This article appeared in:

Mathematical Foundations of Programming Semantics
9th International Conference
New Orleans, USA, April 7-10, 1993
Proceedings

Holomorphic Models of Exponential Types in Linear Logic

R.F. Blute*
Department of Mathematics
University of Ottawa
Ottawa, Canada K1N 6N5
rblute@acadvm1.uottawa.ca

Prakash Panangaden†
School of Computer Science
McGill University
Montreal, Canada H3A 2A7
prakash@cs.mcgill.ca

R.A.G. Seely‡
Department of Mathematics
McGill University
Montreal, Canada H3A 2K6
rags@math.mcgill.ca

Abstract

In this paper we describe models of several fragments of linear logic with the exponential operator ! (called OF COURSE) in categories of linear spaces. We model ! by the *Fock space* construction in Banach (or Hilbert) spaces, a notion originally introduced in the context of quantum field theory. Several variants of this construction are presented, and the representation of Fock space as a space of holomorphic functions is described. This also suggests that the "non-linear" functions we arrive at *via* ! are not merely continuous, but analytic.

Keywords: Fock space, linear logic, Banach spaces, holomorphic functions, quantum field theory.

*Research supported in part by FCAR of Quebec.
†Research supported in part by an NSERC Research Grant.
‡Research supported in part by an NSERC Research Grant and by FCAR of Quebec.

0 Introduction

Linear logic was introduced by Girard [G87] as a consequence of his analysis of the traditional connectives of logic into more primitive connectives. The resulting logic is more resource sensitive; this is achieved by placing strict control over the structural rules of contraction and weakening, introducing a new "modal" operator OF COURSE (denoted $!$) to indicate when a formula may be used in a resource-*insensitive* manner—*i.e.* when a resource is renewable. Without the $!$ operator, the essence of linear logic is carried by the multiplicative connectives; at its most basic level, linear logic is a logic of monoidal-closed categories (in much the same way that intuitionistic logic is a logic of cartesian-closed categories). In modelling linear logic, one begins with a monoidal-closed category, and then adds appropriate structure to model linear logic's additional features. To model linear negation, one passes to the $*$-autonomous categories of Barr [B79]. To model the additive connectives, one then adds products and coproducts. Finally, to model the exponentials, and so regain the expressive strength of traditional logic, one adds a triple and cotriple, satisfying properties to be outlined below. This program was first outlined by Seely in [Se89].

Linear logic bears strong resemblance to linear algebra (from which it derives its name), but one significant difference is the difficulty in modelling $!$. The category of vector spaces over an arbitrary field is a symmetric monoidal closed category, indeed in some sense the prototypical monoidal category, and as such provides a model of the intuitionistic variant of multiplicative linear logic. Furthermore, this category has finite products and coproducts with which to model the additive connectives. It thus makes sense to look for models of various fragments of linear logic in categories of vector spaces. However, modelling the exponentials is more problematic. It is the primary purpose of this paper to present methods of modelling exponential types in categories arising from linear algebra. We study models of the exponential connectives in categories of linear spaces which have monoidal (but generally not monoidal-closed) structure. (We shall also include a model in finite-dimensional vector spaces.)

To model the finer distinctions achieved by linear logic, one ought to consider vector spaces enriched with appropriate additional structure. For example, to model linear negation, one considers vector spaces enriched with an additional topological structure. These are the *linear topologies* of Lefschetz and Barr [Le41, B76a]. The relationship to linear logic is discussed in [Bl93a]. To model the noncommutative [Ab91] or braided

[Bl93b] variants of linear logic, one considers the linear representations of certain Hopf algebras [Bl93a]. Finally, to model the exponentials, it is necessary to consider normed vector spaces.

Vector spaces are inherently finitary structures in the sense that every vector is a finite sum of multiples of basis vectors, and one is allowed only to take finite sums of vectors. To model the notion of infinitely renewable resources, one would like to be able to take infinite sums of vectors. But to do this, one needs a notion of convergence, and to define convergence one needs a notion of topology. The most heavily studied topological vector spaces are Hilbert and Banach spaces which derive their topologies from a norm; either defined indirectly *via* an inner product, as in Hilbert spaces, or directly, as in Banach spaces. Once a vector space is normed, then all of the familiar notions from analysis, such as limit and Cauchy sequence can be defined. What we wish to suggest in this paper is that while the multiplicative and additive fragment MALL of linear logic corresponds to the linear structure of a vector space, the exponentials correspond to its analytic structure.

We begin by introducing the two main notions of complete normed vector space, Banach spaces and Hilbert spaces. The construction which will be used to model the exponential formulas $!A$ arose originally in quantum field theory, and is known as *Fock space.* It was designed as a framework in which to consider many particle states. The key point of departure for quantum field theory was the realization that so-called "elementary" particles are created and destroyed in physical processes and that the mathematical formalism of ordinary quantum mechanics needs to be revised to take this into account. The physical intuitions behind the Fock construction will be sketched in the penultimate section. The formula for Fock space will also be familiar to mathematicians in that it corresponds to the free symmetric algebra on a space. As a free construction, Fock induces a pair of adjoint functors, and hence a cotriple. It is this cotriple which will be used to model $!$. It should be noted that this category of algebras inherits the monoidal structure from the underlying category of spaces but there is no hope that this category could have a monoidal-closed structure.

While Fock space has an abstract representation in terms of an infinite direct sum, physicists such as Ashtekar, Bargmann, Segal and others, see [AM-A80, Ba61, S62] have analyzed concrete representations of Fock space as certain classes of holomorphic functions on the base space. Thus, these models further the intuition that the exponentials correspond to the analytic properties of the space. In fact, there is a clear sense in which morphisms in the Kliesli category for the cotriple can be viewed as generalized holomorphic functions. Thus, there should be an analogy to coherence spaces where the Kliesli category corresponds to the stable maps.

Fock space also has two additional features which correspond to additional structure, not expressible in the syntax of linear logic. These are the *annihilation* and *creation* operators, which are used to model the annihilation and creation of particles in a field. These may give a tighter control of resources not expressible in the pure linear logic. Thus, these models may be closer to the bounded linear logic of Girard, Scedrov and Scott [GSS91].

The results of this paper suggest that analyticity may provide new insights into computability not captured by the traditional notions of continuity. Continuity has been enormously successful in capturing the idea that computable functions process information a finite piece at a time. On the other hand, there are many continuous functions that are not computable. Despite the tremendous clarifications brought about by Scott's ideas, a precise characterization of computability still appeals to notions of encoding from classical recursion theory. With the notion of analytic function one has the notion of convergent power series which represents the function. This is nothing more than an encoding of a continuous function with a discrete string. Thus the notion of encoding *may* be captured by analyticity. Of course, we are far from offering any such theory yet.

Another possible application of this work is that the refined connectives of linear logic may lend insight into certain aspects of quantum field theory. For example, there are two distinct methods of combining particle states. One can superimpose two states onto a single particle, or one can have two particles coexisting. The former seems to correspond to additive conjunction and the latter to the multiplicative. This physical imagery is missing in quantum mechanics, which was specially designed to handle a single particle; it only shows up in quantum field theory.

In this paper, we begin by reviewing the categorical structure necessary to model linear logic, and specifically exponential types. We then give the relevant definitions pertaining to normed vector spaces, as well as a number of examples. We also discuss the monoidal structure of these categories. Then, the various ingredients which go into the construction of Fock space are presented and the resulting adjointness is described. Finally, the holomorphic function representation of Fock space is presented, and a brief description of its physical interpretation is given.

1 Linear Logic and Monoidal Categories

We shall begin with a few preliminaries concerning linear logic. We shall not reproduce the formal syntax of linear logic, nor the usual discussion of its intuitive interpretation or utility—for this the reader is referred to the standard references, such as [G87]. We do recall [Se89] that a categorical semantics for linear logic may be based on Barr's notion of $*$-autonomous categories [B79]. If only to establish notation, here is the definition.

Definition 1 *A category $\mathcal{C}$ is $*$-autonomous if it satisfies the following:*

1. *$\mathcal{C}$ is symmetric monoidal closed; that is, $\mathcal{C}$ has a tensor product $A \otimes B$ and an internal hom $A \multimap B$ which is adjoint to the tensor in the second variable*

$$Hom(A \otimes B, C) \cong Hom(B, A \multimap C)$$

2. *$\mathcal{C}$ has a dualizing object $\perp$; that is, the functor $(\ )^{\perp} : \mathcal{C}^{op} \longrightarrow \mathcal{C}$ defined by $A^{\perp} = A \multimap \perp$ is an involution (*viz.* the canonical morphism $A \longrightarrow ((A \multimap \perp) \multimap \perp)$ is an isomorphism).*

In addition various coherence conditions must hold—a good account of these may be found in [M-OM89]. Coherence theorems may be found in [BCST, Bl91, Bl92]. An equivalent characterization of $*$-autonomous categories is given in [CS91], based on the notion of weakly distributive categories. That characterization is useful in contexts where it is easier to see how to model the tensor $\otimes$, the "par" $⅋$ and linear negation, and the coherence conditions may be expressed in terms of those operations.

The structure of a $*$-autonomous category models the evident eponymous structure of linear logic: the categorical tensor $\otimes$ is the linear multiplicative $\otimes$ and the internal hom $\multimap$ is linear implication. The dualizing object $\perp$ is the unit for linear "par" $⅋$, or equivalently, is the dual of the unit I for the tensor[1].

There are a number of variants of linear logic whose categorical semantics is based on this. First is full "classical" linear logic, which includes the additive operations. These correspond to requiring that the category

[1]In other papers we have used the notation $\top$ for the unit for $\otimes$, and $\oplus$ instead of $⅋$. Here we shall try to avoid controversy by using notation traditional in the context of Banach spaces, and by generally ignoring the "par". So in this paper, $\oplus$ means direct sum, which coincides with Girard's notation. We use $\times$ for cartesian product, corresponding to Girard's &. And we shall use the usual notation for the appropriate spaces when referring to the units.

