\ 1$ and

-1

5. $a)$ Let $w = 2x^2 - y^2 + z^2$. Find the direction and magnitude of the maximum (3)
rate of change (directional derivative) of w at the point $P(3,-2,1)$

$b)$ Find the equation of the tangent line to the curve of intersection of the surfaces
$x^2 + 4y^2 + 2z^2 = 27$ and $x^2 + y^2 - 2z^2 = 11$ at the point $P(3,-2,1)$ (3)

6. Find the critical points of $f(x,y) = x^3 + 3xy^2 + 3y^2 - 15x + 2$ and classify

them as relative maxima, relative minima or saddlepoints.
(5)

7. i) Given $f(x,y) = x^2 \ln\left(\dfrac{y}{x^2}\right)$, Find $\dfrac{\partial^2 f}{\partial x \partial y}$ (10)

ii) If $z = f(x,y)$ where $x = r^2 + s^2$ and $y = 2\,r\,s$ find: $\dfrac{\partial^2 z}{\partial r \partial s}$

iii) If $z = f(x,y)$ is implicitly defined by the equation $e^{xz} + \tan(yz) = xz^2$ find: $\dfrac{\partial z}{\partial x}$.

8. $a)$ Evaluate $\displaystyle\int_0^1 \int_{y^2}^1 \frac{y^3}{\sqrt{y^4 + x^2}}\, dx\, dy$ (8)

$b)$ Combine the following sum into **one** double integral in polar coordinates.

$$\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} dy\,dx \;+\; \int_{1}^{\sqrt{2}} \int_{0}^{x} dy\,dx \;+\; \int_{\sqrt{2}}^{2} \int_{0}^{\sqrt{4-x^2}} dy\,dx$$

9. **Set up** a triple integral needed to find the volume of the solid part of the

(4)

sphere $x^2 + y^2 + z^2 = 16$ cut off by the cylinder $r = 4\sin\theta$.

10. $a)$ Sketch the solid region S that lies in the first octant and is bounded by the

coordinate planes, $z = 4 - x^2$ and

$x + y = 2$ (4)

$b)$ **Set up a** double integral to find the volume of S.

11. Sketch the solid region defined by the limits of integration of the triple integral

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} x\,y\,dz\,dy\,dx$$ and express it

(5)

$a)$ in cylindrical coordinates $b)$ in spherical coordinates

12. Estimate the value of $\int_0^{0.5} \sqrt{4 + x^3}\,dx$ to six decimal place accuracy.

(8)

13. Use power series to evaluate $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2\, e^x}$

(4)

14. $a)$ Find a Taylor polynomial of degree 3 for $f(x) = \sqrt{x}$ centered at $c = 4$

and an expression for Taylor's Remainder term $R_3(x)$

(8) $b)$

Use part (a) to approximate $\sqrt{4.1}$ and to state the accuracy of your approximation

15. Given $\ln(1 + t) = \sum_{n=0}^{\infty} (-1)^n$

$\dfrac{t^{\,n+1}}{n + 1}$ (3)

$a)$ Find the value of the sixth derivative of $\ln(1 + x^2)$ evaluated at $x = 0$

$b)$ Find the sum of the following:

$$\frac{1}{2}-\frac{1}{2}\left(\frac{1}{2}\right)^{2}+\frac{1}{3}\left(\frac{1}{2}\right)^{3}-\frac{1}{4}\left(\frac{1}{2}\right)^{4}+\frac{1}{5}\left(\frac{1}{2}\right)^{5}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

16. Manipulate the known power series for $\frac{1}{1-x}$ to obtain

(7)

i) a power series for $f(x) = \frac{x}{(1-x)^2}$ centered at $c = 0$

ii) a power series for $g(x) = \frac{1}{2x+5}$ centered at $c = 2$

Answers

1. $\frac{dy}{dx} = \frac{1-t}{t}$; $\frac{d^2y}{dx^2} = -\frac{1}{2t^3}$
H.T. at $(-3, 1)$ when $t = 1$
$A = \int_0^2 (4t^2 - 2t^3)dt$ and $\mathcal{L} = 2\int_0^2 \sqrt{2t^2 - 2t + 1}dt$

2. Points of intersection: $(3, \pi/3)$, $(3, 5\pi/3)$ and the pole.
$A = 2\left(\frac{1}{2}\int_0^{\pi/3} 4(1+\cos\theta)^2 d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} 36\cos^2\theta d\theta\right)$
$\mathcal{L} = 2\int_0^{\pi} \sqrt{4(1+\cos\theta)^2 + 4\sin^2\theta}d\theta = 8\int_0^{\pi}\cos(\theta/2)d\theta = 16$

3. $\mathcal{L} = 2$; $a_T = 6t$; $a_N = \sqrt{6}$; $\kappa(t) = \frac{\sqrt{6}}{(3t^2+1)^2}$ and $\kappa(1) = \frac{\sqrt{6}}{16}$.

