

Differential Categories Revisited

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1 Introduction

Differential linear logic [8, 9] introduced a syntactic proof-theoretic approach to differential calculus and, subsequently, differential categories [3] were developed to provide a categorical counterpart for these ideas. In this categorical approach two methods for defining the differentiation were introduced based on, respectively, a deriving transformation and a codereliction. In [10] Fiore proposed an axiomatization for a deriving transformation, which he called a “creation operator”, satisfying additional “strength laws”. These laws were very natural to impose in the presence of a monoidal coalgebra modality and biproducts. However, this introduced an apparently stronger notion of differentiation and left open the question of whether a creation operator was a distinct notion of differentiation. An initial purpose of this paper is to provide a proof that, in the presence of a monoidal coalgebra modality, all these methods of defining differentiation are in fact, equivalent. Thus, there is only one notion of differentiation in linear logic.

In [10], Fiore made another interesting observation, namely, that it is much more convenient to work in a setting with biproducts. Furthermore he argued that, as one can always complete an additive category to have biproducts, one may as well work in a setting with biproducts. Of course, the notion of differentiation in additive categories is tightly coupled to having a coalgebra modality. Thus, when one completes a differential category to have biproducts, one needs to show that this modality can also be extended to the biproduct completion. It is standard that one can extend a *monoidal* coalgebra modality to the biproduct completion using the Seely isomorphisms and Fiore’s work was focused on this case. However, when the modality is *not* monoidal, there is no reason why the modality should extend to the biproduct completion. In fact, the deriving transformations of [3] had not assumed that the coalgebra modality was monoidal. Thus, if one is to entertain these more basic modalities, one must be cautious about assuming the presence of biproducts. A significant aspect of this paper is to point out that there are many natural examples of differential categories in which the coalgebra modality is *not* monoidal, thus exclusively concentrating on monoidal modalities misses an important part of the geography of differentiation (see the Venn diagram in Section 8) .

The paper starts by revisiting the “original” definition of a differential category found in [3]. This relies on the idea of a coalgebra modality and a deriving transformation. Familiarity with linear logic may tempt one to think that this is the “exponential” modality of linear logic but it is not. It is a strictly more general notion as the modality is not assumed monoidal. There are many familiar and important examples of differential categories based on a monoidal coalgebra modality – such as (the opposite of) the free symmetric algebra monad on vector spaces. However, examples of differential categories based on a modality which is not monoidal are less familiar. Two

compelling examples are smooth functions via the free \mathcal{C}^∞ -ring over vector spaces (as mentioned in [3]) and the free Rota-Baxter algebra over modules (see [19]) which we prove, in Proposition 8.8. provides a differential category.

Here we refer to additive categories, with a monoidal coalgebra modality, as “additive linear categories”: their biproduct completion is then an “additive monoidal storage category” [5]. An additive monoidal storage category (which has biproducts) is always an additive linear category (which need not have biproducts) and both always have a monoidal bialgebra modality. Here, to facilitate the proofs, it is convenient to work with a further intermediate notion we called an “additive bialgebra modality”: this is a bialgebra modality which has an additional coherence requirement between the additive and bialgebra structure. It is with respect to additive bialgebra modalities that we prove that deriving transformations and coderelictions are equivalent.

Coderelictions always give deriving transformations, and it was shown, in [3], that a deriving transformation, for a bialgebra modality, is equivalent to a codereliction if and only if the deriving transformation satisfies the ∇ -rule. This latter rule was originally thought to be a completely independent requirement. The key new observation is that, for an additive bialgebra modality with a deriving transformation, the ∇ -rule is, in fact, implied. More specifically, while a deriving transformation is assumed to satisfy five rules – see [d.1]–[d.5] below (which include the linear rule and the Leibniz rule) – over an additive bialgebra modality, we prove that for a deriving transformation which satisfies the linear rule, the ∇ -rule is equivalent to the Leibniz rule.

When an additive symmetric monoidal category has a monoidal coalgebra modality it is straightforward to show that the modality is an additive bialgebra modality. Thus, for additive linear categories deriving transformations and coderelictions are equivalent.

Clearly additive bialgebra modalities and monoidal coalgebra modalities are closely related. In particular, additive bialgebra modalities can always be extended to the biproduct completion (see Section 7) and, furthermore, this biproduct completion has Seely isomorphisms [17]. Thus, additive bialgebra modalities correspond to monoidal storage categories [5] (also called new Seely categories [1, 16]) having the Seely isomorphisms. However, it is well-known (as the name suggests) that monoidal storage categories have a monoidal coalgebra modality! Thus, additive bialgebra modalities are, in fact, monoidal coalgebra modalities.

The above provides an abstract proof of the equivalence of the two notions which relies on results dispersed across a number of papers. In order to bring the result into focus for this paper we provide a direct proof (see Appendices A and B). We make no claim that the resulting proof is more elegant or shorter: it simply has the merit of collecting a complete demonstration of this equivalence under one roof.

This allow us to complete the first purpose of the paper by observing that, in an additive linear category, coderelictions and deriving transformations always satisfy the “strength laws”. Putting all this together one has that deriving transformations and creation operators are, in additive linear categories, completely equivalent.

The final section of the paper, Section 8, provides separating examples for the categorical structures we have introduced. Of particular interest is the example of the modality on differential algebras which we treat in some detail. It is of particular interest as – contrary perhaps to expectations – it is an example (in the dual) of an additive coalgebra modality which *does not* admit a deriving transformation. Furthermore, it does not admit a deriving transformation in the strongest possible sense: assuming that there is a deriving transformation forces the differential ring to be trivial. As far as we know these observations are new. Another interesting example is the Rota-

Baxter modality on a module category: as mentioned above, it is an example of a modality which is not additive yet admits a deriving transformation. Again as far as we know this has not been presented in full detail before.

1.1 Conventions and the Graphical Calculus

We shall use diagrammatic order for composition: explicitly, this means that the composite map fg is the map which first does f then g . Furthermore, to simplify working in symmetric monoidal categories, we will allow ourselves to work in strict symmetric monoidal categories and so will generally suppress the associator and unitor isomorphisms. For a symmetric monoidal category we will use \otimes for the tensor product, K for the unit, and $\sigma : A \otimes B \rightarrow B \otimes A$ for the symmetry isomorphism.