$\mathcal{C}$ have products and coproducts. (If $\mathcal{C}$ is $*$-autonomous, one of these will imply the other by de Morgan duality.) There is also Girard's notion of "intuitionistic" linear logic [GL87], which omits linear negation and "par"—this corresponds to merely requiring that $\mathcal{C}$ be *autonomous*, that is to say, symmetric monoidal closed (with or without products and coproducts, depending on whether or not the additives are wanted). There is an intermediate notion, "full intuitionistic linear logic" due to de Paiva [dP89], in which the morphism $A \longrightarrow A^{\perp\perp}$ need not be an isomorphism. And as mentioned above, there is the notion of weakly distributive category [CS91, BCST], where negation and internal hom are not required.

One classically important class of $*$-autonomous categories are the *compact* categories [KL80] where the tensor is self-dual: $(A \otimes B)^{\perp} \cong A^{\perp} \otimes B^{\perp}$. Linear logicians often regard with derision those models in which "tensor" and "par" coincide, but from some mathematical points of view these are very natural.

In this paper we shall model various fragments of linear logic; we shall describe the fragments in terms of the categorical structure present, without explicitly identifying the fragments.

Finally, in order to be able to recapture the full strength of classical (or intuitionistic) logic, one must add the "exponential" $!$ (and its de Morgan dual $?$). (All our structures will model $!$.) We saw in [Se89] that this amounts to the following.

Definition 2 *A monoidal category $\mathcal{C}$ with finite products admits (Girard)* **storage** *if there is a cotriple* $! : \mathcal{C} \longrightarrow \mathcal{C}$ *(with the usual structure maps* $A \xleftarrow{\epsilon_A} \,!A \xrightarrow{\delta_A} \,!!A$*), satisfying the following:*

1. *for each object $A \in \mathcal{C}$, $!A$ carries (naturally) the structure of a (cocommutative) $\otimes$-comonoid $\top \xleftarrow{e_A} \,!A \xrightarrow{d_A} \,!A \otimes \,!A$ (and the coalgebra maps are comonoid maps), and*

2. *there are natural comonoidal isomorphisms*

$$\mathrm{I} \xrightarrow{\sim} \,!1 \quad \textit{and} \quad !A \otimes \,!B \xrightarrow{\sim} \,!(A \times B)\ .$$

Some remarks: First, it is not hard to see that the first condition above is redundant, the comonoidal structure on $!A$ being induced by the isomorphisms of the second condition. However, the first condition is really the key point here, as may be seen from several generalizations of this definition, to the intuitionistic case without finite products in [BBPH], and to the weakly distributive case, again without finite products, [BCS93].

The main point here is that without products one replaces the second condition with the requirement that the cotriple ! (and the natural transformations ϵ, δ) be comonoidal. And second, one ought not drown in the categorical terminology—terms like "comonoidal" in essence refer to various coherence (or commutativity) conditions which may be looked up when needed. Readers not interested in coherence questions can follow the discussion by just noting the existence of appropriate maps, and believe that all the "right" diagrams will commute. They can regard it as somebody else's business to ensure that this is indeed the case.

In the mid-1980's, Girard studied coherence spaces as a model of system F, and realized the following fact, which led directly to the creation of linear logic. Of course Girard did not put the matter in these categorical terms at the time, but the essential content remains the same—ordinary implication factors through linear implication *via* the cotriple !. (Another way of expressing this is to say that a model of full classical linear logic induces an interpretation of the typed λ-calculus.)

Theorem 1 *If $\mathcal{C}$ is a $*$-autonomous category with finite products admitting Girard storage !, then the Kleisli category $\mathcal{C}_!$ is cartesian closed.*

This result is virtually folklore, but a proof may be found in [Se89].

One of the problems with finding models of linear logic comes from the difficulty of finding well-behaved (in the above sense) cotriples on $*$-autonomous categories. For example, one of the main problems with vector spaces as a model of linear logic is the lack of any natural interpretation of !. (We shall soon return to this point, and indeed, in a sense this is the main point of this paper.) This question seems closely bound up with questions of completeness. Barr [B91] has shown how in certain cases one can get appropriate cotriples (*via* cofree coalgebras) from a subcategory of the Chu construction [B79]. One case where this route works out fairly naturally is if the $*$-autonomous category is compact: in that case, one can construct cofree coalgebras by the familiar formula

$$!\,A = \top \times A \times (A \otimes_s A) \times (A \otimes_s A \otimes_s A) \times \cdots$$

(where the tensors $\otimes_s$ are the symmetric tensor powers). We shall see an echo of this construction in the Fock space construction below.

2 Normed Vector Spaces

As discussed in the introduction, we will be primarily working in normed vector spaces. Normed spaces seem necessary to capture correctly the intuition behind Girard's exponentials. Vector spaces are, in some sense, intrinsically finitary structures. Every vector is a finite sum of multiples of basis vectors, and one is only allowed to take finite sums of arbitrary vectors. It seems likely that to correctly model ! and ?, one should be able to take infinite sums of vectors, thereby capturing the idea of infinitely renewable resource. However, to do this, one needs a notion of convergence. And to define convergence, one needs a notion of norm. Once a space is normed, then it is possible to define limits and Cauchy sequences, and so on. Normed vector spaces, which are the principal objects of study in functional analysis, should be considered as the meeting ground of concepts from linear algebra and analysis. They are also an ideal place to model linear logic.

We will now briefly review the basic concepts of the subject. For more complete discussions, see [KR83, C90, CLM79].

Henceforth all vector spaces are assumed to be over the complex numbers and are allowed to be infinite-dimensional. We will use Greek letters for complex numbers and lower-case Latin letters from the end of the alphabet for vectors.

Definition 3 *A* **norm** *on a vector space V is a function, usually written $\| \; \|$, from V to $\mathbf{R}$, the real numbers, which satisfies*

1. $\| v \| \geq 0$ *for all* $v \in V$,
2. $\| v \| = 0$ *iff* $v = 0$,
3. $\| \alpha v \| = | \alpha | \| v \|$,
4. $\| v + w \| \leq \| v \| + \| w \|$.

For finite dimensional vector spaces the norm usually used is the familiar Euclidean norm. As soon as one has a norm one obtains a metric by the equation $d(u, v) = \| u - v \|$. One can ask whether the resulting space is complete or not as a metric space. It turns out that the spaces that are complete play a central role in functional analysis.

2.1 Banach Spaces

Definition 4 *A* **Banach space** *is a complete, normed vector space.*

Example 1 Consider the space of sequences of complex numbers. We write a for such a sequence, $a = \{a_n\}_{n=1}^{\infty}$ and we write $\| a \|_\infty$ for the supremum of the $| a_i |$.

$$l_\infty = \{a : \| a \|_\infty < \infty\}$$

This is a Banach space with $\| a \|_\infty$ as the norm.

Another norm is obtained on sequences as follows. Define:

$$\| a \|_1 = \Sigma_{n=1}^{\infty} | a_i |$$

Then let:

$$l_1 = \{a : \| a \|_1 < \infty\}$$

More generally, if $p \geq 0$, we may define:

$$l_p = \{a : \| a \|_p \equiv (\Sigma_{n=1}^{\infty} | a |^p)^{1/p} < \infty\}$$

All of these will be examples of Banach spaces. Furthermore, these can be defined not only for sequences of complex numbers, but for sequences obtained from any Banach space.

Example 2 Let X be a compact Hausdorff space. The vector space of complex-valued continuous functions on X is generally denoted $\mathsf{C}(X)$. Since X is compact, such functions must have a supremum, and from this it is straightforward to obtain a norm. Now convergence in this norm is the familiar notion of uniform convergence. As is well known from elementary analysis, sequences of uniformly bounded, continuous functions converge to a bounded continuous function. Thus, we have a Banach space. On the other hand if we looked at functions that vanish outside some closed, bounded interval (the functions of compact support) then we do not get a Banach space since these could converge to a function that does not have compact support.

The following theorem shows one common way in which Banach spaces arise. First we need a definition.

Definition 5 *Suppose that B_1, B_2 are Banach spaces and that T is a linear map from B_1 to B_2. We say that T is* **bounded** *if $sup_{x \neq 0} \dfrac{\| Tx \|}{\| x \|}$ exists. We define the* **norm of** T, *written $\| T \|$, to be this number.*

If T is indeed bounded, then a standard argument [KR83], establishes

Lemma 2 $sup_{\|x\|=1} \| Tx \| = \| T \|$.

Thus one can use vectors of unit norm to calculate the norm of a linear function rather than having to look for the sup over all nonzero vectors.

Linear maps from a Banach space to itself are traditionally called *operators*, and the norm of such maps is called the *operator norm.*

Since a Banach space is also a metric space under the induced metric described above, one can also ask to characterize which linear maps are also continuous. In this regard, we have the following result.

Lemma 3 *A linear map from $f : A \to B$ is continuous if and only if it is bounded.*

The following theorem shows that the category of Banach spaces and bounded linear maps is enriched over itself.

Theorem 4 *If A is a normed vector space and B is a Banach space then the space of bounded linear maps with the norm above is a Banach space.*

We will denote this space $A \multimap B$.

There are several possible categories of interest with Banach spaces as the objects. The most obvious one is the category with bounded linear maps as the morphisms. However, it turns out that the category with *contractive maps*[2] is of greater interest and has nicer categorical properties. These properties are discussed in [B76a] and below.

Definition 6 *A* **contractive map**, *T, from A to B is a bounded linear map satisfying the condition, $\| Tx \| \leq \| x \|$. Equivalently, the contractive maps are those of norm less than or equal to 1.*

We will write $\mathcal{BANCON}$ for the category of Banach spaces and contractive maps and $\mathcal{BANACH}$ for the category of Banach spaces and bounded linear maps. While $\mathcal{BANCON}$ has a richer categorical structure, for the purposes of modelling the exponential types of linear logic, we will be forced to work in $\mathcal{BANACH}$.

2.2 Monoidal Structure of $\mathcal{BANACH}$

We first point out that $\mathcal{BANACH}$ has a canonical symmetric monoidal closed structure. We begin by constructing a tensor product. Let A and B be objects in $\mathcal{BANACH}$. Begin by forming the tensor of A and B, $A \otimes_{\mathbf{C}} B$, as complex vector spaces. We first define a partial norm for elements of the form $a \otimes b$ by the equation:

$$\| a \otimes b \| = \| a \| \| b \|$$

[2]Strictly speaking, they should be called "non-expansive" maps.

