(14) 1. Given the curve C having parametric equations $x = \frac{1}{3}t^3$; $y = t^2 - 2t$

i) Find dy/dx and d^2y/dx^2

ii) Find all points on C where the tangent line is vertical or horizontal

iii) Sketch the graph of C showing the orientation of the curve

iv) **Set up** the integral needed to find: a) the area of the region bounded by C and the x-axis; b) the length of C on the interval $0 \leq t \leq 2$.

(5) 2. Sketch the curve $r = 1 + 2\cos\theta$ and **set up** the integral needed to find the area of the region inside the bigger loop but outside the smaller loop.

(11) 3. Sketch the space curve C defined by $\vec{r}(t) = \langle \sqrt{2}\cos t, \sin t, \sin t \rangle$ and find the following:

i) the length of C on the interval $0 \leq t \leq \pi$

ii) the unit tangent and principal unit normal vectors T and N

iii) the equation of the tangent line to the curve at $t = \frac{\pi}{3}$

iv) the curvature at $t = \frac{\pi}{3}$

(6) 4. a) Sketch and give the name of the surface $z = \sqrt{x^2 + 4y^2 - 1}$
b) Convert $\rho = 4\sin\phi\cos\theta$ into both Cartesian and cylindrical coordinates. Sketch and give the name of this surface.

(9) 5. Suppose $F(x, y, z) = x^2y + y^2z + \cos(xz)$
a) Find the directional derivative of F at the point $P(0, 2, 1)$ in the direction of $\vec{v} = \langle 1, -1, 2 \rangle$.
b) For the level surface $F(x, y, z) = 5$, find the equation of the tangent plane at the point $P(0, 2, 1)$.
c) On the level surface $F(x, y, z) = 5$, find $\frac{\partial z}{\partial y}$ (Hint: use implicit differentiation)

(6) 6. Find and classify the critical points of $f(x, y) = 4xy - x^2y - y^3$

(4) 7. If $z = f(x, y)$ where $x = r\cos\theta$ and $y = r\sin\theta$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial^2 z}{\partial\theta\partial r}$

(4) 8. Evaluate $\displaystyle\int_0^1 \int_{4x}^4 e^{-y^2}\, dy dx$

(4) 9. Sketch the solid part of the sphere $x^2 + y^2 + z^2 = 4a^2$ cut off by the cylinder $x^2 - 2ax + y^2 = 0$ and **set up** an integral needed to find its volume.

(7) 10. a) Sketch the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the hemisphere $z = 2 + \sqrt{4 - x^2 - y^2}$

b) Set up triple integrals in both Cartesian and spherical coordinates needed to find the volume of S

c) Evaluate the easier of the two integrals in part (b).

(5) 11. Evaluate $\displaystyle\iint_R (x + y)e^{x^2 - y^2}\, dA$ where R is the region enclosed by the lines: $x - y = 0$, $x - y = 2$, $x + y = 0$ and $x + y = 3$.

(10) 12. a) Use the Binomial Series to find the Maclaurin series for $f(x) = \sqrt{1 + x}$. b) Using the answer in part (a), find the Maclaurin series for $g(x) = \sqrt{4 + x^2}$. c) Find $g^{(6)}(0)$ without differentiating 6 times (No decimals please!). d) Using the answer in part (b), find an approximation for $\sqrt{4.25}$ correct to 3 decimal places.

(5) 13. Given $S(x) = \displaystyle\int_0^x \frac{t - \sin t}{t^3}\, dt$ find a power series for $S(x)$ about $c = 0$ and give its radius of convergence.

(6) 14. a) Find a Taylor Polynomial of degree 2 for $f(x) = \sqrt[3]{x}$ centered at $c = 8$. b) Use part (a) to approximate $\sqrt[3]{9}$ and give an upper bound for the error in your approximation, using Lagrange's form of the remainder.