We shall make extensive use of the graphical calculus [13] for symmetric monoidal categories as this makes proofs easier to follow. Note, however, that our diagrams are to be read down the page – from top to bottom – and we shall often omit labelling wires with objects. We refer the reader to [18] for an introduction to the graphical calculus in monoidal categories and its variations, and to [3] for the graphical calculus of a differential category.

We will be working with coalgebra modalities: these are based on a comonad $(!, \delta, \varepsilon)$ where $!$ is the functor, δ is the comultiplication and ε is the counit. As in [3], we will use functor boxes when dealing with string diagrams involving the functor $!$: a mere map $f : A \rightarrow B$ will be encased in a circle while $!(f) : !A \rightarrow !B$ will be encased in a box:

$$f = \begin{array}{c} A \\ | \\ \circlearrowleft f \\ | \\ B \end{array} \quad !(f) = \begin{array}{c} !A \\ | \\ \boxed{f} \\ | \\ !B \end{array}$$

2 Differential Categories

Tensor differential categories are structures over additive symmetric monoidal categories with a coalgebra modality. We begin by recalling the components of this structure starting with the notion of an additive category. Here we mean “additive” in the sense of being commutative monoid enriched: we do not assume negatives nor do we assume biproducts (this differs from the usage in [15] for example). This allows many important examples such as the category of sets and relations or the category of modules for a commutative rig¹.

Definition 2.1 An **additive category** is a commutative monoid enriched category, that is, a category in which each hom-set is a commutative monoid with an addition operation $+$ and a zero 0 , and such that composition preserves the additive structure, that is: $k(f+g)h = kfh + kgh$, $0f = 0$ and $f0 = 0$. An **additive symmetric monoidal category** is a symmetric monoidal category which is also an additive category in which the tensor product is compatible with the additive structure in the sense that $k \otimes (f+g) \otimes h = k \otimes f \otimes h + k \otimes g \otimes h$, $0 \otimes h = 0$, and $h \otimes 0 = 0$.

¹Rigs are also known as a semirings: they are rings without negatives.

In a symmetric monoidal category, a cocommutative comonoid is a triple (C, Δ, e) consisting of an object C , a map $\Delta : C \rightarrow C \otimes C$ called the comultiplication and a map $e : C \rightarrow K$ called the counit such that the following diagrams commute:

$$\begin{array}{ccc}
 C & \xrightarrow{\Delta} & C \otimes C \\
 \Delta \downarrow & & \downarrow \Delta \otimes 1 \\
 C \otimes C & \xrightarrow{1 \otimes \Delta} & C \otimes C \otimes C
 \end{array}
 \qquad
 \begin{array}{ccc}
 & C & \\
 & \swarrow \Delta & \searrow \Delta \\
 C & \xleftarrow{e \otimes 1} & C \otimes C \xrightarrow{1 \otimes e} C \\
 & & \swarrow \Delta \quad \searrow \sigma \\
 & & C \otimes C
 \end{array}$$

A morphism of comonoids $f : (C, \Delta, e) \rightarrow (D, \Delta', e')$ is a map $f : C \rightarrow D$ which preserves the comultiplication and counit, that is, the following diagrams commute:

$$\begin{array}{ccc}
 C & \xrightarrow{\Delta} & C \otimes C \\
 f \downarrow & & \downarrow f \otimes f \\
 D & \xrightarrow{\Delta'} & D \otimes D
 \end{array}
 \qquad
 \begin{array}{ccc}
 C & \xrightarrow{f} & D \\
 e \searrow & & \downarrow e' \\
 & & K
 \end{array}$$

Definition 2.2 A **coalgebra modality** [3] on a symmetric monoidal category is a quintuple $(!, \delta, \varepsilon, \Delta, e)$ consisting of a comonad $(!, \delta, \varepsilon)$, a natural transformation $\Delta_A : !A \rightarrow !A \otimes !A$, and a natural transformation $e_A : !A \rightarrow K$ such that for each object A , $(!A, \Delta_A, e_A)$ is a cocommutative comonoid and δ preserves the comultiplication, that is, $\delta \Delta = \Delta(\delta \otimes \delta)$.

Note that we do not assume that the functor $!$ of a coalgebra modality is a monoidal functor – this will come later. The coKleisli maps for the comonad are important: these maps are of the form $f : !A \rightarrow B$; amongst these are the **linear maps** $\varepsilon g : !A \rightarrow B$ where $g : A \rightarrow B$.

Requiring that Δ and e be natural transformations is equivalent to asking that for each map $f : A \rightarrow B$, $!(f) : !A \rightarrow !B$ is a comonoid morphism. This can be used to show that δ is a comonoid morphism.

Lemma 2.3 For any coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$, δ also preserves the counit, that is, $\delta e = e$, and so δ is a comonoid morphism.

PROOF: By the naturality of e and that $(!, \delta, \varepsilon)$ is a comonad, we obtain that:

$$\delta e = \delta!(\varepsilon)e = e$$

□

Definition 2.4 A **differential category** is an additive symmetric monoidal category with a coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ which comes equipped with a **deriving transformation** [3], that is, a natural transformation $d_A : !A \otimes A \rightarrow !A$, which is represented in the graphical calculus as:

$$d := \begin{array}{c}
 \begin{array}{ccc}
 & !A & A \\
 & \cup & \\
 & \text{---} & \\
 & \cup & \\
 & !A &
 \end{array}
 \end{array}$$

such that d satisfies the following equations:

[d.1] Constant Rule: $d e = 0$

[d.2] Leibniz Rule:

$$d\Delta = (\Delta \otimes 1)(1 \otimes \sigma)(d \otimes 1) + (\Delta \otimes 1)(1 \otimes d)$$

[d.3] Linear Rule: $d\varepsilon = e \otimes 1$

[d.4] Chain Rule:

$$d\delta = (\Delta \otimes 1)(\delta \otimes d)d$$

[d.5] Interchange Rule: $(d \otimes 1)d = (1 \otimes \sigma)(d \otimes 1)d$

The first axiom [d.1] states that the derivative of a constant map is zero. The second axiom [d.2] is the Leibniz rule or the product rule for differentiation. The third axiom [d.3] says that the derivative of a linear map is constant. The fourth axiom [d.4] is the chain rule. The last axiom [d.5] is the interchange law, which naively states that differentiating with respect to x then y is the same as differentiation with respect to y then x . It should be noted that [d.5] was not originally a requirement in [3] but was later added to the definition to ensure that the coKleisli category of a differential category was a Cartesian differential category [4].