We would like to extend this partial norm to a norm on all of $A \otimes_{\mathbf{C}} B$. Such a norm is called a *cross norm.* It turns out that there are many such cross norms, a number of which were discovered by Grothendieck. The one we will use in this paper is called the *projective cross norm.* It is in some sense the least such. A detailed discussion of these issues is contained in [T79]. The projective cross norm is defined for an arbitrary element, x, of $A \otimes_{\mathbf{C}} B$ by the following formula:

$$\| x \| = inf\{\| a \| \| b \| \text{ such that } x = \Sigma a \otimes b\}$$

One can verify that this is in fact a cross norm on $A \otimes_{\mathbf{C}} B$. Now, the resulting normed space will not be complete in general, so one obtains a Banach space by completing it. This will act as the tensor product in the category $\mathcal{BANACH}$. It will be denoted simply by $A \otimes B$. Furthermore, we have the following adjunction in $\mathcal{BANACH}$.

Lemma 5 *The functor $B \otimes (\ )$ is left adjoint to $B \multimap (\ )$.*

Corollary 6 *$\mathcal{BANACH}$ is a symmetric monoidal closed category.*

Analogously, $\mathcal{BANCON}$ is also a monoidal closed category. Note that although one only uses contractive maps in this category, the internal hom is still given by all bounded linear maps.

As such, they are models of (at least) the multiplicative fragment of intuitionistic linear logic. To obtain a model of the classical linear logic, one possibility is the topological construction of Barr in [B76a]. See also [Bl93a]. The idea is to add an additional topological structure to the space, and then only consider maps which are also continuous with respect to this topology. If the topology is chosen carefully, one obtains a large class of reflexive objects, *i.e.* objects which are isomorphic to their double dual space. Such objects can be used to model the negation of classical linear logic.

2.3 Completeness Properties of $\mathcal{BANCON}$ and $\mathcal{BANACH}$

The main advantage of studying the category of contractions is in its completeness properties. While $\mathcal{BANACH}$ has very weak completeness properties, $\mathcal{BANCON}$ is complete and cocomplete. These constructions exist in $\mathcal{BANACH}$ but some lose the universal property. We will describe some of these universal properties. We begin with finite coproducts.

Definition 7 *Let A and B be Banach spaces. The direct sum, $A \oplus B$, is the Cartesian product equipped with the norm $\| a \oplus b \| = \| a \| + \| b \|$.*

Then we have the distributivity property of $\otimes$ over $\oplus$.

Proposition 7 $A \otimes (B \oplus B') \cong (A \otimes B) \oplus (A \otimes B')$.

We now discuss finite products.

Definition 8 *The product of two Banach spaces, $A \times B$, has as its underlying space $A \oplus B$, but now with norm given by:*

$$\| a \oplus b \| = max\{\| a \|, \| b \|\}$$

As a category of vector spaces, $\mathcal{BANCON}$ is fairly unique in this respect. While most such categories model the additive fragment of linear logic, they invariably equate the two connectives, since finite products and coproducts coincide. In other words, $\mathcal{BANCON}$ does not share the familiar property of being an additive category.

We now present countably infinite products and coproducts.

Definition 9 *Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of Banach spaces. Define $\Pi(A_i)$ to be those sequences which converge in the l_∞ norm,* i.e. *bounded sequences equipped with the obvious norm.*

Define $\Sigma(A_i)$ to be all sequences which converge in the l_1 norm.

This gives countable products and coproducts in $\mathcal{BANCON}$. Similar constructions can be applied for uncountable products and coproducts.

Equalizers in $\mathcal{BANCON}$ correspond to equalizers in the underlying category of vector spaces. The fact that bounded maps are continuous implies that the subspace will be complete. Coequalizers are obtained as a quotient, with the induced norm being the infimum of the norms of the elements of the equivalence class. See [C90] for a discussion of quotients of Banach spaces.

Theorem 1 *$\mathcal{BANCON}$ is complete and cocomplete.*

All of the above constructions exist in $\mathcal{BANACH}$, but some of them will lose their universal property. $\mathcal{BANACH}$ is an additive category, with sums and products given by the coproducts in $\mathcal{BANCON}$. (Note that the two spaces $A \times B$ and $A \oplus B$ are isomorphic in $\mathcal{BANACH}$, but not in $\mathcal{BANCON}$.) In $\mathcal{BANACH}$, the above infinite products and coproducts exist, but do not share the universal property. They only have this property for bounded families of maps. Equalizers and coequalizers are as in $\mathcal{BANCON}$.

2.4 Hilbert Spaces

An alternate approach to defining a norm on a vector space is *via* an inner product. An inner product has the property that it induces a norm on the underlying space.

Definition 10 *Given a complex vector space, V, an* **inner product** *for V is a function from $V \times V$ to the complex numbers which is conjugate linear in its first argument and linear in its second argument. This is written $\langle u|v\rangle$.*

Furthermore, an inner product must have the following properties.

- $\langle x|x\rangle \geq 0$
- $\langle x|y\rangle = \overline{\langle y|x\rangle}$
- *if* $\langle x|x\rangle = 0$, *then* $x{=}0$

Here, $\bar{z}$ refers to complex conjugation. Real Hilbert spaces are defined analogously, with conjugation being taken to be the identity.

Given an inner product we immediately get a norm by $\| x \| = (\langle x|x\rangle)^{1/2}$. As with Banach space what turns out to be crucial is the property of being complete.

Definition 11 *A* **Hilbert space** *is a vector space equipped with an inner product such that the vector space is complete in the induced norm.*

Example 3 The space l_2 of all sequences of complex numbers such that:

$$\Sigma_{i=1}^{\infty} | a_i |^2 \leq \infty$$

One defines an inner product by:

$$\langle x|y\rangle = \Sigma_{i=1}^{\infty} x_i \overline{y_i}$$

Every finite dimensional complex vector space is a Hilbert space with the usual inner product.

The category of Hilbert spaces and bounded linear maps will be denoted by $\mathcal{HILBERT}$. This category has a tensor product which can be constructed in a manner analogous to the construction for Banach spaces. $\mathcal{HILBERT}$ also has finite products and coproducts, in both cases these are given by direct sum, with the evident inner product. $\mathcal{HILBERT}$ does not have very many infinite limits or colimits.

3 Symmetric and Antisymmetric Tensors

We introduce two further constructions in the category $\mathcal{BANACH}$. These will be quotients of the tensor product. Since the category has coequalizers such quotients will be well-defined.

3.1 Symmetric Tensor Products

First, we introduce the symmetric tensor product of a Banach space with itself.

Definition 12 *Let A be a Banach space. The Banach space $A \otimes_s A$ is defined to be the following coequalizer:*

$$A \otimes A \underset{\tau}{\overset{id}{\rightrightarrows}} A \otimes A \longrightarrow A \otimes_s A$$

Note that τ is the twist map, $a \otimes b \mapsto b \otimes a$.

This is the general definition of symmetrized tensor. It turns out that in categories of vector spaces, this quotient is canonically isomorphic to the equalizer of these two maps, and that this equalizer is split by the map:

$$a \otimes b \mapsto \frac{1}{2}(a \otimes b + b \otimes a)$$

We will frequently use this representation in the sequel.

The n^{th} symmetric power is defined analogously. The Banach space $\bigotimes^n A$ has $n!$ canonical endomorphisms, and the Banach space $\bigotimes_S^n$ is the coequalizer of all of these. Again, it is isomorphic to the equalizer, and there is a splitting, as above. A good way to view the symmetrized tensor is to observe that the symmetric group acts on the space $\bigotimes^n A$, and that the symmetrized tensor is the invariant subspace. As such, an appropriate notation for the symmetrized tensor is:

$$\frac{\bigotimes^n A}{n!}$$

We will also freely use this representation, as well.

3.2 Antisymmetric Tensor Products

This will be defined in a similar fashion. Again, we first define the antisymmetric tensor of a Banach space B with itself. It will be denoted $B \otimes_A B$. It is the coequalizer of the following diagram:

$$B \otimes B \underset{-\tau}{\overset{id}{\rightrightarrows}} B \otimes B \longrightarrow B \otimes_A B$$

Here, $-\tau$ is the map $a \otimes b \mapsto -b \otimes a$.

Members of this space can canonically viewed as elements of the ordinary tensor product, of the form:

$$x = a \otimes b - b \otimes a$$

The n^{th} antisymmetric power is defined analogously.

4 Fock space and categories of algebras

4.1 Fock space

We are now ready to define the Fock spaces. They are traditionally defined in $\mathcal{HILBERT}$; we will, however, define them in $\mathcal{BANACH}$.

Definition 13 *Let B be a Banach space. The* **symmetric Fock space** *of B is the infinite direct sum of the spaces $\bigotimes_s^n B$, where, when n is zero we use the complex numbers. The* **antisymmetric Fock space** *of B is the infinite direct sum of the spaces $\bigotimes_A^n B$.*

$$\mathcal{F}(B) = C \oplus B \oplus \cdots \oplus \bigotimes\nolimits_s^n B \oplus \cdots$$

$$\mathcal{F}_A(B) = C \oplus B \oplus \cdots \oplus \bigotimes\nolimits_A^n B \oplus \cdots$$

Since Fock is defined using infinite direct sums and coequalizers it is clear that Fock defines a functor.

We can think of an element of $\mathcal{F}(B)$ as an infinite sequence $\langle c, v_1, v_2, \ldots \rangle$ where c is a complex number and $v_i \in B_i$.

Now we check that the Fock space actually satisfies all the properties that need to be satisfied by an OF COURSE type, *i.e.* satisfies the properties of [Se89], discussed in Section 1. This consists of two parts, verifying that Fock spaces form a cotriple on the category of Banach algebras and verifying the so-called exponential law, *viz.* $!(A \times B) \cong\ !A \otimes\ !B$. We check the former by displaying a suitable adjunction in the next subsection.

Proposition 8 *Let A and B be Banach spaces.*

$$\mathcal{F}(A \times B) \cong \mathcal{F}(A) \otimes \mathcal{F}(B).$$

Here the product is what is called the direct sum by analysts. The isomorphism is in the category $\mathcal{BANACH}$.

