(Marks)

(6) 1. Let $g(x) = \int_0^x \frac{1 - e^{-t^3}}{t}\, dt,\ x \geq 0.$

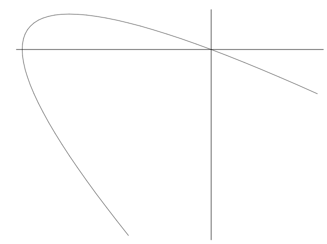
(a) What is the Maclaurin series for $g(x)$?

(b) Estimate $g(0.2)$ correctly to 7 decimal places.

(8) 2. Use the Binomial series to obtain the Maclaurin series for $f(x) = \sqrt{1+x}$; what is its radius of convergence?
Using this series, find the Maclaurin series for $\sqrt{9+x}$; what is its radius of convergence?
Use the first four terms of this series to approximate $\sqrt{9.5}$

(4) 3. Find the smallest $n$ so that the Maclaurin polynomial $p_n(x)$ for $f(x) = e^x$ approximates $e^x$ to within $\pm 5 \times 10^{-6}$ on the interval $|x| \leq 0.5$

(8) 4. Consider the curve given by parametric equations $\begin{cases} x = t^2 - 4 \\ y = 2t - t^2 \end{cases}$, whose graph is given at the right.



(a) Find the $x$ and $y$ intercepts and the coordinates of the point where the tangent is horizontal.

(b) Find $\frac{d^2y}{dx^2}$ and deduce where the graph changes concavity.

(c) Find the area of the region above the $x$-axis and below the curve.

(4) 5. Find the arclength of the cardioid $r = 1 - \cos\theta$.

(4) 6. Set up (*but do not evaluate*) the integral needed to find the area of the region common to both $r = 3 - 2\cos\theta$ and $r = 2$.

(6) 7. Identify and sketch the following surfaces. Show all your work, including appropriate traces and intercepts.

(a) $\rho = \sin\theta \sin\varphi$

(b) $x^2 + y^2 = z^2 + 1$

(13) 8. A particle moves along the curve $\boldsymbol{r}(t) = e^{-2t}\,\boldsymbol{i} + e^{2t}\,\boldsymbol{j} + 2\sqrt{2}\,t\,\boldsymbol{k}$

(a) Find the velocity, speed, and acceleration at $t = 0$.

(b) Find the unit tangent vector and the equation of the tangent line to the curve at $t = 0$.

(c) Find the tangential and normal components of acceleration $a_T, a_N$ at $t = 0$

(d) Find the curvature $\kappa$ at $t = 0$.

(e) What is the length of the curve from $(1, 1, 0)$ to $(e^{-2}, e^2, 2\sqrt{2})$?

(3) 9. Show that the following function is not continuous at the origin. Be sure to justify your answer.

$$f(x) = \begin{cases} \frac{-3xy}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(4) 10. Find the point of maximum curvature for the curve $\begin{cases} x = 5\cos t \\ y = 3\sin t \end{cases}$.

(4) 11. If $z = f(x^2 - y^2,\ y^2 - x^2)$, for $f$ a differentiable function, then show that $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$.

(Marks)

(8) 12. Given the surface $g(x,y,z) = x^3 + y^3 + z^3 - xyz = 0$:

(a) Find the directional derivative of $g$ at the point $P_0(0,-1,1)$ in the direction $\boldsymbol{v} = \langle 2,-1,2\rangle$.

(b) Find the equation of the tangent plane to the surface at the point $P_0$.

(c) Show that the curve $\boldsymbol{r} = \left\langle \dfrac{t^3}{4} - 2\,,\ \dfrac{4}{t} - 3\,,\ \cos(t-2)\right\rangle$ is tangent to this surface at the point $P_0$.

(5) 13. Find and classify the critical points of $f(x,y) = 2x^2 + 4xy^2 - x^3 - 8y^2$.

(5) 14. Find the extreme values of $f(x,y) = x^2 + 2y^2 - 2x + 3$ subject to the constraint $x^2 + y^2 = 10$.

