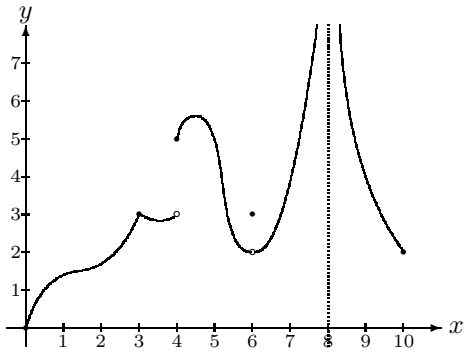
(Marks)

(7) 1. Below is the graph of a function $y = f(x)$ with domain contained in the closed interval $[0, 10]$. Open circles represent points not on the graph of f and solid circles represent points that are on the graph.



(a) For what values of x does the function have;
i. jump discontinuities? ii. infinite discontinuities?
iii. removable discontinuities?

(b) Determine;
i. $\lim_{x\to 4^-} f(x)$ ii. $\lim_{x\to 4^+} f(x)$
iii. $\lim_{x\to 4} f(x)$ iv. $\lim_{x\to 8} f(x)$

(4 × 2) 2. Calculate the following limits if possible. Note that a limit is allowed to be ∞ or $-\infty$. State *dne* (does not exist) if the limit is neither $\pm\infty$ nor a definite real number.

(a) $\lim_{x\to -2} \dfrac{2x^2 + x - 6}{x + 2}$ (b) $\lim_{x\to 3^-} \dfrac{x^2 + 1}{x^2 - 9}$ (c) $\lim_{x\to\infty} \dfrac{x + 3}{\sqrt{x^2 + 1}}$ (d) $\lim_{x\to 2^-} \left(\dfrac{1}{x - 2} - \dfrac{4}{x^2 - 4}\right)$

(3) 3. Find the derivative of the function $f(x) = \sqrt{2x}$ using the *limit* definition of the derivative.

(6 × 4) 4. Find the derivative of each of the following functions. *Do not simplify your answers.*

(a) $y = x\sqrt{x} + \dfrac{1}{x^2\sqrt{x}} - \sin\sqrt{x}$ (b) $y = \dfrac{\mathrm{e}^{5x}}{5x + \mathrm{e}^{5x}} + \mathrm{e}^{5\pi}$ (c) $y = \csc x + \mathrm{e}^{\cos x}\cos x$

(d) $f(x) = \ln(x^3 + 3x)$ (e) $f(x) = \sec(\tan 3x)$ (f) $f(x) = (x^2 + 1)^{\sin x}$

(3) 5. The curve $y = \dfrac{1}{1 + x^2}$ is called a **witch of Maria Agnesi**. Find an equation of the tangent line to this curve at the point $(-1, \frac{1}{2})$.

(3) 6. Give the second derivative of the following function. $h(t) = \mathrm{e}^t \sin t$.

(3) 7. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y'$ by *implicit* differentiation for $\dfrac{x + 3}{y} = 4x + y^2$.

(6) 8. A particle moves along a straight line with equation of motion $s = f(t) = t^3 - 9t^2 + 15t + 10$, where s is measured in meters and t is measured in seconds.

(a) Find the acceleration when $t = 2$. (b) When is the particle at rest?

(c) When is the particle moving in the positive direction?

(d) When does the particle reach a velocity of -6 m/s?

(4) 9. Suppose a forest fire spreads in a circle with the radius changing at the rate of 2 meters per minute. When the radius reaches 80 meters, at what rate is the area of the burning region increasing?

(4) 10. Let $f(x) = x^3 + x^2 - 8x - 1, \quad -3 \le x \le 4$. Find the absolute maximum value and the absolute minimum value of f and state where they are attained.

(4) 11. Consider the following problem:
An open rectangular box with a square base must have a volume of $13,500\text{cm}^3$. Find the dimensions of the box that minimise the amount of material used.