Proof – We need to exhibit maps in both directions and show that all the conditions required for an isomorphism are satisfied. The isomorphism is based on the following "formal calculation".

$$
\begin{aligned}
\mathcal{F}(A\times B) &= \mathcal{F}(A\oplus B)\\
&= \mathbf{C}\oplus(A\oplus B)\oplus\tfrac{1}{2}((A\oplus B)\otimes_s(A\oplus B))\cdots\\
&= \mathbf{C}\oplus A\oplus B\oplus\tfrac{1}{2}(A\otimes_s A)\oplus\tfrac{1}{2}(B\otimes_s B)\oplus(A\otimes_s B)\cdots\\
&= \mathcal{F}(A)\otimes\mathcal{F}(B).
\end{aligned}
$$

The rigorous argument is as follows. We call an element of $\mathcal{F}(B)$ a *pure tensor* if it is of the form $\langle 0,0,\ldots,v,0,0,\ldots\rangle$ and a *finite-rank* tensor if it is of the form $\langle v_0,v_1,\ldots,v_n,0,0,\ldots\rangle$; *i.e.* zero after some finite stage. Now the pure tensors form a basis for $\mathcal{F}(B)$. In order to define the iso from $\mathcal{F}(A\times B)$ to $\mathcal{F}(A)\otimes\mathcal{F}(B)$ we need only specify the map on the pure tensors. A pure tensor, p, in $\mathcal{F}(A\times B)$ looks like $p=\Sigma x_1\otimes\ldots\otimes x_n$ where $x_i=y_i+z_i, y_i\in A, z_i\in B$. Using distributivity of $\otimes$ over $+$ we have

$$p=\Sigma[(y_1+z_1)\otimes\ldots\otimes(y_n+z_n)]$$

$$=\Sigma[y_1\otimes\ldots\otimes y_n+\ldots+(y_{i_1}\otimes\ldots\otimes y_{i_k})\otimes(z_{j_1}\otimes\ldots\otimes z_{j_l})+\ldots+z_1\otimes\ldots\otimes z_n]$$

The last expression is a sum of elements of $\mathcal{F}(A)\otimes\mathcal{F}(B)$. The iso in the other direction is obtained by viewing the pure elements of $\mathcal{F}(A)$ and $\mathcal{F}(B)$ as polynomials and carrying out polynomial multiplication. ▮

The units are easily identified.

Lemma 9 *The complex numbers,* $\mathbf{C}$*, viewed as a Banach space form a unit for tensor product. The one point space, written* $\mathbf{0}$*, is the unit for the direct sum.*

The effect of $\mathcal{F}$ on the units is given below. The proofs are immediate from the definitions. Equality means isomorphism in $\mathcal{BANACH}$.

Lemma 10 *1.* $\mathcal{F}(\mathbf{0})=\mathbf{C}$.

2. $\mathcal{F}(\mathbf{C})=l_1$.

Proof – The proof of the first assertion is immediate. For the second assertion, note that, since $\mathbf{C}$ is the unit for tensor all the terms in the infinite direct sum are just **C**. Thus we have infinite sequences of members of **C** with the same convergence criterion as for l_1. ∎

This lemma shows that one cannot use this construction in categories of finite-dimensional spaces.

Now we consider the antisymmetrized Fock space[3]. It turns out that one gets a model of the exponential types in the category of finite-dimensional vector spaces using the antisymmetrized Fock space.

Proposition 11 *If V is a finite-dimensional vector space of dimension n, then $\mathcal{F}_A(V)$ is also a finite-dimensional vector space with dimension 2^n.*

Proof – Consider the vector space $\bigotimes_A^p V$ with $p > n$. We claim that this space is the zero vector space. Since $\otimes$ is adjoint to internal hom in $\mathcal{VEC}_{fd}$, the space $\bigotimes_A^p V$ is isomorphic to the space of completely antisymmetric p-linear maps from V to the scalars. Let f denote such a map. Since V is only n-dimensional one cannot have p linearly independent arguments to such maps. Thus one of the arguments must be a linear combination of the others. Thus on any arguments f becomes a combination of terms of the form $f(\ldots, u, \ldots, u, \ldots)$ where two arguments must be equal. But antisymmetry makes such a term zero. Thus f is the zero vector and the vector space $\bigotimes_A^p V$ is the one-point space. Thus the infinite direct sum becomes a finite direct sum. Now consider $p \leq n$. It is clear that one can only choose C_p^n sets of p linearly independent vectors given a basis. Thus the dimensionality of the space $\bigotimes_A^p V$ is C_2^n and hence, adding the dimensions to get the dimension of the direct sum, we conclude that the dimension of $\mathcal{F}_A(V)$ is 2^n. ∎

The exponential law for the antisymmetric case can be argued similarly. The detailed verification can be found in [BSZ92] in Section 3.2 on exponential laws.

4.2 Categories of algebras

In this section we shall review some basic facts about categories of algebras, and see in particular how these fit into the current context. (See [M71] for a review of the basic categorical facts, and [L65] for the basic algebra, for instance.) For reference, we do give the following definition here.

[3]The arguments below are well-known to differential geometers. Prakash Panangaden would like to thank Steve Vickers for reminding him about these facts.

Definition 14 *A triple consists of a functor $F: \mathcal{B} \longrightarrow \mathcal{B}$, together with natural transformations $\eta: id \longrightarrow F$ and $\mu: FF \longrightarrow F$, such that $\mu \circ \eta F = \mu \circ F\eta = id$ and $\mu \circ \mu T = \mu \circ T\mu$.*

One simple point to recall is that categories of algebras and of coalgebras are closely connected to the existence of triples and cotriples. Given a triple $F: \mathcal{B} \longrightarrow \mathcal{B}$, (with structure morphisms η, μ), an F-algebra is an object B and a morphism $h: F(B) \longrightarrow B$ (subject to two commutativity conditions, corresponding to the associative and unit laws). (This notion can be generalized to arbitrary functors.) There is a canonical category of such algebras, the Eilenberg-Moore category $\mathcal{C}^F$, and an adjunction $\mathcal{C} \rightleftarrows \mathcal{C}^F$. Any adjunction canonically induces a triple, and this one canonically induces the original triple. The category of free F-algebras is the Kleisli category $\mathcal{C}_F$ of the triple; again, there is a canonical adjunction $\mathcal{C} \rightleftarrows \mathcal{C}_F$ which induces the original triple. Of course this dualizes for cotriples, with the corresponding notion of coalgebras. (We shall avoid the unpleasant use of terms like "coEilenberg-Moore" and "coKleisli".)

Usually mathematicians have been more interested in the Eilenberg-Moore category of a triple (or cotriple) than in the Kleisli category; although there has been some interest in Kleisli categories recently (for instance in the context of linear logic, as mentioned earlier in this paper), we shall follow this tradition and shall work in Eilenberg-Moore categories. Indeed, it is there that we shall find some of our models. One reason for this is quite practical: it is often simpler to recognize the category of algebras and so derive the triple (similarly, once one has a candidate for a triple, it is often simpler to construct the category of algebras and verify the adjunction than to directly show the original functor is a triple). But there is another reason: we want to show that the Fock space functor is a cotriple (so as to model $!$), but on the categories of spaces we consider, this is not the case—rather it is a triple. By passing to the algebras, we can fix this, because of the following fact:

Fact Given an adjunction $\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D}$, $F \dashv U$, the composite UF is a triple on $\mathcal{C}$, and so (dually) the composite FU is a cotriple on $\mathcal{D}$.

So we obtain our model of $!$ on the category of algebras.

4.2.1 Algebras for the symmetric (bosonic) Fock space construction

We begin with a more traditional notion of algebra; the connection between these comes *via* the triple induced by the adjunction given by the free algebra construction, as outlined above. In other words, the category of (traditional) algebras is equivalent to the category of UF algebras.

Definition 15 *An algebra* A *is a space* A *equipped with morphisms*

$$m: A \otimes A \longrightarrow A \text{ and } i: \mathbf{C} \longrightarrow A$$

satisfying

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{C} \otimes A \arrow[r, "i \otimes id"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "m"] & A \otimes \mathbf{C} \arrow[l, "id \otimes i"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

Here we are supposing the base field to be $\mathbf{C}$; otherwise replace $\mathbf{C}$ with the base field k. If in addition the following diagram commutes, then the algebra A is said to be *symmetric* or commutative. (τ is the canonical "twist" morphism.)

$$\begin{tikzcd}
A \otimes A \arrow[r, "\tau"] \arrow[dr, "m"'] & A \otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$

An example of such an algebra comes from the Fock space of a Banach space: the multiplication m is defined by "multiplication of series" in an evident manner. The use of the symmetrized tensor in the definition of Fock space guarantees that this will indeed be a symmetric algebra, and it is standard that this description gives the free such algebra. In other words, we have the following proposition.

Proposition 12 *Given a Banach space B, the Fock space $\mathcal{F}(B)$ canonically carries an algebra structure, and indeed is the free symmetric algebra generated by B.*

It follows from this (or rather from the adjunction $\mathcal{BANCON} \rightleftarrows \mathcal{SALG}$) that we have a cotriple on the category $\mathcal{SALG}$ of symmetric algebras, given by taking the Fock algebra on the underlying space of an algebra. As the details of this are both standard and similar to the case of the antisymmetric Fock space construction, which we shall discuss in more detail next, we shall leave the details here to the reader.

4.3 Algebras for the antisymmetric (fermionic) Fock space construction

Recall that we work in the context of $\mathcal{VEC}_{fd}$ finite dimensional vector spaces when considering the antisymmetric Fock construction. This category is self-dual, and is compact with biproducts: the product and coproduct coincide. This duality also implies that a triple is also a cotriple, so we can model ! in the category of spaces. However, to show that the Fock space construction defines a triple (or cotriple), it is again simpler to consider the category of algebras. Although we are not familiar with any previous consideration of this category of algebras as such, the context is familiar: the antisymmetric Fock space construction is usually called (when thought of as an algebra) the Grassman algebra, or the "alternating" or "interior" algebra; the multiplication defined on it is called the "wedge product" (a term derived from the usual notation for this product).

