Calculus III Final Examination

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Exhibit your work . You may assume the following:

$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

1. Establish a power series representation for $y = \arctan x$. (4 %)

2. Find the first four non-zero terms of the Maclaurin series for $y = e^{-x} \cos x$ (3 %)

3. Consider $f(x) = \int_0^x \sin(t^2)\, dt$

   (a) Represent $f(x)$ as a power series . (4%)

   (b) Compute $f^{(7)}(0)$. (1%)

   (c) Estimate $f(1)$ to 3 decimal places . (4 %)

4. Consider $f(x) = \sqrt{1+x}$

   (a) Write the first two terms of the Taylor Expansion with $c = 8$ (4 %)

   (b) Suppose one uses that expansion to estimate $f(8.01)$.
   Estimate the error $R_n(8.01)$. (3%)

5. Write integrals for each of the following: (3.5% each)

DO NOT ATTEMPT TO SOLVE THE INTEGRALS

(a) The perimeter of the hypocycloid
$x = \cos^3\theta$, $y = \sin^3\theta \quad \theta \in [0, 2\pi]$

(b) The area of the region bounded by the coordinate axes
and $x = t^3$, $y = t + 2$.

(c) The area of one petal of the rose $r = 3\cos(3\theta)$

(d) The surface area of the apple-shaped object obtained by revolving the top half of the cardioid $r = 2(1 - \cos\theta)$ about the $X - axis$.

(e) The volume of smaller part cut from the sphere $\rho = 2$
by the plane $z = 1$.

(f) The moment of inertia about the $Z - axis$ of the solid region [shaped like an ice cream cone (1 scoop)] bounded by the sphere $x^2 + y^2 + z^2 = R^2$ and the cone $z = \sqrt{x^2 + y^2}$.
The density of the solid is $\delta(x, y, z) = (z + 3)^2$

6. Consider the curve generated by the parametric equations:
$x = t^2 - 1$, $y = t + 1$

(a) Draw a rough sketch of its graph for $-2 \leq t \leq 2$ (2%)

(b) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. (3%)

(c) Determine all points at which the tangent is horizontal or vertical.
(2%)

7. Consider a particle moving on a circular path of radius b
$\overrightarrow{R}(t) = (b\cos\omega t\,, b\sin\omega t\,)$ where $\omega = \dfrac{d\theta}{dt}$ is the constant angular velocity.

(a) Find the velocity vector and show that it is orthogonal to $\overrightarrow{R}(t)$.
(3%)

(b) Find the speed of the particle at any time t. (2%)

(c) Find the magnitude of the acceleration vector. (3%)

(d) Demonstrate that the acceleration vector is always directed toward the centre of the circle. (3%)

(e) Compute the curvature. (3%)

8. Verify that $w = \ln(2x + 2ct)$ is a solution of the wave equation (3%)
$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}$$

9. Write the equation of the tangent plane at $(1, -1, 13)$ to $f(x, y) = x^3 + y^3 - 3x - 12y + 4$ (3 %)

10. Consider $f(x, y) = \cos \pi xy + xy^2$.

(a) Compute the directional derivative of f at $(-1, -1)$ in the direction of $\mathbf{i}+\mathbf{j}$. (2%)

(b) Estimate, using the directional derivative, the change

in f if (x, y) is moved from $(-1, -1)$ a distance of 0.1 units in the direction of $\mathbf{i}+\mathbf{j}$. (2%)

11. Find and classify the critical points of $z = f(x, y) =$ $x^3 + y^3 - 2xy + 6$ (6 %)

12. Determine the largest value of $f(x, y) = e^{x^2 - y^2}$ on and in the triangle in the first quadrant bounded by the coordinate axes and the line $x + 5y = 6$. (7 %)

13. Find the largest and smallest values of $f(x, y) = x^2 + 4y^3$ on the ellipse $x^2 + 2y^2 = 1$. (6%)

14. Evaluate $\int_0^{16} \int_{\sqrt{x}}^{4} \cos(y^3)\, dy\, dx$ (4%)

15. Determine the volume of the solid cut from the first octant by the surface $z = 4 - x^2 - y$. (5%)

SOLUTIONS

1) $\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k$; $\frac{1}{1+t} = \sum_{k=0}^{\infty} (-t)^k$; $\frac{1}{1+t} = \sum_{k=0}^{\infty} (-1)^k\ t^k$

$\frac{1}{1+t^2} = \sum_{k=0}^{\infty} (-1)^k t^{2k}$; $\arctan x = \int_0^x \frac{1}{1+t^2} dt =$

$\sum_{k=0}^{\infty} (-1)^k \int_0^x t^{2k} dt = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$

2) $e^{-x} \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

$= (1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - +....)(1 - \frac{x^2}{2} + \frac{x^4}{24} - +....)$

$= 1 - x + \frac{x^3}{3} - \frac{x^4}{6} + -....$

3a) $\sin t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}$; $\sin t^2 = \sum_{k=0}^{\infty} \frac{(-1)^k t^{4k+2}}{(2k+1)!}$