(4) 15. a) Find a power series for $f(x) = \dfrac{x}{(1 - x)^2}$. b) Use part (a) to find the sum of $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Answers

1. $\frac{dy}{dx} = \frac{2t-2}{t^2}$; $\frac{d^2y}{dx^2} = \frac{4-2t}{t^5}$
   V.T. at $(0,0)$ and H.T. at $(\frac{1}{3}, -1)$
   $A = -\int_0^2 (t^4 - 2t^3)dt$ and $\mathcal{L} = \int_0^2 \sqrt{t^4 + 4(t-1)^2}dt$

2. $A = \int_0^{2\pi/3} (1 + 2\cos\theta)^2 d\theta - \int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^2 d\theta$

3. Note $x^2 + y^2 + z^2 = 2$ and $y = z$. So the curve is intersection of the the given sphere and plane. Alternatively it can be considered as the curve of intersection of the cylinder $\frac{x^2}{2} + y^2 = 1$ and the plane $y = z$.
   $\mathcal{L} = \sqrt{2}\pi$; $\vec{T}(t) = \langle -\sin t, \frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\cos t\rangle$; $\vec{N}(t) = \langle -\cos t, -\frac{1}{\sqrt{2}}\sin t, -\frac{1}{\sqrt{2}}\sin t\rangle$
   $\langle x, y, z\rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\rangle + t\langle -\sqrt{6}, 1, 1\rangle$; $t \in \mathbf{R}$
   $\kappa = \frac{1}{\sqrt{2}}$ (curvature at any point)

4. a) Hyperboloid of one sheet, top part only. b) Sphere of center $(2,0,0)$ and radius 2.

5. a) Let $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$ then $D_{\vec{u}}F(0,2,1) = \frac{4}{\sqrt{6}}$
   b) $y + z = 3$
   c) $\dfrac{\partial z}{\partial y} = -\dfrac{x^2 + 2yz}{y^2 - x\sin(xz)}$

6. $(0,0)$ and $(4,0)$ are saddle points; $(2, -\frac{2}{\sqrt{3}})$ is a local minimum while $(2, \frac{2}{\sqrt{3}})$ is a local maximum.

7. $\frac{\partial z}{\partial r} = f_x \cos\theta + f_y \sin\theta$
   $\frac{\partial^2 z}{\partial\theta\partial r} = -r(f_{xx}\cos\theta + f_{yx}\sin\theta)\sin\theta + r(f_{xy}\cos\theta + f_{yy}\sin\theta)\cos\theta - f_x\sin\theta + f_y\cos\theta$

8. $\frac{1}{8}(1 - e^{-16})$

9. $V = \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} \int_{-\sqrt{4a^2-r^2}}^{\sqrt{4a^2-r^2}} r\,dz\,dr\,d\theta = \frac{16\pi a^3}{3}$

10. $V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2+\sqrt{4-x^2-y^2}} dz\,dy\,dx$
    $V = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{4\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 8\pi$

11. $I = \frac{1}{4}(e^6 - 7)$

12. $(1+x)^{1/2} = 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} \frac{(-1)^n (1)(3)\cdots(2n-3)x^n}{2^n n!}$
    $\sqrt{4+x^2} = 2\left(1 + \frac{x^2}{4}\right)^{1/2} = 2 + \frac{1}{4}x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n (1)(3)\cdots(2n-3)x^{2n}}{2^{3n-1} n!}$
    $g^{(6)}(0) = \frac{45}{32}$
    $\sqrt{4.25} \simeq 2 + \frac{1}{16} - \frac{1}{1024} \simeq 2.062$; $|error| \leq a_3 \simeq \frac{1}{2^{15}} = 0.0000305$

13. $S(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+3)!}$ $\quad R = \infty$

14. $T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$
    $R_2(9) = \frac{5}{81(z^{8/3})}$ so $|R_2(9)| \leq \frac{5}{81(8^{8/3})} \simeq 0.000241$ (since $8 < z < 9$)
    $\sqrt[3]{9} \simeq 2 + \frac{1}{12} - \frac{1}{288} \simeq 2.07986$

15. Starting with the geometric series one can show $\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$
    put $x = 1/2$ to get $\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$