Definition 16 *An alternating algebra* A *is a graded algebra* A *(with unit) whose multiplication map satisfies the property that, if x, y are of degree m, n respectively, then $xy = (-1)^{nm} yx$ (which by the grading must be of degree $n + m$).*

Note that the unit must be of degree 0. Morphisms of alternating algebras are just homomorphisms as algebras.

Proposition 13 *There is a canonical alternating algebra structure on $\mathcal{F}_A(V)$, for any finite dimensional vector space V. The antisymmetric Fock construction is left adjoint to the forgetful functor $U\colon \mathcal{VEC}_{fd} \underset{U}{\overset{\mathcal{F}_A}{\rightleftarrows}} \mathcal{AALG}$, where $\mathcal{AALG}$ is the category of alternating algebras. As a consequence, $\mathcal{F}_A$ defines a triple (and so cotriple) on $\mathcal{VEC}_{fd}$.*

Proof – (Sketch) The multiplication on $\mathcal{F}_A(V)$ is the standard "wedge" product [L65], which to elements $x_1 \otimes_A \ldots \otimes_A x_n, y_1 \otimes_A \ldots \otimes_A y_m$ gives the product $x_1 \otimes_A \ldots \otimes_A x_n \otimes_A y_1 \otimes_A \ldots \otimes_A y_m$. Here $x \otimes_A y$ means the equivalence class of $x \otimes y$ in $A \otimes_A A$. (Essentially this is the same "multiplication of power series" we had in the symmetric case, with the alternating product used in place of the usual tensor.) For a vector space V, define $\eta\colon V \longrightarrow U\mathcal{F}_A(V)$ as the canonical injection. Given an alternating algebra A, define $\epsilon\colon \mathcal{F}_A(UA) \longrightarrow A$ by "adding the terms of the series": $\langle x_0, x_1, x_2^1 \otimes_A x_2^2, \ldots\rangle \mapsto i(x_0) + x_1 + m(x_2^1, x_2^2) + \cdots$, where i, m are the algebra maps.

To verify that we have an adjunction we must show the following commute:

$$\begin{tikzcd} \mathcal{F}_A(V) \arrow[r, "\mathcal{F}_A(\eta)"] \arrow[dr, "1"'] & \mathcal{F}_A(U\mathcal{F}_A(V)) \arrow[d, "\epsilon\mathcal{F}_A"] \\ & \mathcal{F}_A(V) \end{tikzcd}$$

$$\begin{tikzcd} UA \arrow[r, "\eta U"] \arrow[dr, "1"'] & U\mathcal{F}_A(UA) \arrow[d, "U\epsilon"] \\ & UA \end{tikzcd}$$

The second diagram is obvious; to verify the first, notice that $\mathcal{F}_A(\eta(x))$ maps

$$\begin{array}{lll}\langle x_0, x_1, x_2^1 \otimes_A x_2^2, \ldots\rangle & \mapsto & \langle x_0, \\ & & \langle 0, x_1, 0, \ldots\rangle, \\ & & \langle 0, 0, x_2^1, 0, \ldots\rangle \otimes_A \langle 0, 0, x_2^2, 0, \ldots\rangle, \\ & & \vdots \\ & & \rangle\end{array}$$

and it is clear that "adding up this series" just returns the original term. ∎

It now follows that we can model $!$ in $\mathcal{VEC}_{fd}$ with $\mathcal{F}_A$, *via* the formula $!\,V = (\mathcal{F}_A(V^\perp))^\perp$.

5 The Holomorphic-Function Representation of Fock Space

A possible reaction to the results of the last section is that the Fock space construction works purely fortuitously, in the sense that the proper notions of tensor products and infinite direct-sums happen to exist and conspire to make the construction of internal comonoids possible. In the present section we argue that in fact this construction is linked to much deeper mathematics. The symmetrized Fock space on a Banach space B, turns out to be a space of holomorphic functions (analytic functions) on B, properly defined. This hints at possible deeper connections between analyticity and computability which need to be explored.

The ideas here stem from early work by Bargmann [Ba61] on Hilbert spaces of analytic functions in quantum mechanics. This was extended by Segal [S62, BSZ92] to quantum field theory and Segal's extension was used by Ashtekar and Magnon [AM-A80] to develop quantum field theory in curved spacetimes. (A brief summary of the ideas is contained in an appendix to [P80] and in [P79].) The latter work involved making sense of the familiar Cauchy-Riemann conditions on infinite-dimensional spaces.

We quickly recapitulate the basic notion of analytic function in terms of one complex variable before presenting the infinite-dimensional case. A very good elementary reference is *Complex Analysis* by Ahlfors [Ah66]. Given the complex plane, $\mathbf{C}$, one can define functions from $\mathbf{C}$ to $\mathbf{C}$. Let z be a complex variable; we can think of it as $x + iy$ and thus one can think of functions from $\mathbf{C}$ to $\mathbf{C}$ as functions from $\mathbf{R}^2$ to $\mathbf{R}^2$. An *analytic* or *holomorphic* function is one that is everywhere differentiable. In the notion of differentiation, the limit being computed, *viz.*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

allows h to be an arbitrary complex number and hence this limit is required to exist no matter in what direction h approaches 0. This much more stringent requirement makes complex differentiability much stronger than the usual notion of differentiability. If a complex function is differentiable at a point it can be represented by a convergent power series in a suitable open region about the point. If one uses the fact that h can approach zero along either axis one can derive the *Cauchy-Riemann* equations for a complex valued function $f = u(x, y) + iv(x, y)$ of the complex variable $z = x + iy$,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

What is remarkable about complex functions is that this definition of analyticity yields the result that a complex-analytic function can be expressed by a convergent power-series in a region of the complex plane. This is remarkable because only one derivative is involved in the Cauchy-Riemann equations whereas the statement that a power-series representation exists is stronger, for real-valued functions, even than requiring infinite differentiability. In real analysis one has examples of functions that are infinitely differentiable at a point, but do not have a power series representation in any neighbourhood of that point. A function may have a power series representation that is valid everywhere, a so-called *entire* holomorphic function; the complex exponential function is an example.

There is a formal perspective, due to Wierstrass, that is rather more illuminating. Think of a complex variable $z = x + iy$ and its conjugate $\overline{z} = x - iy$ as being, formally, independent variables. A function could depend on z and on its complex conjugate, $\overline{z}$, for example, the function that maps each z to $z\overline{z} + iz\overline{z}$. An *analytic* or *holomorphic* function is one which has no dependance on $\overline{z}$. This is expressed formally by $df/d\overline{z} = 0$. When expressed in terms of the real and imaginary parts of f and z, this equation becomes the familiar Cauchy-Riemann equations. Thus this reinforces the view that a holomorphic function is properly thought of as a single complex-valued function of a single variable rather than as two real-valued functions of two real variables.

The theory of functions of finitely many complex variables is a nontrivial extension of the theory of functions of a single complex variable.

Entirely new phenomena occur, which have no analogues in the theory of a single complex variable. An excellent recent text is the three volume treatise by Gunning [Gu90]. For our purposes we need only the barest beginnings of the theory. Given $\mathbb{C}^n$, we can have functions from $\mathbb{C}^n$ to $\mathbb{C}$. One can introduce complex coordinates on $\mathbb{C}^n$, $z_1, \ldots, z_n$. One can define a holomorphic function here as one having a convergent power-series expansion in $z_1, \ldots, z_n$. The key lemma that allows one to mimic some of the results of the one-dimensional case is Osgood's lemma[4].

Lemma 14 *If a complex-valued function is continuous in an open subset D of $\mathbb{C}^n$ and is holomorphic in each variable seperately, then it is holomorphic in D.*

From this one can conclude that a holomorphic function in n variables satisfies the Cauchy-Riemann equations $\frac{\partial f}{\partial \bar{z}_i} = 0$. One is free to take either one of (a) satisfying Cauchy-Riemann equations or (b) having convergent power-series representations as the definition of holomorphicity.

Now we describe how to define holomorphic functions on infinite-dimensional, complex, Banach spaces. The basic intuition may be summarized thus. One starts with subspaces of finite *codimension*. Thus the quotient spaces are isomorphic to some $\mathbb{C}^n$. One can define what is meant by a holomorphic function on these quotient spaces as in the preceding paragraph. By composing a holomorphic function with the canonical surjection from the original Banach space to the quotient space we get a function on the original Banach space. These functions can all be taken to be holomorphic.

$$\begin{tikzcd} & B \arrow[d] \arrow[dr, dotted] & \\ B/\sim \; = & \mathbb{C}^n \arrow[r] & \mathbb{C} \end{tikzcd}$$

Intuitively these are the functions that are constant along all but finitely many directions, and holomorphic in the directions along which they do vary. These functions are called *cylindric holomorphic functions.* Because the sequence of coefficients of a power-series is absolutely convergent, we can define an l_1 norm on these functions in terms of the power-series. Finally the collection of all holomorphic funcitons is defined by taking the l_1-norm completion of the cylindric holomorphic functions.

[4]There is a considerably harder theorem, called Hartog's theorem, which drops the requirement of continuity.

Given a Banach space B, let U be a subspace with finite codimension n, *i.e.* the quotient space B/U is an n complex-dimensional vector space. The space B/U is isomorphic to $\mathbf{C}^n$. Let $\phi : B/U \to \mathbf{C}^n$ be an isomorphism; such a map defines a choice of complex coordinates on B/U. Let π_U be the canonical surjection from B to B/U.

Definition 1 *A* **cylindric holomorphic** *function on B is a function of the form $f \circ \phi \circ \pi_U$, where U, π_U and ϕ are as above and f is a holomorphic function from $\mathbf{C}^n$ to $\mathbf{C}$.*

We need to argue that the choice of coordinates does not make a real difference. Of course which functions get called holomorphic does depend on the choice of coordinates, but the space of holomorphic functions has the same structure[5]. Suppose that U and V are both subspaces of B and that U is included in V. Suppose that both these spaces are spaces of finite codimension, say n and m respectively. Clearly $n \geq m$. Now we have a linear map $\pi_{UV} : B/U \to B/V$ given by $x + U \mapsto x + V$; clearly this is a surjection. Now given coordinate functions $\phi : B/U \to \mathbf{C}^n$ and $\psi : B/V \to \mathbf{C}^m$ we can define a function $\alpha : \mathbf{C}^n \to \mathbf{C}^m$, given by $\psi \circ \pi_{UV} \circ \phi^{-1}$, which makes the diagram commute. Thus we do not have to impose "coherence" conditions on the choice of coordinates, we can always translate back and forth between different coordinate systems.