4. (a) Note $x^2 + y^2 = z^2$ and $z = t$. So the curve spirals upward on the boundary of the cone $x^2 + y^2 = z^2$
(b) $x^2 + y^2 - z^2 = 9$, and $z \geq 0$ Hyperboloid of one sheet, top part only.
(c) Three parabolas, $x = y^2$, $x = y^2 + 1$ and $x = y^2 - 1$.

5. (a) Maximum rate of change$=\|\nabla w(3, -2, 1)\| = 2\sqrt{41}$ in the direction of $\nabla w(3, -2, 1)$, or in the direction of the unit vector $\frac{1}{\sqrt{41}}\langle 6, 2, 1\rangle$.
(b) Its direction vector $\vec{v}$ is parallel to $\nabla F(3, -2, 1) \times \nabla G(3, -2, 1)$ where $F(x, y, z) = x^2 + 4y^2 + 2z^2$ and $G(x, y, z) = x^2 + y^2 - 2z^2$.
$L:\ \langle x, y, z\rangle = \langle 3, -2, 1\rangle + t\langle 10, 6, 9\rangle;\quad t \in \mathbf{R}$

6. $(-1, -2)$ and $(-1, 2)$ are saddle points; $(\sqrt{5}, 0)$ is a local minimum while $(-\sqrt{5}, 0)$ is a local maximum.

7. (i) $\frac{\partial f}{\partial y} = \frac{x^2}{y}$ and $\frac{\partial^2 f}{\partial x \partial y} = \frac{2x}{y}$
(ii) $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}(2s) + \frac{\partial z}{\partial y}(2r)$
$\frac{\partial^2 z}{\partial r \partial s} = 4rs\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x^2}\right) + 4(r^2 + s^2)\frac{\partial^2 z}{\partial y \partial x} + 2\frac{\partial z}{\partial y}$
(iii) Let $F(x, y, z) = e^{xz} + \tan(yz) - xz^2$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{z(z - e^{xz})}{xe^{xz} + y\sec^2(yz) - 2xz}$$

8. (a) $I = \frac{1}{4}(\sqrt{2} - 1)$ (Change the order of integration)
(b) $I = \int_0^{\pi/4}\int_1^2 r dr d\theta$

9. $V = \int_0^{\pi}\int_0^{4\sin\theta}\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r dz dr d\theta$

10. (b) $V = \int_0^2\int_0^{2-x}(4 - x^2)dydx$

11. (a) $I = \int_0^{2\pi}\int_0^2\int_{r^2}^4 (r^2\cos\theta\sin\theta) r dz dr d\theta$
(b) $I = \int_0^{2\pi}\int_0^{\arctan(1/2)}\int_0^{4/\cos\phi}(\rho^2\sin^2\phi\cos\theta\sin\theta)\rho^2\sin\phi d\rho d\phi d\theta +$
$\int_0^{2\pi}\int_{\arctan(1/2)}^{\pi/2}\int_0^{\cot\phi\csc\phi}(\rho^2\sin^2\phi\cos\theta\sin\theta)\rho^2\sin\phi d\rho d\phi d\theta$

12.
$$\lim_{x\to 0}\frac{x^2(\frac{1}{2} - \frac{x^2}{4!} + \cdots)}{x^2(1 + x + \frac{x^2}{2!} + \cdots)} = \frac{1}{2}$$

13. $\sqrt{4+x^3} = 2(1+\frac{x^3}{4})^{1/2} = 2(1+\frac{1}{2}(x^3/4) + \sum_{n=2}^{\infty} \frac{(-1)^{(n-1)}(1)(3)\cdots(2n-3)x^{3n}}{2^{3n}n!})$
$\int_0^t \sqrt{4+x^3} = 2(t + \frac{t^4}{32} - \frac{t^7}{2^7(7)} + \frac{t^{10}}{2^{10}(10)} - \cdots)$
$\int_0^{0.5} \sqrt{4+x^3} \simeq 1 + \frac{1}{2^8} - \frac{1}{2^{13}(7)} \simeq 1.003889$
$|error| \leq \frac{1}{2^{19}(10)} = 0.2 \times 10^{-6}$

14. (a) $T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$
$R_3(x) = \frac{-15(x-4)^4}{16(4!)z^{7/2}}$
(b) $T_3(4.1) = 2 + \frac{1}{4}(0.1) - \frac{1}{64}(0.1)^2 + \frac{1}{512}(0.1)^3 \simeq 2.0248457$
$|R_2(4.1)| \leq \frac{(15)(0.1)^4}{16(4!)(4^{7/2})} = \frac{(15)(0.1)^4}{2^{11}(4!)} \simeq 3.0518 \times 10^{-8}$ (since $4 < z < 4.1$)

15. (i) Starting with the geometric series one can show $\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$ with $R = 1$
(ii)
$$\frac{1}{2x+5} = (1/9)\left(\frac{1}{1+(2/9)(x-2)}\right) = (1/9)\sum_{n=0}^{\infty}(-2/9)^n(x-2)^n = \sum_{n=0}^{\infty}\frac{(-2)^n(x-2)^n}{3^{2n+2}}$$
where $R = 9/2$

16. (a) $f^{(6)}(0) = \frac{6!}{3} = 240$ (b) $\ln(3/2)$