(5) 15. Given $\displaystyle\iiint_{\mathcal{S}} \sqrt{x^2+y^2}\, dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{1} \sqrt{x^2+y^2}\, dz\, dy\, dx$
sketch the region $\mathcal{S}$, and convert the integral to cylindrical and to spherical coordinates.
(*Do not evaluate the integral.*)

16. Evaluate:

(4) (a) $\displaystyle\int_0^1\int_{y^2}^1 y\sin(x^2)\, dx\, dy$

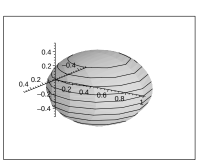
(4) (b) $\displaystyle\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\, dx\, dy$

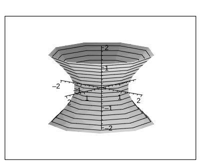
(5) 17. Use the transformation $\{\, u = xy\,,\ v = x^2 - y^2 \,\}$ to evaluate the integral $\displaystyle\iint_{\mathcal{R}} (x^2+y^2)\cos(xy)\, dx\, dy$, where $\mathcal{R}$ is the region to the right of the $y$-axis bounded by the hyperbolas $xy = -3$, $xy = 3$, $x^2 - y^2 = 1$, and $x^2 - y^2 = 9$.

## Answers

**1.**(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{3n}}{n!3n}$ (b) $0.00266133 \pm 0.000000009$ **2.** $\displaystyle\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!2^n}x^n$ R of C = 1. $\displaystyle\sqrt{9+x} = 3 + \frac{x}{6} + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{3\cdot 1\cdot 3\cdot 5\cdots(2n-3)}{n!18^n}x^n$ R of C = 9. $\sqrt{9.5} \simeq 3.082208 \pm 0.0000011$
**3.** $n = 6$ (Note: this *must* be done with Taylor's Inequality.) **4.**(a) $x$-int: $(-4,0)$, $(0,0)$; $y$-int: $(0,-8)$, $(0,0)$; HT: $(-3,1)$. (b) $\dfrac{d^2y}{dx^2} = \dfrac{-4}{8t^3}$; PI: $(-4,0)$ (c) $\frac{8}{3}$. **5.** 8 **6.** $\int_0^{\pi/3}(3 - 2\cos\theta)^2\, d\theta + \int_{\pi/3}^{\pi} 4\, d\theta$
**7.**(a) Sphere (radius: $\frac{1}{2}$, center: $(0,\frac{1}{2},0)$) (b) Hyperboloid of one sheet (central axis: $z$-axis) [Graphs below] **8.**(a) $\boldsymbol{v}(0) = \langle -2, 2, 2\sqrt{2}\rangle$, $v(0) = 4$, $\boldsymbol{a}(0) = \langle 4,4,0\rangle$ (b) $\boldsymbol{T}(0) = \langle -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\sqrt{2}\rangle$, TL: $\{x = 1-t\,,\ y = 1+t\,,\ z = \sqrt{2}t\}$. (c) $a_T(0) = 0$, $a_N(0) = 4\sqrt{2}$. (d) $\kappa(0) = \frac{1}{4}\sqrt{2}$ (e) length: $e^2 - e^{-2}$.
**9.** Along the path $y = x$ the limit is $-\frac{3}{2} \neq 0$. **10.** $(\pm 5, 0)$ (calculate it!, don't just guess.) **11.** LHS $= 2xy(f_u - f_v + f_v - f_u) = 0$ **12.**(a) $\frac{5}{3}$ (b) $x + 3y + 3z = 0$ (c) Because $(v)(2)\cdot\nabla g(P_0) = 0$ **13.** 3 saddle pts: $(0,0), (2,\pm 1)$; local max: $(\frac{4}{3}, 0)$ **14.** Max's: $f(-1,\pm 3) = 24$; min: $f(\sqrt{10},0) = 12 - 2\sqrt{10}$
**15.** $\int_0^{2\pi}\int_0^1\int_{r^2}^1 r^2\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\varphi}\rho^3\sin^2\varphi\, d\rho\, d\varphi\, d\theta + \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\cot\varphi\csc\varphi}\rho^3\sin^2\varphi\, d\rho\, d\varphi\, d\theta$ [Graph below] **16.**(a) $\frac{1}{4} - \frac{1}{4}\cos(1)$ (b) $\frac{\pi}{8}\ln 5$ **17.** $8\sin(3)$

**The Graphs:**

14(a) 

14(b) 

14(c)