Our first revision of differential categories is that the constant rule [d.1] is in fact derivable:

Lemma 2.5 *For a coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ on additive symmetric monoidal category, any natural transformation $d_A : !A \otimes A \rightarrow !A$ satisfies the constant rule [d.1].*

PROOF: By naturality of e and d , and the additive structure, we have the following equalities:

$$d e = d!(0)e = (!(0) \otimes 0)d e = 0 d e = 0$$

□

Corollary 2.6 *For a coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$ on additive symmetric monoidal category, the following are equivalent for a natural transformation $d_A : !A \otimes A \rightarrow !A$:*

- (i) d is a deriving transformation;
- (ii) d satisfies the product rule [d.2], the linear rule [d.3], the chain rule [d.4], and the interchange rule [d.5].

3 Bialgebra Modalities, Coderelictions and the ∇ -Rule

In [3], Blute, Cockett and Seely observed that if the coalgebra modality came equipped with a bialgebra structure and a codereliction then one could obtain a deriving transformation. We recall these ideas, starting with the notion of a bialgebra modality. In a symmetric monoidal category, a (commutative) bialgebra is a quintuple $(A, \nabla, u, \Delta, e)$ such that (A, ∇, u) is a commutative monoid (dual of a cocommutative comonoid), (A, Δ, e) is a cocommutative comonoid, and the following diagrams commute:

$$\begin{array}{c}
 A \otimes A \xrightarrow{\nabla} A \\
 \searrow e \otimes e \quad \downarrow e \\
 \quad \quad \quad K
 \end{array}
 \quad
 \begin{array}{c}
 K \xrightarrow{u} A \\
 \searrow u \otimes u \quad \downarrow \Delta \\
 \quad \quad \quad A \otimes A
 \end{array}
 \quad
 \begin{array}{c}
 K \xrightarrow{u} A \\
 \searrow \quad \downarrow e \\
 \quad \quad \quad K
 \end{array}
 \quad
 \begin{array}{ccc}
 A \otimes A & \xrightarrow{\Delta \otimes \Delta} & A \otimes A \otimes A \otimes A \\
 \downarrow \nabla & & \downarrow 1 \otimes \sigma \otimes 1 \\
 A & \xrightarrow{\Delta} & A \otimes A \otimes A \otimes A \\
 & & \downarrow \nabla \otimes \nabla \\
 & & A \otimes A
 \end{array}$$

Definition 3.1 A **bialgebra modality** [3] on an additive symmetric monoidal category is a septuple $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ consisting of a coalgebra modality $(!, \delta, \varepsilon, \Delta, e)$, a natural transformation $\nabla_A : !A \otimes !A \rightarrow !A$, and a natural transformation $u_A : K \rightarrow !A$ such that for each object A , $(!A, \nabla_A, u_A, \Delta_A, e_A)$ is a bialgebra, and ε is compatible with ∇ in the following sense: $\nabla \varepsilon = \varepsilon \otimes e + e \otimes \varepsilon$.

By the naturality of ∇ , Δ , u and e we note that for every map f , $!(f)$ is both a monoid and comonoid morphism. In the original definition of a bialgebra modality in [3] it was also required that $u\varepsilon = 0$; however this is provable:

Lemma 3.2 For a bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$, $u\varepsilon = 0$.

PROOF: By the naturality of u and ε , and the additive structure we have the following:

$$u\varepsilon = u!(0)\varepsilon = u\varepsilon 0 = 0$$

□

The key ingredient required to obtain a deriving transformation for a bialgebra modality is:

Definition 3.3 A **codereliction** [3] for a bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ is a natural transformation $\eta_A : A \rightarrow !A$, such that the following equalities hold:

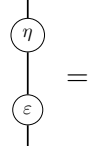
[dC.1] Constant Rule: $\eta e = 0$

$$\begin{array}{c}
 | \\
 \circlearrowleft \eta \\
 | \\
 \circlearrowleft e \\
 |
 \end{array}
 = 0$$

[dC.2] Product Rule: $\eta \Delta = \eta \otimes u + u \otimes \eta$

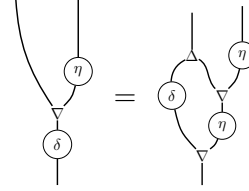
$$\begin{array}{c}
 | \\
 \circlearrowleft \eta \\
 | \\
 \cup \Delta
 \end{array}
 =
 \begin{array}{c}
 | \\
 \circlearrowleft \eta \\
 |
 \end{array}
 \begin{array}{c}
 | \\
 \circlearrowleft u \\
 |
 \end{array}
 +
 \begin{array}{c}
 | \\
 \circlearrowleft u \\
 |
 \end{array}
 \begin{array}{c}
 | \\
 \circlearrowleft \eta \\
 |
 \end{array}$$

[dC.3] Linear Rule: $\eta\varepsilon = 1$



[dC.4] Chain Rule:

$$(1 \otimes \eta)\nabla\delta = (\Delta \otimes \eta)(1 \otimes \nabla)(\delta \otimes \eta)\nabla$$



As for the constant rule for the deriving transformation, the constant rule [dC.1] for a coderelection can be derived:

Lemma 3.4 *For a bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$, any natural transformation $\eta_A : A \rightarrow !A$ satisfies the constant rule [dC.1].*

PROOF: By naturality of e and η , and the additive structure, we have the following equalities:

$$\eta e = \eta!(0)e = 0\eta e = 0$$

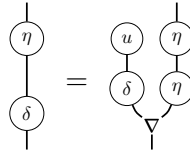
□

Corollary 3.5 *For a bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$, the following are equivalent for a natural transformation $\eta_A : A \rightarrow !A$:*

- (i) η is a coderelection;
- (ii) η satisfies the product rule [dC.2], the linear rule [dC.3], and the chain rule [dC.4].

