Calculus III Final Examination

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Exhibit your work . You may assume the following :

$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

1. Establish a power series representation for $y = \arctan x$. (6%)
2. Consider $f(x) = \int_0^x \sin(t^2)\, dt$
   (a) Represent $f(x)$ as a power series . (7%)
   (b) Compute $f^{(7)}(0)$. (3%)
   (c) Estimate $f(1)$ to 3 decimal places . (5%)
3. Find the first four non-zero terms of the Maclaurin series for $y = e^x \cos x$ (5%)
4. Draw a rough sketch of the polar curve $r = 1 - 2\cos\theta$. Find analytically two (distinct) points at which the tangent is horizontal . (8%)
5. Write integrals for each of the following :
   DO NOT ATTEMPT TO SOLVE THE INTEGRALS
   (a) The perimeter of the hypocycloid
   $x = \cos^3\theta \;\; y = \sin^3\theta \quad \theta \in [0, 2\pi]$ (3%)

(b) The area of one petal of the rose $r = 3\cos(3\theta)$ (3%)

(c) the volume of the solid region above the XY plane which lies inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 1$ in
   (a) rectangular coordinates. (2%)
   (b) cylindrical coordinates. (2%)
   (c) spherical coordinates. (2%)

6. Consider a particle moving on a circular path of radius b $\overrightarrow{R}(t) = (b\cos\omega t\,, b\sin\omega t\,)$ where $\omega = \dfrac{d\theta}{dt}$ is the constant angular velocity.

   (a) Find the velocity vector and show that it is orthogonal to $\overrightarrow{R}(t)$. (3%)

   (b) Find the speed of the particle. (2%)

   (c) Find the magnitude of the acceleration vector. (2%)

   (d) Demonstrate that the acceleration vector is always directed toward the centre of the circle. (2%)

   (e) Compute the curvature. (3%)

7. $w = f(x, y)$ has continuous partial derivatives. Suppose that we substitute the polar coordinates r and θ .

   Prove $\dfrac{1}{r}\dfrac{\partial w}{\partial \theta} = -f_x \sin\theta + f_y \cos\theta$. (3%)

8. Find a unit vector in the direction in which $f(x, y) = \cos\pi xy + xy^2$ increases most rapidly at $(\frac{1}{2}, \mathbf{1})$ (3%)

9. Find the extreme value(s) of the function
$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$ (5%)

10. Find the (relative) maximum and minimum values of $f(x, y) = x^2 y$ on the line $x + y = 3$. <u>USE LAGRANGE MULTIPLIERS</u> (7%)

11. Compute the volume of the tetrahedron in the first octant bounded by the coordinate planes and the plane $6x + 3y + 3z = 6$. (6%)

12. Evaluate $\int_0^1 \int_y^1 \frac{\sin x}{x} dx dy$ (5%)

13. Calculate the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and also bounded above by the paraboloid $z = x^2 + y^2$ and below by the XY plane. (5%)

14. Calculate $\int\int\int_E z^2 \, dx \, dy \, dz$ where E is the solid region
$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \leq 1$ (8%)