We will suppress these translation functions in what follows and assume that the coordinates have been serendipitously chosen to make the form of the functions simple. In other words, we can fix a family of subspaces $\{W_n | n \in N\}$ with W_n having codimension n and $W_{n+1} \subset W_n$. The coordinates can be chosen so that the space B/W_n has coordinates $z_1, \ldots, z_n$.

Suppose that f is a cylindric holomorphic function on B. This means that there is a finite-codimensional subspace W, and a holomorphic function f_W, from W to $\mathbf{C}$, such that $f = f_W \circ \pi_W$. The function f_W regarded as a function of n complex variables has a power-series representation

$$f_W(z_1, \ldots, z_n) = \Sigma a_{i_1 \ldots i_k} z_1^{i_1} \ldots z_k^{i_k}$$

and furthermore we have the following convergence condition

$$\Sigma |a_{i_1 \ldots i_k}| < \infty.$$

[5]This happens even in the one dimensional case. The function $\overline{z}$ is considered anti-holomorphic traditionally, but one could have called it holomorphic by interchanging the role of z and $\overline{z}$.

Thus with each such cylindric holomorphic function we can define the sum of the absolute values of the coefficients in the power-series expansion as the norm of the function. Viewing the sequences of coefficients as the elements of a complex vector space, we have an l_1 norm. We write $\| f \|$ for this norm of a cylindric holomorphic function.

Definition 2 *An* l_1**-holomorphic** *function on* B *is the limit of a sequence of cylindric holomorphic function in the above norm.*

The l_1 emphasizes that the holomorphic functions are obtained by a particular norm completion. In the corresponding theory of holomorphic functions on Hilbert spaces, one uses the inner-product to define polynomials and then perform a completion in the L_2 norm. A key difference is that our norm is defined on the sequence of coefficients whereas in the Hilbert space case, one uses the L_2 norm which is defined in terms of integration.

In the resulting Banach space there are several formal entities that were adjoined as part of the norm-completion process. We need to discuss in what sense these formally-defined entities can be regarded as bona-fide functions. Let $W_1, \ldots, W_r, \ldots$ be an infinite sequence of subspaces of B, each embedded in the previous. Assume, in addition, that all these spaces have finite codimension. Now assume that there is a sequence of cylindric holomorphic functions, f_n, on B obtained from a holomorphic function, $f^{(n)}$ on each of the quotient spaces B/W_i. Finally, assume that the sequence $\| f_n \|$ of (real) numbers is convergent. Such a sequence of cylindric holomorphic functions defines a holomorphic function on B. We call this function f. We need to exhibit f as a map from B to $\mathbf{C}$. Accordingly, let x be a point of B. For each of the functions f_n we have $|f_n(x)| \leq \| f_n \|$. Since the sequence of norms converges we have the sequence $f_n(x)$ converges absolutely and hence converges. Thus the function f *qua* function is given at each x of B by $\lim_{n\to\infty} f_n(x)$. However, in order to use the word "function" we need to show that the power-series has a domain of convergence. Unfortunately, it may not have a non-trivial domain of convergence but, in a sense to be made precise, it comes close to having a non-trivial domain of convergence.

The power-series representation of the function f is given as follows. It depends, in general, on infinitely many variables but each term in the power series will be a monomial in finitely many variables. Consider the coefficient of $z_{i_1}^{j_1} \ldots z_{i_k}^{j_k}$ in the expansion of f. In all but finitely many of the f_n all the indicated variables will appear in their power-series expansions. Consider the coefficients of this term in each power series; this forms a sequence of complex numbers α_n where α_n is 0 if there is no such term in the expansion of f_n. Since $|\alpha_n| \leq \| f_n \|$ the sequence α_n converges absolutely and hence converges to, say, α. This is the coefficient of $z_{i_1}^{j_1} \ldots z_{i_k}^{j_k}$ in the power-series expansion of f.

Consider the coordinates $z_1, \ldots, z_n$. This defines an n-dimensional subspace of the Banach space, which we call U_n. Now consider the power-series for f. It defines a family of holomorphic functions f^n where f^n is defined on the subspace U_n and is obtained by retaining only those terms in the power-series expansion of f which involve variables among $z_1, \ldots, z_n$. These are analytic functions on the U_n and, as such, have non-trivial domains of convergence. However, as n increases the radii of convergence could tend to 0. So we have the slightly weaker statement than the usual finite-dimensional notion; instead of having a non-zero radius of convergence in the Banach space we have a non-zero radius of convergence on every finite-dimensional subspace. If one uses *entire* functions, rather than analytic functions, at the starting point of the construction, then one can show that the resulting functions are entire; see page 67, theorem 1.13, of the book by Baez, Segal and Zhou [BSZ92]. Unfortunately when using the representation of elements of Fock space one may carry out simple operations that do not produce entire functions so we cannot just choose to work with entire functions. Nevertheless, many common functions, most notably the exponential, are entire.

Given a bona fide holomorphic function one can express it as a power series. The coefficients are calculated in the usual way, *viz.* by using Taylor's theorem

$$f = \Sigma_n \Sigma_{i_1+\ldots+i_k=n} \frac{1}{i_1! \ldots i_k!} \left(\frac{\partial^n}{\partial^{i_1} z_1 \ldots \partial^{i_k} z_k}\right) z_1^{i_1} \ldots z_k^{i_k}$$

Since the mixed partial derivatives commute (the functions are holomorphic and hence certainly differentiable enough) the partial derivatives are, concretely speaking, symmetric arrays. Abstractly speaking this just means that they are elements of the symmetrized tensor product.

We can write this as follows.

Theorem 15 *A holomorphic function can be represented by its power-series expansion where the n^{th} term in the power-series expansion is a symmetrized n^{th} derivative:*

$$f = \Sigma(1/k!) D^{(k)} f$$

where the notation $D^{(k)} f$ means symmetrized k^{th} derivative of f.

The symmetrized derivatives live in the symmetrized tensor products of B with itself. One thus has a correspondence with the standard Fock representation and the notion of holomorphic function since in each case one has a string of symmetrized vectors.

5.1 A Digression on Complex Structures

This section can be skipped on a first reading[6]; however, the reader who feels queasy about all the explicit coordinate dependence in the definitions so far may find this section comforting. There is no need to start with complex vector spaces. One could have used real vector spaces from the outset. In order to sketch this briefly we begin with the notion of a complex structure on a vector space.

Definition 17 *Let V be a real vector space. A complex structure is a linear operator $J : V \to V$ such that $J^2 = -I$.*

An example of a complex structure on $\mathbf{R}^2$ is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

It is immediate that V must be even-dimensional or infinite-dimensional if a complex structure exists on it. A given vector space may have several different complex structures defined on it.

One can go back and forth between real vector spaces equipped with complex structures and complex vector spaces in the following way. Suppose that (V, J) is a real vector space equipped with a complex structure. Now we can formally define the "complexification" of V as a vector space $V_{\mathbf{C}} = V' \oplus V''$ where V' and V'' are copies of V and multiplication by complex numbers is given by $(x + iy) * \langle a, b\rangle = \langle xa - by, xb + ay\rangle$. Now we can define a linear operator P on $V_{\mathbf{C}}$ by the formula $P\langle a, b\rangle = (1/2)\langle a + Jb, b - Ja\rangle$. It is easy to verify that P defines a projection operator on $V_{\mathbf{C}}$. It defines a subspace of $V_{\mathbf{C}}$, which is, as a real vector space, isomorphic to V. Similarly, given a complex vector space W we can construct a real vector space which is isomorphic to W, as a real vector space, and equip it with a complex structure which will give us back W when we apply the construction above to it. We first form the direct sum $W \oplus W$. Now we define a complex structure on this space by $J\langle a, b\rangle = \langle ia, -ib\rangle$. It is easy to check the claims made. The upshot is that one can talk about

[6] and every subsequent reading as well.

real vector spaces equipped with complex structures or about complex vector spaces interchangably.

The final piece of mathematics that we need is the *Lie derivative* from classical differential geometry. Developing the definitions from scratch would involve a long digression. Fortunately the ideas are simple so we will give an intuitive account. In what follows the word "smooth" is meant to signify infinitely differentiable. Consider a smooth manifold; a curved surface is an excellent model to keep in mind. Suppose that one has a smooth vector field on this manifold; that is to say a smooth assignment of a vector at every point on the manifold. Classical results from differential equations say that there is a family of nonintersecting curves that fill the manifold and such that the curves are everywhere tangent to the given vector field. Now these curves are all parametrized by a real parameter say t. If we fix a value for t, we can define a smooth bijective map ψ_t of the manifold to itself (a so-called "diffeomorphism") which is defined by moving each point t units along the unique curve passing through it. We can make the map ψ_t act on functions defined on V as follows: $\psi_t^*(f) = f \circ \psi_t$, for f a complex-valued function defined on V. We can now define the *Lie derivative of f along the vector field u at the point p* as the limit

$$\mathcal{L}_v f = \lim_{t \to 0} \frac{\psi_t^*(f)(p) - f(p)}{t}$$

This gives another function from V to the complex numbers. Intuitively we imagine that the given vector field, u, defines a flowing fluid. The vector at each point defines the velocity of the fluid locally and the streamlines of the fluid give the family of curves mentioned above. The Lie derivative measures changes that an observer flowing with the fluid would see.

For us the Lie derivative tells us how to define changes seen "when travelling along the direction defined by a vector field". Now recall what is meant by an analytic function in ordinary complex analysis. A complex-valued function of two real variables, x and y, is analytic if it depends only on the complex variable $z = x + iy$ and not on the conjugate variable $\overline{z} = x - iy$. The Cauchy-Riemann equations say this precisely. The Lie derivative is what we need in order to do this in the infinite-dimensional case.