In [10] an alternative axiom for the chain rule [dC.4] is used:

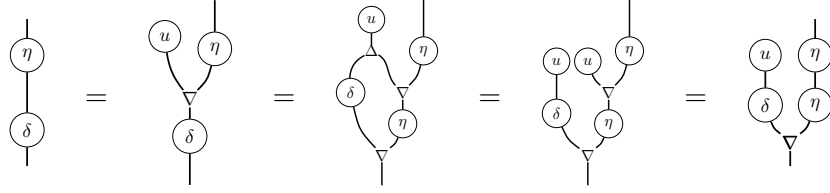
[dC.4'] Alternative Chain Rule: $\nabla\delta = (u \otimes \eta)(\delta \otimes \eta)\nabla$



In a monoidal storage category – the setting assumed in [10] – [dC.4] and [dC.4'] are equivalent. However in the setting of a mere bialgebra modality, it is clear that [dC.4] implies [dC.4']: the reverse implication, however, does not appear to hold. Thus, at this stage we prove the implication in one direction:

Lemma 3.6 *For a bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$, any natural transformation $\eta_A : A \rightarrow !A$ which satisfies the chain rule [dC.4] also satisfies the alternative chain rule [dC.4'].*

PROOF: The bialgebra structure gives the following chain of equalities:

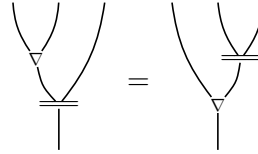


□

Let us now consider the relation between deriving transformations and bialgebra modalities. This is captured by the ∇ -rule [3]:

Definition 3.7 For a bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$, a natural transformation $d_A : !A \otimes A \rightarrow !A$ is said to satisfy the ∇ -rule if:

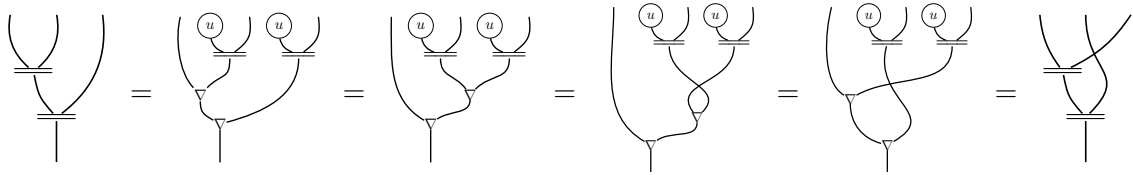
[d.∇] ∇ -Rule: $(\nabla \otimes 1)d = (1 \otimes d)\nabla$



Notice this implies that $d = (1 \otimes u \otimes 1)(\nabla \otimes 1)d = (1 \otimes u \otimes 1)(1 \otimes d)\nabla$. This has the consequence that the ∇ -rule implies the interchange rule:

Lemma 3.8 For a bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$, any natural transformation $d_A : !A \otimes A \rightarrow !A$ which satisfies the ∇ -rule, [d.∇], also satisfies the interchange rule, [d.5].

PROOF: By [d.∇] and the commutative monoid structure, we have the following equality:



□

Corollary 3.9 For a bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$, the following are equivalent for a natural transformation $d_A : !A \otimes A \rightarrow !A$:

- (i) d is a deriving transformation;
- (ii) d satisfies the product rule [d.2], the linear rule [d.3], the chain rule [d.4], and the ∇ -rule [d.∇].

which is reminiscent of the identities satisfied by the projection and injection maps of a biproduct. These maps will be key to the proof of Lemma 4.3 below. For an additive bialgebra modality this means that we have:

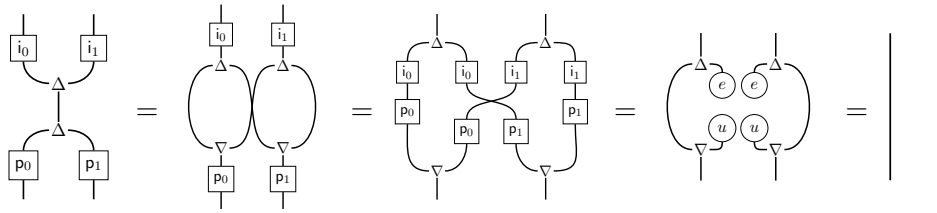
$$!(i_j)!(p_k) = \begin{cases} eu & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

This allows the derivation of the following useful identity:

Lemma 4.2 *For an additive bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ and a natural transformations $\eta_A : A \rightarrow !A$ which satisfies the linear rule [dC.3], the following identity holds:*

$$!(i_0) \otimes !(i_1) \nabla \Delta !(p_0) \otimes !(p_1) = 1$$

PROOF: By the additive bialgebra modality, we have the following equality:



□

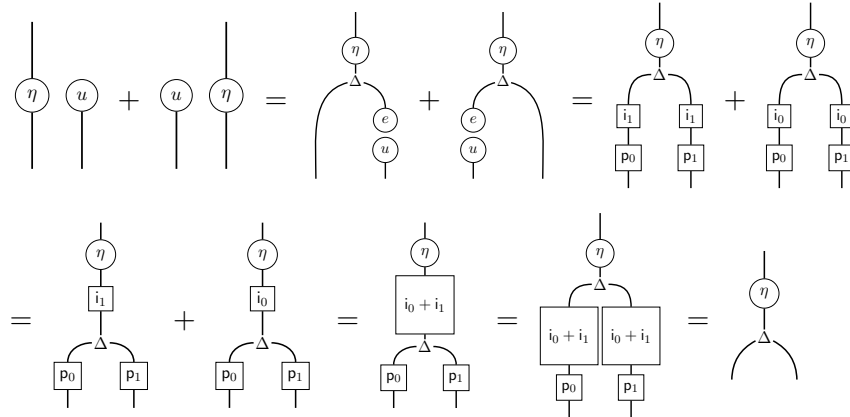
For an additive bialgebra modality, the linear rule [dC.3] implies the product rule [dC.2]:

Proposition 4.3 *For an additive bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$, any natural transformation $\eta_A : A \rightarrow !A$ which satisfies the linear rule [dC.3], also satisfies the product rule [dC.2].*

PROOF: Notice that by naturality of η , we have that:

$$\eta!(f + g) = (f + g)\eta = f\eta + g\eta = \eta!(f) + \eta!(g)$$

Then using i_j and p_k , we have the following:



□

Corollary 4.4 For an additive bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$, the following are equivalent for a natural transformation $\eta_A : A \rightarrow !A$:

- (i) η is a codereliction;
- (ii) η satisfies the linear rule [dC.3] and the chain rule [dC.4].

