(5) 1. Expand $f(x) = \sqrt[3]{1+3x}$ as a binomial series. Express your answer in $\sum$ form and state the radius of convergence.

(7) 2. Use a power series to approximate the given definite integral correct to 6 decimal places:

$$\int_0^{0.4} x\ln(1+x^4)dx$$

(2) 3. Find the exact sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$$

(4) 4. How many terms of the Maclaurin series for e^x are needed to approximate $\sqrt{e}$ correct to 2 decimals? Justify clearly.

(12) 5. Given the curve $\mathcal{C}$ having parametric equations: $x = 10 - t^2$; $y = t^3 - 12t$

i) find dy/dx and d^2y/dx^2

ii) find all points on $\mathcal{C}$ where the tangent line is vertical or horizontal

iii) sketch the graph of $\mathcal{C}$ showing the orientation of the curve

iv) **set up, but do not evaluate,** the integral needed to find the area of the region bounded by $\mathcal{C}$ and the x-axis on the interval $0 \le t \le 2$.

(7) 6. Given the polar curves $r = \sin\theta$ and $r = \sin(2\theta)$, do the following:

i) sketch both graphs on the same axes

ii) find all the points of intersection for $\theta \in [0, \pi]$

iii) **set up, but do not evaluate,** the integral needed to find the area of the region common to both curves.

(12) 7. Sketch the space curve $\mathcal{C}$ defined by $\vec{r}(t) = \langle 3t, 3\cos t, 3\sin t\rangle$ and find the following:

i) the length of $\mathcal{C}$ on the interval $0 \le t \le \pi/2$

ii) the unit tangent vector $\vec{T}$ and the principal unit normal vector $\vec{N}$

iii) an equation of the tangent line to the curve at $t = \frac{\pi}{4}$

iv) the curvature κ

(6) 8. Sketch and give the name of the following surfaces:

i) $\frac{x^2}{9} - y^2 - z^2 = 1$

ii) $4x^2 + y^2 + 3z^2 = 2y$

(3) 9. Approximate $f(0.98, 1.03)$, where $f(x, y) = x^5 - 2y^3$ using differentials.

(4) 10. For the level surface $x^2y + y^2z + z^2x = 5$, find the equations of the tangent plane and normal line at the point $P(1, -1, 2)$.

(3) 11. If $z = xy + f(x^2 + y^2)$, show that

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y^2 - x^2$$

(6) 12. Find and classify the critical points of $f(x, y) = x^3 + y^3 + 3x^2 - 18y^2 + 81y$.

(4) 13. Use the method of Lagrange multipliers to find the **minimum** of $f(x, y, z) = x^2 + y^2 + z^2$ **subject to** $3x + 2y + z = 6$

(3) 14. Let $f(x, y)$ be a continuous function for all x and y. Reverse the order of integration in

$$\int_1^2 \int_x^{x^3} f(x, y)dydx + \int_2^8 \int_x^8 f(x, y)dydx$$

(5) 15. **Set up, but do not evaluate,** a triple integral needed to calculate the volume of the solid S that is bounded by the surfaces $y^2 + z = 4$ and $x^2 + 3y^2 = z$. Sketch the region.

(6) 16. Find the volume of the solid region S bounded above by $z = 8 - (x^2 + y^2)$ and below by $z = x^2 + y^2$.

(6) 17. let S be the solid region above the xy-plane, inside the cylinder $x^2 + y^2 = 4$ and outside the cone $z^2 = 3(x^2 + y^2)$. Evaluate (using spherical coordinates)

$$\iiint_S \frac{1}{x^2 + y^2 + z^2} dV$$

(5) 18. Evaluate $\iint_R \left(\frac{2x + y}{x - 2y + 5}\right)^2 dA$ where R is the region enclosed by the lines: $y = -2x$, $x = 2y$, $2x + y = 5$ and $x - 2y = 5$ (Hint: Use the transformation: $u = \frac{2x+y}{5}$ and $v = \frac{x-2y}{5}$).

ANSWERS

1. $(1 + 3x)^{1/3} = 1 + x + \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n(2)(5)\cdots(3n - 4)}{n!} x^n$

2. $\displaystyle\int_0^{0.4} x \ln(1 + x^4)dx \simeq \frac{(0.4)^6}{6} - \frac{(0.4)^{10}}{20} = 0.0006775$

   $|\text{error}| < \frac{(0.4)^{14}}{42} \simeq 6.4 \times 10^{-8}$

4. Use Lagrange's remainder to show $n = 3$, i.e., 4 terms are needed.

3. the sum is equal to $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

5. $\dfrac{dy}{dx} = \dfrac{-3t^2 + 12}{2t}$; $\dfrac{d^2y}{dx^2} = \dfrac{3t^2 + 12}{4t^3}$

   V.T. at $(10, 0)$ and H.T. at $(6, \pm 16)$

   $A = \displaystyle\int_0^2 (t^3 - 12t)(-2t)dt$

6. Intersection of a four-leaved rose and a circle; Points of intersection are: $(\frac{\sqrt{3}}{2}, \frac{\pi}{3})$, $(\frac{\sqrt{3}}{2}, \frac{2\pi}{3})$ and the pole.

   $A = \int_0^{\pi/3} \sin^2\theta d\theta + \int_{\pi/3}^{\pi/2} \sin^2(2\theta) d\theta$

7. The curve is a circular helix lying on the boundary of the cylinder $y^2 + z^2 = 9$.

   $\mathcal{L} = \frac{3\sqrt{2}\pi}{2}$; $\vec{T}(t) = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\cos t\rangle$; $\vec{N}(t) = \langle 0, -\cos t, -\sin t\rangle$

   $\langle x, y, z\rangle = \langle \frac{3\pi}{4}, \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\rangle + t\langle 3, -\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\rangle$; $t \in \mathbf{R}$

   $\kappa = \frac{1}{6}$ (curvature at any point)

8. i) Hyperboloid of two sheets with x axis as its axis.

   (ii) Ellipsoid $4x^2 + (y-1)^2 + 3z^2 = 1$ (its center is $(0, 1, 0)$)

9. $f(0.98, 1.03) \simeq f(1, 1) + df|_{(1,1)} = -1.28$

10. The tangent plane is $2x - 3y + 5z = 15$ and the Normal line is $\langle x, y, z\rangle = \langle 1, -1, 2\rangle + t\langle 2, -3, 5\rangle$; $t \in \mathbf{R}$

11. Let $u = x^2 + y^2$ then $\frac{\partial z}{\partial x} = y + \frac{\partial f}{\partial u}(2x)$ and $\frac{\partial z}{\partial y} = x + \frac{\partial f}{\partial u}(2y)$ leading to the the result.

12. $(0, 3)$ and $(-2, 9)$ are saddle points; $(0, 9)$ is a local minimum and $(-2, 3)$ is a local maximum.

13. Minimum is attained at the point $(9/7, 6/7, 3/7)$; the minimum value of f is $18/7$.

14. $I = \int_1^8 \int_{\sqrt[3]{y}}^{y} f(x, y) dx dy$ (sketch the region)

15. One possible way to set up the integral is:

    $V = \int_{-2}^{2} \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} \int_{x^2+3y^2}^{4-y^2} dz dy dx$

16. Use cylindrical coordinates; $V = 16\pi$

17. $I = \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^{2/\sin\phi} \rho^2 \sin\phi d\rho d\phi d\theta = \frac{4\pi^2}{3}$

18. $I = \int_0^1 \int_0^1 \left(\frac{u}{v+1}\right)^2 (5) du dv = \frac{5}{6}$