Definition 18 *Let B be a Banach space over the complex numbers. Now let J be a complex structure on this space. We call a vector v* **holomorphic** *if $Jv = iv$ and* **anti-holomorphic** *if $Jv = -iv$. If we have a* *real vector space V, equipped with a complex structure, we can define a holomorphic or an anti-holomorphic vector in the same way.*

A holomorphic vector plays roughly the role of a complex variable while an anti-holomorphic vector plays the role of a complex-conjugate variable. Now we can state the infinite-dimensional analogue of the Cauchy-Riemann conditions.

Definition 19 *A function $f : V \to \mathbf{C}$ is* **holomorphic** *if (i) it is differentiable and (ii) for every anti-holomorphic vector field v we have $\mathcal{L}_v f = 0$. An equivalent condition is $\mathcal{L}_{Jv} f = i\mathcal{L}_v f$ for holomorphic vector fields v.*

It is easy to check that the latter form of the second condition gives the usual Cauchy-Riemann equations in the one-dimensional case by choosing the vector fields appropriately.

6 The Physical Origin of Fock Space

The Fock space constructions described in the previous sections were independently invented by physicists and mathematicians. The symmetric Fock space (called the bosonic Fock space by physicists) is well known to mathematicians as the symmetric tensor algebra whereas the antisymmetric Fock space (fermionic Fock space) was invented by Grassman, at least in the finite-dimensional case, under the name of exterior algebra or alternating algebra. In this section we describe the role of Fock space in quantum field theory. In order to prevent intolerable regress in definitions we assume that the reader has an at least intuitive grasp of differential equations, the definition of a smooth manifold and associated concepts like that of a smooth vector field[7]

We begin with a brief discussion of quantum mechanics and classical mechanics. In classical mechanics one has systems which vary in time. The role of theory is to describe the temporal evolution of systems. Such temporal evolution is governed by a differential equation. The fact that one uses differential equations says something fundamental about the local nature of the dynamics of physical systems, at least according to conventional classical mechanics. In dealing with differential equations one has to distinguish between quantities that are determined and quantities that may be freely specified: the so called "initial conditions". Experiment tells one that systems are described by second-order differential

[7] Remarks requiring a more sophisticated vocabulary will appear as footnotes.

equations and hence that the functions being described and their first derivatives, at a given point of time, are part of the initial conditions. The space of all possible initial conditions is called the space of possible states or "phase" space, and is the kinematical arena on which dynamical evolution occurs[8]. A fundamental mathematical assumption is that the phase space is a $2n$ dimensional smooth manifold[9]. The points of phase space are called states. If the system is a collection of, say 7, particles, the states will correspond to the 42 numbers required to specify the positions and the velocities of each of the particles in three-dimensional space.

Through each point in phase space is a vector giving rise to a smooth vector field called the Hamiltonian vector field. One can draw a family of curves such that at every point there is exactly one curve passing through that point and the Hamiltonian is tangent to the curve at that point. Roughly speaking, the vector field defines a differential equation and the curves represent the family of solutions where each point represents a possible specification of initial conditions. An *observable* is a physical quantity that is determined by the state. As such it corresponds to a real-valued function on phase space. A typical example is the total energy of a system. Most of experimental mechanics is aimed at determining the Hamiltonian. In the formal development of analytical mechanics there is a special antisymmetric 2-form called the *symplectic form* which plays a fundamental mathematical role but is hard to describe in an intuitive or purely physical way.

In quantum mechanics, the above picture changes in the following fundamental ways. The observables become the fundamental physical entities. These are defined to form a particular subalgebra of an algebraic structure called a C^*-algebra. The key point is that this algebra is not commutative, unlike the algebra of smooth functions on a manifold. Furthermore, the failure of commutativity is directly linked to the symplectic form; this was Dirac's contribution to the theory of quantum mechanics. Thus, structures available at the classical level provide guidance as to what the "correct" C^*-algebra should be.

There is a representation of this algebra as the algebra of operators on a Hilbert space. The space of states acquires the structure of a Hilbert space and becomes the carrier of the representation of the C^*-algebra. One presentation of this abstract Hilbert space is as the space of square-

---

[8]Sometimes one has a more complicated situation in which the phase space is constrained in such a way that it cannot be simply defined as a manifold. These are called non-holonomic constraints and correspond to such familiar situations as skating and rolling.

[9]Actually it has the structure of the cotangent bundle of a smooth n-manifold.

integrable complex-valued functions on a suitable underlying space; for example the space of possible configurations of a system. The space of states has acquired linear structure; this means that one can add states reflecting the intuition that in quantum mechanics a system can be in the superposition of two (or more) states. The inner product measures the extent to which two states resemble each other. Finally the fact that one has complex functions is strongly suggested by the observation of interference phenomena in nature.

An observable is a self-adjoint operator. The link between the mathematics and experiment is the following. If one attempts to measure the observable O for a system in state ψ one will obtain an eigenvalue of O. Self-adjoint operators have real eigenvalues so we will get a real-valued result. If ψ is an eigenvector with eigenvalue α, then, with no indeterminacy or uncertainty, one will obtain the value α. If ψ is not an eigenvector, one can express ψ as a linear combination of eigenvectors in the form $\psi = \Sigma a_i \psi_i$ where the ψ_i are assumed to be eigenvectors with eigenvalues α_i. The result of measuring O will be α_i with probability $|a_i|^2$. It is important to keep in mind that the absolute squares of the a_i correspond to probabilities but it is the a_i themselves that enter into the linear combinations of states. This interplay between the complex coefficients and the interpretation of their squares as probabilities is what distinguishes the probabilistic aspects of quantum mechanics from statistical mechanics which also has a probabilistic aspect but where one directly manipulates probabilities.

The dynamics of systems is described by a *first-order* differential equation called Schroedinger's equation. Thus, the evolution of states in quantum mechanics is determinate, just as in classical mechanics. The indeterminacy usually associated with quantum mechanics appears in the fact that the state of a system may not be an eigenstate of the observable being measured so the outcome of the measurement may be indeterminate.

Quantum mechanics is designed to handle systems in which the number of interacting entities (usually called "particles") is fixed. On the other hand, experiment tells us that at sufficiently high energies particles may be created or destroyed. Quantum field theory was invented to account for such processes. The original formulations of this theory due to Dirac, Heisenberg, Fock, Jordan, Pauli, Wigner and many others was quite heuristic. Now a reasonably rigourous theory is available; see the book by Baez, Segal and Zhou [BSZ92] for a recent exposition of quantum field theory.

The first need in a many-particle theory is a space of states which can describe variable numbers of particles; this is what Fock space is [Ge85]. The second ingredient is the availability of operators that can describe the creation and annihilation of particles. Of course, there is much more that needs to be said in order to see how all this formalism translates into calculations of realistic physical processes but that would require a very thick book which, in any case, has been written many times over.

Given a Hilbert space H in quantum mechanics representing the states of a single particle one can construct a many-particle Hilbert space as $\mathcal{F}(H)$. Suppose that $\psi, \phi \in H$; one interprets the element $\psi \otimes_s \phi$ of $H \otimes_s H$ as a two-particle state with one particle in the state ψ and the other in the state ϕ. Similarly for the other summands of $\mathcal{F}(H)$. The reason for the symmetrization is that one is dealing with indistinguishable particles so that the n-particle states have to carry representations of the permutation group. Thus one could have particle states that were symmetric or antisymmetric under interchange leading to the bosonic or fermionic Fock spaces respectively. It is a remarkable fact that both types of particles are observed in nature. Notice that $\psi \wedge \psi$ is identically zero hence one cannot have many-particle states in the antisymmetric Fock space in which both particles are in the same one-particle state. This is observed in nature as the exclusion principle. Fock space is the space of states for quantum field theory and is constructed from the space of states for quantum mechanics.

The following interesting operators are defined on Fock space. Let $\psi = \langle \psi_0, \psi_1, \psi_2, \ldots, \psi_n, \ldots \rangle$ be an element of $\mathcal{F}(H)$. Now let σ be an element of H. We define the operator $C(\sigma)$ by

$$C(\sigma)\psi = \langle 0, \psi_0\sigma, \sqrt{2}\,\psi_1 \otimes_s \sigma, \ldots, \sqrt{n+1}\,\psi_n \otimes_s \sigma, \ldots \rangle$$

This operator creates a particle in the state σ. There is an analogous operator $A(\sigma)$ which destroys a particle in state σ. These two operators are adjoint to each other. The fundamental algebraic relation between them is $A(\sigma)C(\sigma) - C(\sigma)A(\sigma) = I$ where I is the identity operator. From these two we can define the operator $N(\sigma) = C(\sigma)A(\sigma)$. Let v_n be a state with n particles in the state σ and with no other particles. For the rest of the paragraph we drop explicit mention of σ. Now $A\,v_n = \sqrt{n}\ v_{n-1}$ and $C\,v_n = \sqrt{n+1}\ v_{n+1}$, hence we have $N\,v_n = n\,v_n$. Thus v_n is an eigenstate of N with eigenvalue n; for this reason N is called the number operator. Now we also have $N A\,v_n = (AC - I)A\,v_n = A(CA - I)v_n = A(N - I)v_n = (n-1)A\,v_n$. In other words, $A\,v_n$ is also an eigenstate of N with eigenvalue $(n-1)$. This justifies the name "annihilation" operator. A similar calculation can be done for the creation operator. If we are successful in developing a theory of reduction of proof nets in terms of operator algebras, in the sense of Girard's geometry of interaction, we will have the A and C operators available. We hope that these can be used to give a quantitative handle on resource consumption during computation.

The presentation of Fock space above emphasized the concept of many-particle states. Mathematically, however, $\mathcal{F}(H)$ is just a Hilbert space and can be presented differently. As we have shown in the last section, it can be presented as the space of holomorphic functions of a Hilbert space (the details are somewhat different from the Banach space case but the ideas are essentially the same). The space of holomorphic functions has as its inner product

$$\langle g, f\rangle = \frac{1}{2\pi i}\int f(z)g(\overline{z})e^{-z\overline{z}}dz\,d\overline{z}.$$

(See [IZ80] page 435, for example.) What do the creation and annihilation operators look like from this perspective? For simplicity, let us look at power series in a single variable z. This amounts to only looking at the many-particle states of the form σ tensored with itself. The creation operator is just $z*(.)$ while the annihilation operator is just $d(.)/dz$. One can easily check that $(AC - CA)f = d(z*f)/dz - z*df/dz = f$; in other words the basic algebraic relation holds. Furthermore one can ask what the eigenstates of A and C look like. Clearly the eigenstate of C is just the zero vector. The eigenstate of A is the state represented by the holomorphic function e^z. These states actually exist in nature and are called "coherent" states; they occur, for example, in lasers. The key point about coherent states is that they "look classical"; one can remove a particle without changing the state. As such they bear a superficial resemblance to the role of ! formulas in linear logic.