$\int_0^x \sin(t^2) dt = \sum_{k=0}^{\infty} \frac{(-1)^k t^{4k+3}}{(2k+1)!(4k+3)}$

b) $\frac{f^{(7)}(0)}{7!} = \frac{(-1)^1}{(2(1)+1)!(4(1)+3)} = -\frac{1}{42}$; $f^{(7)}(0) = -120$

c) $k = 2$: $\frac{1}{(120)(11)} \approx .0008 > (5)10^{-4}$

$k = 3$: $|\frac{-1}{(5040)(15)}| \approx .00001 < (5)10^{-4}$

By the $A.S.T.$,as the truncation error $<$ the first term neglected,

$f(1) = \sum_{k=0}^{2} \frac{(-1)^k (1)^{4k+3}}{(2k+1)(4k+3)} = \frac{1}{3} - \frac{1}{42} + \frac{1}{(120)(11)} \approx .310$

4a) $f^{(0)}(8) = \sqrt{1+8} = 3$ $\quad f^{(1)}(8) = \frac{1}{2\sqrt{1+8}} = \frac{1}{6}$

$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(8)(x-8)^k}{k!} = 3 + \frac{x-8}{6} + ...$

b) $R_n(x) \leq |\frac{f^{(n+1)}(z)(x-8)^{n+1}}{(n+1)!}|$ $\quad 8 < z < 8.01$

$|f^{(2)}(z)| = |\frac{1}{2}(-\frac{1}{2})(1+z)^{\frac{-3}{2}}| = |\frac{-1}{4(1+z)^{\frac{3}{2}}}| \leq \frac{1}{4(9)^{\frac{3}{2}}} = \frac{1}{108}$

$R_1(x) \leq |\frac{\frac{1}{108}(8.01-8)^{1+1}}{(1+1)!}| \approx 4.62(10^{-7})$

5a) $s = \int_0^{2\pi} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\ d\theta$

$= \int_0^{2\pi} \sqrt{(-3\cos^2\theta\sin\theta)^2 + (3\sin^2\theta\cos\theta)^2}\ d\theta$

$= 3\int_0^{2\pi} |\cos\theta\sin\theta|\sqrt{\cos^2\theta + \sin^2\theta}\ d\theta = 3\int_0^{2\pi} |\cos\theta\sin\theta|\ d\theta$

b) $A = \int_{-8}^{0} y dx = \int_{-2}^{0}(t+2)(3t^2)dt$

c) $A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}(3\cos 3\theta)^2\ d\theta$

d) $S.A = 2\pi\int_0^{\pi} y ds = 2\pi\int_0^{\pi}(r\sin\theta)\sqrt{r^2 + (\frac{dr}{d\theta})^2}\ d\theta$

$= 2\pi\int_0^{\pi} 2(1-\cos\theta)(\sin\theta)\sqrt{4(1-\cos\theta)^2 + 4\sin^2\theta}\ d\theta$

$= 8\sqrt{2}\pi\int_0^{\pi}(1-\cos\theta)(\sin\theta)\sqrt{1-\cos\theta}\ d\theta$

e) $V = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\sec\phi}^{2}\rho^2\sin\phi d\rho d\phi d\theta$

f) $I_z = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{R}(\rho\cos\phi + 3)^2\rho^4\sin^3\phi d\rho d\phi d\theta$

6a) $x + 1 = t^2 = (y-1)^2$; $x = (y-1)^2 - 1$

parabola opening to the right , vertex (-1,1)

b) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{2t}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{1}{2t}) = \frac{d}{dt}(\frac{1}{2t})\frac{dt}{dx} = -\frac{1}{2t^2}\ (\frac{1}{\frac{dx}{dt}}) = -\frac{1}{4t^3}$

c) A necessary condition for a horizontal tangent is $0 = \frac{dy}{dx} = \frac{1}{2t}$,impossible.

The tangent is vertical at $t = 0$ $i.e.(-1,1)$.

7) $\mathbf{v}(t) = b\omega(-\sin\omega t, \cos\omega t)$

$\mathbf{v}(t) \bullet \mathbf{R}(t) = b\omega(-\sin\omega t, \cos\omega t) \bullet b(\cos\omega t, \sin\omega t) = 0$

b) $||\mathbf{v}(t)|| = b\omega\sqrt{(-\sin\omega t)^2 + (\cos\omega t)^2} = b\omega$

c) $\mathbf{a}(t) = -b\omega^2(\cos\omega t, \sin\omega t)$, $||\mathbf{a}(t)|| = b\omega^2$

d) $\mathbf{a}(t) = -\text{constant}(\mathbf{R}(t))$

e) $\kappa = \dfrac{||\mathbf{v}x\mathbf{a}||}{||\mathbf{v}||^3} = \dfrac{||b\omega(-\sin\omega t, \cos\omega t, 0)x(-b\omega^2(\cos\omega t, \sin\omega t, 0))||}{(b\omega)^3}$

$\dfrac{||b^2\omega^3(0,0,-1)||}{(b\omega)^3} = \dfrac{1}{b}$

8) $\dfrac{\partial w}{\partial x} = \dfrac{1}{x+ct}$; $\dfrac{\partial^2 w}{\partial x^2} = -\dfrac{1}{(x+ct)^2}$

$\dfrac{\partial w}{\partial t} = \dfrac{c}{x+ct}$; $\dfrac{\partial^2 w}{\partial t^2} = -\dfrac{c^2}{(x+ct)^2}$

Combining the above yields the result.