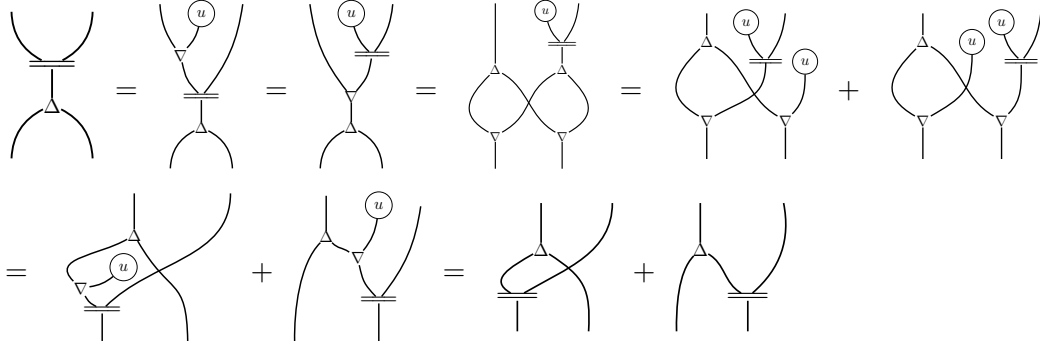
Turning our attention to deriving transformations for additive bialgebra modalities, we begin by noticing that satisfying the Leibniz rule is equivalent to satisfying the ∇ -rule:

Proposition 4.5 For an additive bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$, the following are equivalent for a natural transformation $d_A : !A \otimes A \rightarrow !A$ which satisfies the linear rule [d.3]:

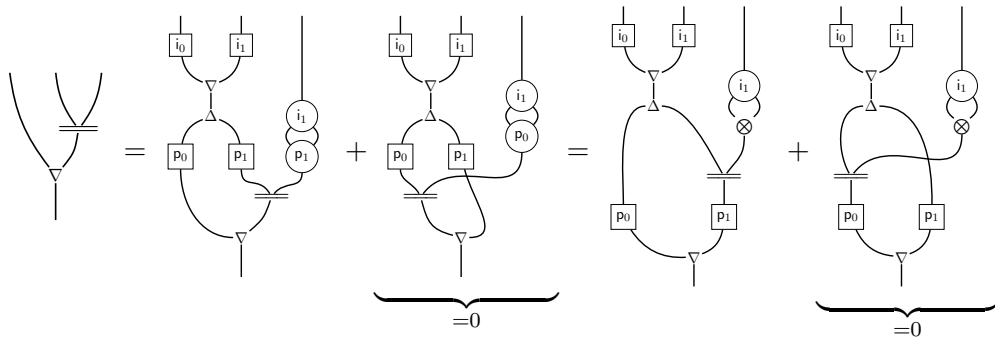
- (i) d satisfies the product rule [d.2];
- (ii) d satisfies the ∇ -rule [d. ∇].

PROOF:

[d. ∇] \Rightarrow [d.2]: It is easy to see that since d satisfies [d.3] that $(u \otimes 1)d$ satisfies [dC.3], the linear rule for coderelictions. However, by Lemma 4.3, this implies that $(u \otimes 1)d$ satisfies [dC.2], the product rule for coderelictions. Therefore, we have:



[d.2] \Rightarrow [d. ∇]: By the properties of i_j and p_k , and the identity of Lemma 4.2, we have:



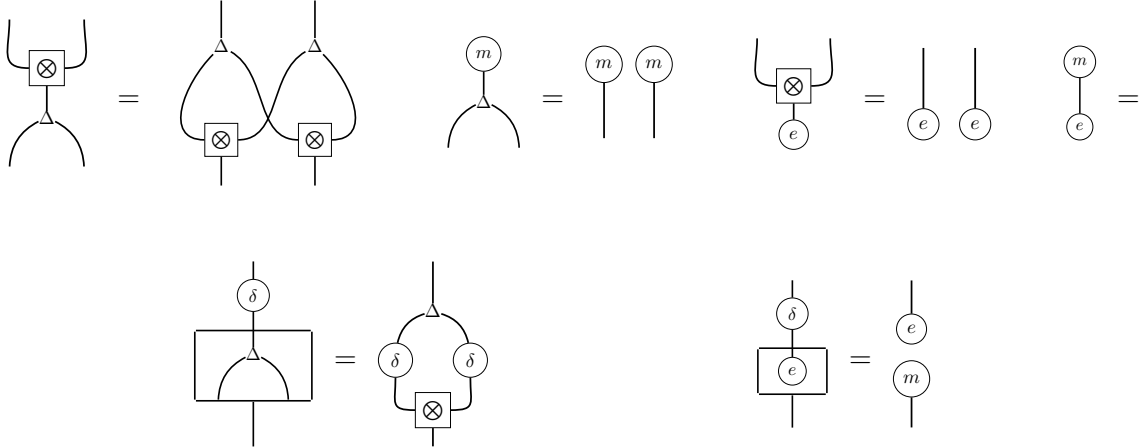
(i) Δ and e are monoidal transformations (or equivalently m_\otimes and m_K are comonoid morphisms), that is, the following diagrams commute:

$$\begin{array}{ccc}
 & & K \\
 & \nearrow m_\otimes & \downarrow e \\
 !A \otimes !B & \xrightarrow{m_\otimes} & !(A \otimes B) \\
 \downarrow \Delta \otimes \Delta & & \downarrow \Delta \\
 !A \otimes !A \otimes !B \otimes !B & & \\
 \downarrow 1 \otimes \sigma \otimes 1 & & \\
 !A \otimes !B \otimes !A \otimes !B & \xrightarrow{m_\otimes \otimes m_\otimes} & !(A \otimes B) \otimes !(A \otimes B)
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & !K \\
 & \nearrow m_K & \downarrow e \\
 K & \xrightarrow{m_K} & !K \\
 \downarrow m_K \otimes m_K & & \downarrow \Delta \\
 & & !K \otimes !K
 \end{array}$$