7 Conclusion

To summarize the results we have claimed in this paper, we have produced models of the following fragments of linear logic. First, in finite-dimensional vector spaces we have a complete model of classical linear logic, albeit with a compact category, so that the tensor and par are identified, as are ! and ?. In the category of symmetric Banach *algebras* we have a model of the $\otimes, \times, \oplus,$! fragment. This category cannot be endowed with closed structure, since $Hom(X,Y) = Hom(\mathrm{I}\otimes X, Y) = Hom(\mathrm{I}, X \multimap Y)$; in this category the unit I for $\otimes$ is also the initial object so the last hom set would have to be a singleton, clearly not the case for arbitrary X, Y. The closely related $\mathcal{BANCON}$ has several very pleasant features as a model of the multiplicative and additive fragment of linear logic—it is a rare example of a category of linear spaces which is neither additive nor compact—but unfortunately it is not possible to extend this to a model of ! on the algebra category as we did with $\mathcal{BANACH}$ as the exponential isomorphism fails there. In $\mathcal{HILBERT}$ we get results analagous to those with $\mathcal{BANACH}$, modelling the $\otimes, \times, \oplus, !$ fragment in the category of algebras. In addition to producing these models, we have described a mathematical representation for ! using holomorphic functions which suggests that one might profitably think of computability in terms of analyticity rather than continuity. Furthermore the mathematical structures described in this paper arise from quantum field theory and are suggestive of links with that subject.

It is crucial that one appreciate the differences between our work and that of Girard in [G89]. He has also used Banach algebras but all proofs are represented in a single Banach algebra, whereas we model formulas as individual algebras, with proofs as algebra homomorphisms. That is to say, we work in the category of Banach algebras, rather than inside a particular algebra. His major achievement is modelling cut elimination in terms of operator algebras. We on the other hand model provability in the appropriate fragment of linear logic.

Our next goal is to model the proof theory of linear logic in the spirit of Geometry of Interaction. Rather than following Girard, we will be guided by the following intuitions which are suggested by the physical interpretation of Fock space. We think of formulas as representing *states*, that is to say elements of a Fock space; a proof represents the process of interaction between particles in the initial state resulting in the particles observed in the final state. Mathematically the process is described by a combination of creation and annihilation operators. Proof normalization transforms processes into "observably equivalent" processes. In particular, we hope that our version of such a theory will permit a sharper analysis of complexity of computations.

Acknowledgements

We wish to thank several collegues for helpful conversations: John Baez, Mike Barr, Jim Otto, Steve Vickers, and especially Abhay Ashtekar.

References

[Ab91] Abrusci, V.M. "Phase semantics and sequent calculus for pure noncommutative classical linear propositional logic", Journal of Symbolic Logic **56** (1991) 1403–1451.

[Ah66] Ahlfors, L.V. *Complex Analysis.* McGraw-Hill, New York, 1966, second edition. (First edition 1953).

[AM-A80] Ashtekar, A. and A. Magnon-Ashtekar "A geometrical approach to external potential problems in quantum field theory", Journal of General Relativity and Gravitation **12** (1980) 205-223.

[BSZ92] Baez, J.C., I.E. Segal and Z. Zhou *Introduction to algebraic and constructive quantum field theory.* Princeton University Press, Princeton, New Jersey, 1992.

[Ba61] Bargmann, V. "On a Hilbert space of analytic functions and an associated integral transform", Communications in Pure and Applied Mathematics **14** (1961) 187-214.

[B76a] Barr, M. "Duality of vector spaces", Cahiers de Topologie et Géometrie Differentielle **17** (1976) 3–14.

[B76b] Barr, M. "Duality of Banach spaces", Cahiers de Topologie et Géometrie Differentielle **17** (1976) 15–52.

[B79] Barr, M. *$*$-Autonomous Categories.* Lecture Notes in Mathematics **752**, Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[B91] Barr, M. "$*$-autonomous categories and linear logic", Mathematical Structures in Computer Science **1** (1991) 159–178.

[BBPH] Benton, B.N., G. Bierman, V. de Paiva, and M. Hyland "Term assignment for intuitionistic linear logic (preliminary report)", Preprint, University of Cambridge, 1992.

[Bl91] Blute, R.F. "Proof nets and coherence theorems", in *Categories and Computer Science*, Paris 1991, Lecture Notes in Computer Science **530**, Springer-Verlag, Berlin, Heidelberg, New York, 1991.

[Bl92] Blute, R.F. “Linear Logic, Coherence and Dinaturality”, to appear in Theoretical Computer Science.

[Bl93a] Blute, R. “Linear Topology, Hopf Algebras and $*$-autonomous Categories”, preprint, McGill University (1993)

[Bl93b] Blute, R. “Braided Proof Nets and Categories”, in preparation, (1993).

[BCST] Blute, R.F., J.R.B. Cockett, R.A.G. Seely, and T.H. Trimble “Natural deduction and coherence for weakly distributive categories”, Preprint, McGill University, 1992. (Submitted to JPAA)

[BCS93] Blute, R.F., J.R.B. Cockett, and R.A.G. Seely “**!** and **?** for weakly distributive categories: Storage as tensorial strength”, preprint, McGill University, in preparation.

[CLM79] Cigler, J., V. Losert, P. Michor *Banach Modules and Functors on Categories of Banach Spaces.* Dekker, 1979

[CS91] Cockett, J.R.B. and R.A.G. Seely “Weakly distributive categories”, in M.P. Fourman, P.T. Johnstone, A.M. Pitts, eds., *Applications of Categories to Computer Science*, London Mathematical Society Lecture Note Series **177** (1992) 45–65.

[C90] Conway, J. *A Course in Functional Analysis.* Springer Verlag Graduate Texts in Mathematics **96**, (1990)

[F64] Freyd, P.J. *Abelian Categories: An introduction to the theory of functors.* Harper and Row, New York, 1964.

[Ge85] Geroch, R. *Mathematical Physics.* University of Chicago Press, 1985.

[G87] Girard, J.-Y. “Linear logic”, Theoretical Computer Science **50** (1987) 1–102.

[G89] Girard, J.-Y. “Geometry of interaction I: Interpretation of system F”, Proceedings of ASL meeting, Padova, 1988.

[GL87] Girard, J.-Y. and Y. Lafont “Linear logic and lazy computation”, in TAPSOFT’87, **2**, Lecture Notes in Computer Science **250**, Springer-Verlag, Berlin, Heidelberg, New York, 1987, 52–66.

[GSS91] Girard, J.-Y., A. Scedrov, and P.J. Scott "Bounded linear logic", Theoretical Computer Science **97** (1992) 1–66.

[Gu90] Gunning, R. *Introduction to Holomorphic functions of several variables, Volume 1: Function theory.* Mathematics Series, Wadsworth and Brooks/Cole, Belmont CA, 1990.

[IZ80] Itzykson, J, and J. Zuber *Introduction to Quantum Field Theory.* McGraw-Hill, New York, 1980.

[KR83] Kadison, R.V. and J.R. Ringrose *Fundamentals of the Theory of Operator Algebras.* Academic Press, New York, 1983

[KL80] Kelly, G.M. and M. La Plaza, "Coherence for Compact Closed Categories", Journal of Pure and Applied Algebra **19** (1980) 193–213.

[L65] Lang, S. *Algebra.* Addison-Wesley Pub. Co., Reading, Menlo Park, London, Don Mills, 1965.

[Le41] Lefschetz, S. *Algebraic Topology.* AMS Colloquium Publications **27**, 1941.

[M71] Mac Lane, S. *Categories for the Working Mathematician.* Graduate Texts in Mathematics **5**, Springer-Verlag, Berlin, Heidelberg, New York, 1971

[M-OM89] Martí-Oliet, N. and J. Meseguer "From Petri nets to linear logic", in D.H. Pitt *et al.*, eds. *Category Theory and Computer Science*, Manchester 1989, Lecture Notes in Computer Science **389**, Springer-Verlag, Berlin, Heidelberg, New York, 1989.

[dP89] de Paiva, V.C.V. "A Dialectica-like model of linear logic", in D.H. Pitt *et al.*, eds., *Category Theory and Computer Science*, Lecture Notes in Computer Science **389**, Springer-Verlag, Berlin, Heidelberg, New York, 1989.

[P79] Panangaden, P. "Positive and negative frequency decompositions in curved spacetimes", Journal of Mathematical Physics **20** (1979) 2506–2514.

[P80] Panangaden, P. "Propagators and Renormalization of Quantum Field Theory in Curved Spacetimes.", Ph.D. Thesis, Physics Department, University of Wisconsin-Milwaukee,

1980. Available through University Microfilms International, Ann Arbor.

[Se89] Seely, R.A.G. "Linear logic, $*$-autonomous categories and cofree coalgebras", in J. Gray and A. Scedrov (eds.), *Categories in Computer Science and Logic*, Contemporary Mathematics **92** (Am. Math. Soc. 1989).

[S62] Segal, E. "Mathematical characterization of the physical vacuum for linear Bose-Einstein fields", Illinois Journal of Mathematics **6** (1962) 500–523.

[T79] Takesaki M. *Theory of Operator Algebras.* Springer-Verlag, Berlin, Heidelberg, New York, 1979.

The following correction appeared in "Fock Space: a Model of Linear Exponential Types" (Blute-Panangaden-Seely):

Remark 16 *We wish to point out an error in an earlier draft of this paper [BPS]. In that paper, it is stated that the Fock construction is functorial on the larger category of Banach spaces and bounded linear maps. In fact, when one applies the Fock construction to a map of norm greater than* 1*, one might obtain a divergent expression. Thus, we are forced to work in the smaller category of contractions.*