9) $\nabla f = (3x^2 - 3, 3y^2 - 12)$; $\nabla f(1,-1) = (0,-9)$

$0(x-1) - 9(y-(-1)) - (z-13) = 0$ *or* $z + 9y = 4$

10) $\nabla f = (-\pi y\sin(\pi xy) + y^2, -\pi x\sin(\pi xy) + 2xy)$

$\nabla f(-1,-1) = (1,2)$; $D_u f(-1,-1) = (1,2).\frac{(1,1)}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$

b) $\Delta f \approx (D_u f).\Delta s = \frac{3\sqrt{2}}{2}(0.1) \approx 0.21$

11) $f_x = 3x^2 - 2y = 0$ *if* $y = \dfrac{3x^2}{2}$,

$f_y = 3y^2 - 2x = 3(\dfrac{3x^2}{2})^2 - 2x = 0$ if $x = 0$ or $\frac{2}{3}$

The critical points are $(0,0)$ and $(\frac{2}{3}, \frac{2}{3})$.

$f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -2$

$D = f_{xx}\, f_{yy} - (f_{xy})^2 = 36xy - 4$

$D(0,0) = -4 \Rightarrow$saddle point

$D(\frac{2}{3}, \frac{2}{3}) = 12 > 0$, $f_{xx}(\frac{2}{3}, \frac{2}{3}) > 0 \Rightarrow$rel. min.

A saddle point occurs at $(0,0)$ and a relative minimum at $(\frac{2}{3}, \frac{2}{3})$.

12) $f_x = 2xe^{x^2-y^2}$, $f_y = -2ye^{x^2-y^2}$ Thus $\nabla f = 0$ only at the origin.
(0,0)$\rightarrow (6,0)$ $f = e^{x^2}$;obvious max at $x = 6$ $i.e.$ e^{36}
(0,0)$\rightarrow (0,\frac{6}{5})$ $f = e{-}^{y^2}$;obvious max at $y = 0$ $i.e.$ $e^0 = 1$
On the oblique line $x = 6 - 5y$ $(0 \leq x \leq 6\ , 0 \leq y \leq \frac{6}{5})$:
$f = e^{(6-5y)^2-y^2}$ which is maximized when its exponent is $i.e.$
at $y = \frac{5}{4} > \frac{6}{5}$ $i.e$ max at endpoint .$f|_{y=0} = 1$; $f|_{y=\frac{6}{5}} = e^{\frac{36}{25}}$
f attains its maximum in the trianglular region of e^{36} at $(6,0)$.

13) $(2x, 12y^2) = \lambda(2x, 4y)$
$g = x^2 + 2y^2 - 1 = 0$ $\boxed{1}$, $2x = \lambda(2x)$ $\boxed{2}$,$12y^2 = \lambda 4y$ $\boxed{3}$
Case I : $x = 0$ From $\boxed{1}$, $y = \pm\frac{1}{\sqrt{2}}$
Case II : $x \neq 0$ From $\boxed{2}$, $\lambda = 1$ From $\boxed{3}$, $y = 0$ or $\frac{1}{3}$.
If $y = 0$, then from $\boxed{1}$, $x = \pm 1$
If $y = \frac{1}{3}$, then $\boxed{1}$ yields $x = \pm\frac{\sqrt{7}}{3}$
The critical points are $(0,\pm\frac{1}{\sqrt{2}})$, $\pm(1,0)$ and $(\pm\frac{\sqrt{7}}{3}, \frac{1}{3})$.
$f(0,\pm\frac{1}{\sqrt{2}}) = \pm\sqrt{2}$; $f(\pm 1, 0) = 1$; $f(\pm\frac{\sqrt{7}}{3}, \frac{1}{3}) = \frac{25}{27}$
The maximum of f of $\sqrt{2}$ is attained at $(0,\frac{1}{\sqrt{2}})$.
The minimum of f of -$\sqrt{2}$ is attained at $(0,-\frac{1}{\sqrt{2}})$.

14) $Integral = \int_0^4 \int_0^{y^2} \cos(y^3)\, dxdy = \int_0^4 y^2 \cos(y^3)\, dy$
$= \frac{1}{3}\sin(y^3)|_0^4 = \frac{1}{3}\sin 64 \approx 0.306675346$

15) $Volume = \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz\, dydx = \frac{376}{30}(units)^3$