(ii) Δ and e are $!$ -coalgebra morphisms, that is, the following diagrams commute:

$$\begin{array}{ccc}
 !A & \xrightarrow{\delta} & !!A \\
 \downarrow \Delta & & \downarrow !(\Delta) \\
 !A \otimes !A & \xrightarrow{\delta \otimes \delta} & !!A \otimes !!A \xrightarrow{m_\otimes} !(A \otimes !A)
 \end{array}
 \qquad
 \begin{array}{ccc}
 !A & \xrightarrow{\delta} & !!A \\
 \downarrow e & & \downarrow !(e) \\
 K & \xrightarrow{m_K} & !(K)
 \end{array}$$

In string diagrams, that Δ and e are both monoidal transformations and $!$ -colagebra morphisms is expressed as follows:



Definition 5.2 A **linear category** [1, 5] is a symmetric monoidal category with a monoidal coalgebra modality; an **additive linear category** is a linear category which is also an additive symmetric monoidal category.

A monoidal coalgebra modality on an additive linear category induces an additive bialgebra

modality. The multiplication and unit are respectively:

$$\nabla := !A \otimes !A \xrightarrow{\delta \otimes \delta} !!A \otimes !!A \xrightarrow{m_\otimes} !(A \otimes !A) \xrightarrow{!(\varepsilon \otimes e + e \otimes \varepsilon)} !A$$

$$K \xrightarrow{m_K} !K \xrightarrow{!(0)} !A$$

Proposition 5.3 *The monoidal coalgebra modality of an additive linear category is an additive bialgebra modality.*

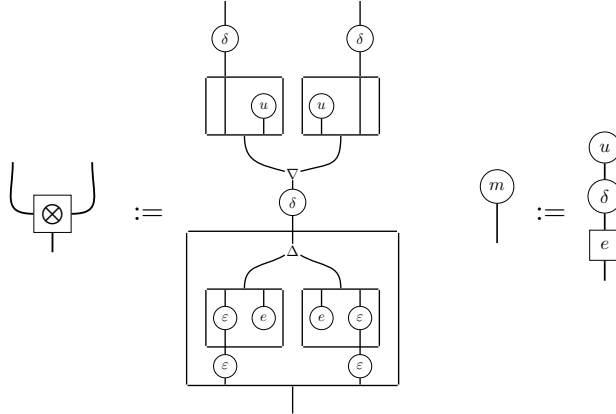
PROOF: See Appendix A. □

Conversely an additive bialgebra modality induces a monoidal coalgebra modality. The monoidal structure m_\otimes and m_K are defined respectively as:

$$\begin{array}{ccccccc} !A \otimes !B & \xrightarrow{\delta \otimes \delta} & !A \otimes !B & \xrightarrow{!(1 \otimes u) \otimes !(u \otimes 1)} & !(A \otimes !B) \otimes !(A \otimes !B) & \xrightarrow{\nabla} & !(A \otimes !B) \\ m_\otimes \downarrow := & & & & & & \downarrow \delta \\ !(A \otimes B) & \xleftarrow{!(\varepsilon \otimes \varepsilon)} & !(A \otimes !B) & \xleftarrow{!(\varepsilon \otimes e) \otimes !(e \otimes \varepsilon)} & !(A \otimes !B) \otimes !(A \otimes !B) & \xleftarrow{!(\Delta)} & !(A \otimes !B) \end{array}$$

$$\begin{array}{ccc} K & \xrightarrow{u} & !K \\ m_K \downarrow := & & \downarrow \delta \\ !(K) & \xleftarrow{!(e)} & !!K \end{array}$$

Represented in string diagrams as:



Proposition 5.4 *Every additive bialgebra modality is a monoidal coalgebra modality.*

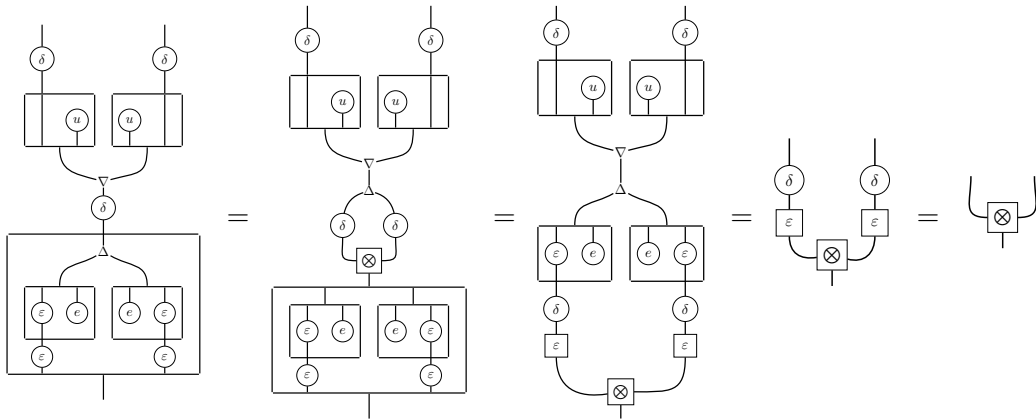
PROOF: See Appendix B. □

These constructions, between additive bialgebra modalities and monoidal coalgebra modalities, are in fact inverses of each other.

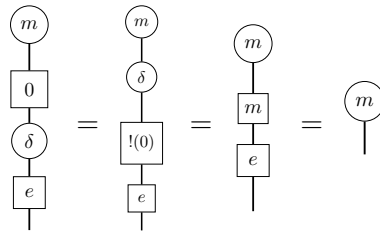
Theorem 5.5 *For an additive symmetric monoidal category, monoidal coalgebra modalities correspond bijectively to additive bialgebra modalities. Therefore, the following are equivalent:*

- (i) *An additive linear category;*
- (ii) *An additive symmetric monoidal category with an additive bialgebra modality.*

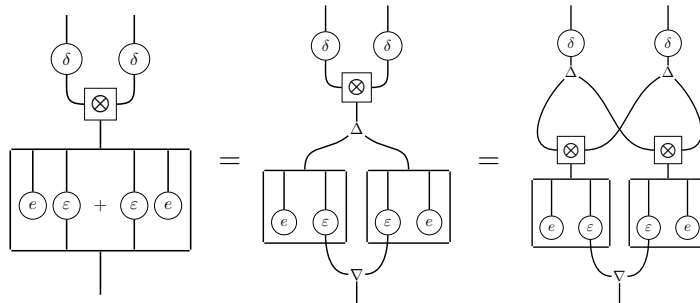
PROOF: Starting from an monoidal coalgebra modality, we first check that we re-obtain m_{\otimes} :