(a) produce an explicit function depending on one variable only whose optimisation solves the problem.
(b) state what the variable represents and any restrictions on it. (c) STOP

(9) 12. Let $f(x) = \dfrac{(2x + 3)(x - 3)^2}{x^3}$. Then $f'(x) = 9\dfrac{(x - 3)(x + 3)}{x^4}$ and $f''(x) = -18\dfrac{x^2 - 18}{x^5}$.

Sketch a graph of f including all asymptotes, intercepts, local extrema and points of inflection labelled as such. (*Note:* $f(3\sqrt{2}) \approx 0.25$ & $f(-3\sqrt{2}) \approx 3.75$.)

(Marks)

(4) 13. Find $f(x)$ given $f'(x) = \mathrm{e}^x - 2\sin x$, and $f(0) = -3$.

(4 × 3) 14. Evaluate the following integrals.

(a) $\displaystyle\int \left(5x^2 - \frac{5}{x^2} - \sqrt[5]{x^2} - 2^5\right) \mathrm{d}x.$ (b) $\displaystyle\int \frac{\cos x}{1-\cos^2 x}\,\mathrm{d}x.$ (c) $\displaystyle\int \frac{(x-3)^2}{x}\,\mathrm{d}x.$

(d) $\displaystyle\int_{\pi/6}^{\pi/3} (12 - \sec^2 x)\,\mathrm{d}x.$ *Exact answer please, no decimals.*

(3) 15. Find the area of the region between $y = 4x + x^2$ and the x-axis from $x = 1$ to $x = 3$.

(3) 16. Evaluate $\displaystyle\int_{-1}^{4} |x-3|\,\mathrm{d}x$ by interpreting it in terms of area.

ANSWERS

1. (a) 4, 8, 6 (b) 3, 5, DNE, ∞

2. (a) -7 (b) $-\infty$ (c) 1 (d) 1/4

3. $f'(x) = \displaystyle\lim_{h\to 0} \frac{\sqrt{2(x+h)} - \sqrt{2x}}{h} = \frac{1}{\sqrt{2x}}$

4. (a) $\displaystyle\frac{3}{2}x^{1/2} - \frac{5}{2}x^{-7/2} - (\cos\sqrt{x})\frac{1}{2\sqrt{x}}$ (b) $\displaystyle\frac{5\mathrm{e}^{5x}(5x+\mathrm{e}^{5x}) - \mathrm{e}^{5x}(5+5\mathrm{e}^{5x})}{(5x+\mathrm{e}^{5x})^2}$

(c) $-\csc x\cot x + \mathrm{e}^{\cos x}(-\sin x)\cos x + \mathrm{e}^{\cos x}(-\sin x)$ (d) $\displaystyle\frac{3x^2+3}{x^3+3x}$

(e) $3\sec(\tan 3x)\tan(\tan 3x)\sec^2 3x$ (f) $(x^2+1)^{\sin x}\left(\cos x\ln(x^2+1) + \sin x\frac{2x}{x^2+1}\right)$

5. $y = \frac{1}{2}x + 1$ 6. $h''(t) = 2\mathrm{e}^t\cos t$ 7. $\displaystyle\frac{y - 4y^2}{x + 3 + 2y^3}$

8. (a) -6 m/s (b) $t = 1$s or 5s (c) $t < 1$ and $t > 5$ (d) $t = 3 \pm \sqrt{2}$s

9. $320\pi\mathrm{m}^2/\mathrm{s}$ 10. MAX = 47, MIN = -203/27

11. (a) $A(x) = x^2 + 54000/x$ (b) where x (> 0) is the length of a side of the base.

12. See graph

13. $f(x) = \mathrm{e}^x + 2\cos x - 6$

14. (a) $\displaystyle\frac{5}{3}x^3 + \frac{5}{x} - \frac{5}{7}x^{7/5} - 32x - C$ (b) $-\csc x + C$

(c) $\displaystyle\frac{x^2}{2} - 6x + 9\ln|x| + C$ (d) $2\pi - 2/\sqrt{3}$.

15. 74/3 16. 17/2