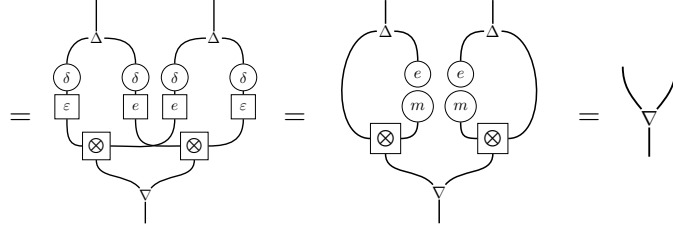


Next we check that we get back m_K :

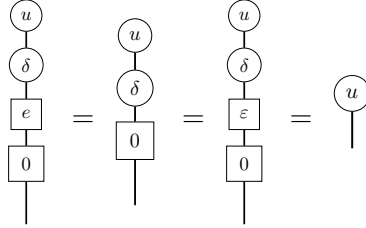


Starting with an additive bialgebra modality, we first check that we re-obtain the multiplication:





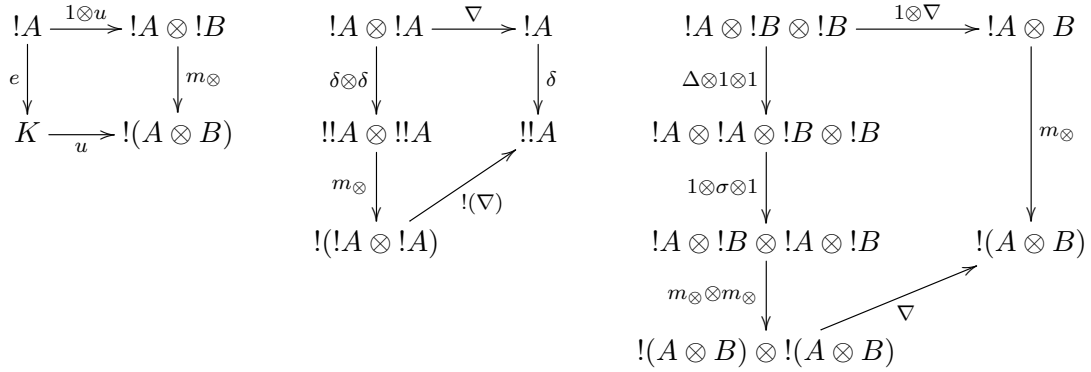
Finally we prove that we re-obtain the unit:



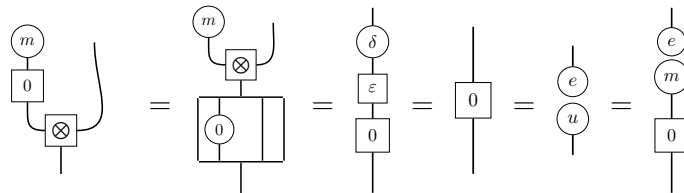
□

The identities expressing compatibility between the bialgebra modality and the monoidal coalgebra, which were observed in [10] (Theorem 3.1) to hold in an additive monoidal storage category, also hold in an additive linear category:

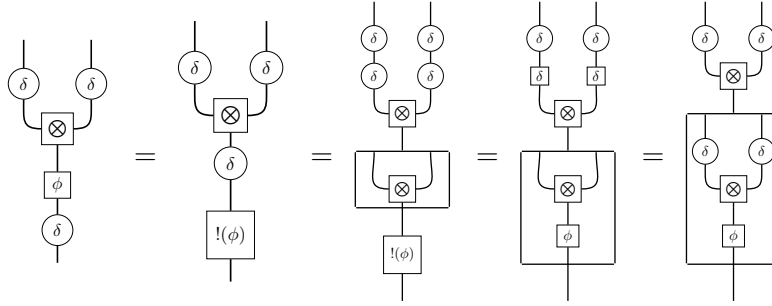
Proposition 5.6 *In an additive linear category, the following diagrams commute:*



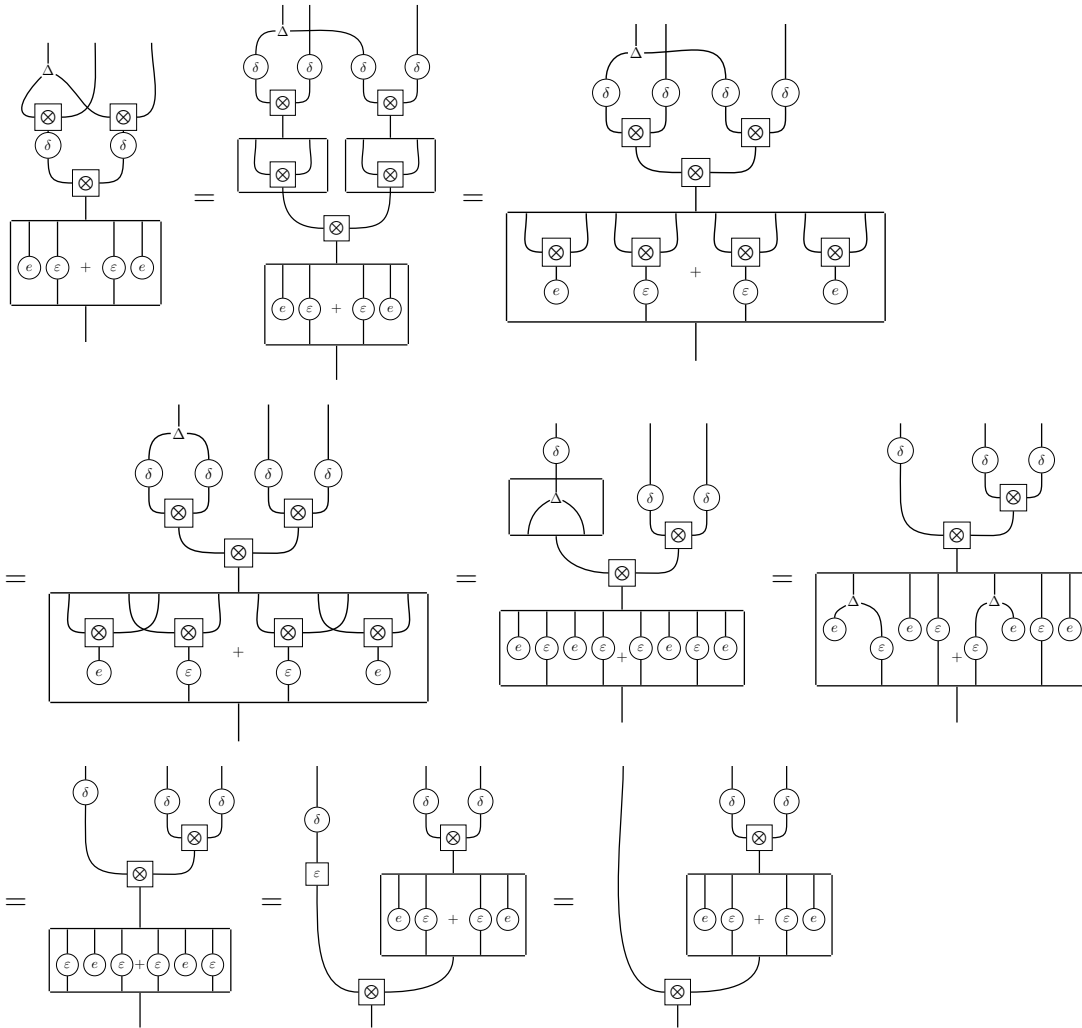
PROOF: For the first square on the left we use the naturality of m_\otimes , the unit laws of the monoidal functor and that $!(0)$ splits:



For the middle square we use the naturality of m_{\otimes} , the comonad δ square and that δ is a monoidal transformation:



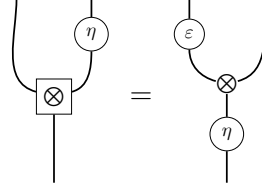
For the square on the right we use that δ is a monoidal transformation, naturality of m_{\otimes} , associativity and symmetry of m_{\otimes} , that e and ε are monoidal transformations and that Δ is a !-coalgebra morphism:



□

We turn our attention to the relation between the monoidal structure and the differential structure. In [10] Fiore introduced another axiom relating η to the monoidal structure:

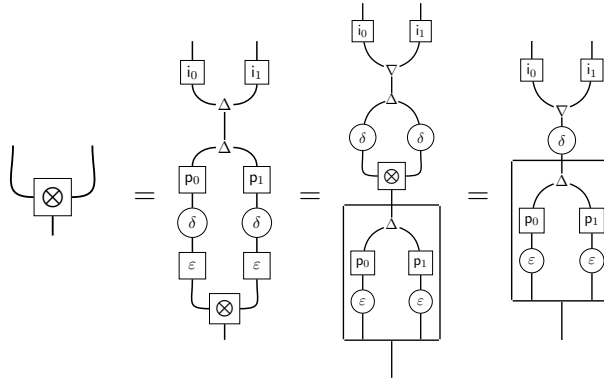
[dC.m] Monoidal Rule: $(1 \otimes \eta)m_\otimes = (\varepsilon \otimes 1)\eta$



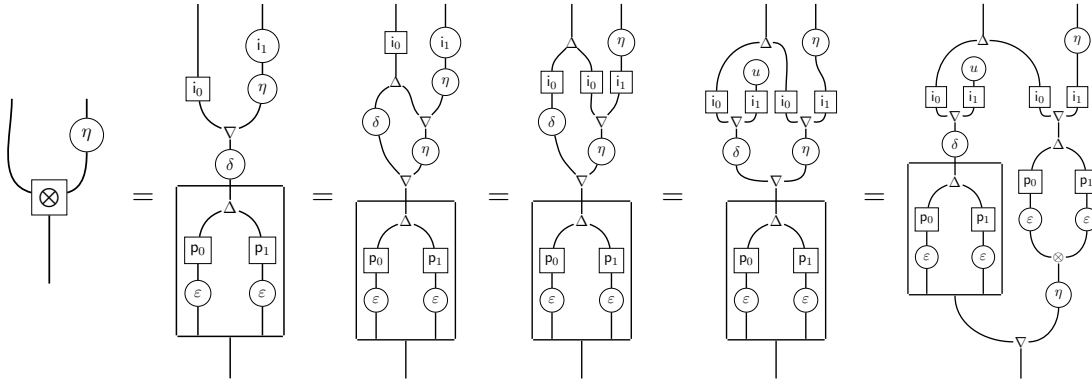
It turns out that coderelictions for the additive bialgebra modalities always satisfy the monoidal rule [dC.m]:

Proposition 5.7 *With respect to the induced bialgebra modality, in an additive linear category all coderelictions satisfy the monoidal rule [dC.m].*

PROOF: By Lemma 4.2 and the fact that Δ is a $!$ -coalgebra morphism, we first have that:



where i_j and p_k are defined as in the previous section. Expressing m_\otimes as above, then by the linear rule [dC.3], chain rule [dC.4], the naturality of u and ∇ , and the first diagram of Lemma 5.6 we have the following equality :



□

Conversely, the alternative chain rule [dC.4'] and the monoidal rule [dC.m] imply the chain rule [dC.4].

Lemma 5.8 *In an additive linear category, any natural transformation $\eta_A : A \rightarrow !A$ which satisfies the alternative chain rule [dC.4'] and the monoidal rule [dC.m], on the the induced additive bialgebra modality also satisfies the chain rule [dC.4].*

PROOF: By the compatibility relations of Proposition 5.6 and the coalgebra modality requirements we have:

□

Corollary 5.9 *For an additive linear category the following are equivalent for a natural transformation $\eta_A : A \rightarrow !A$ and the induced additive bialgebra modality:*

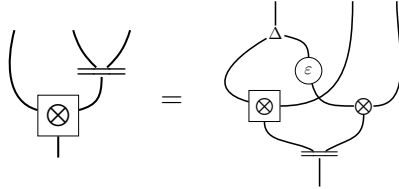
- (i) η is a codereliction;
- (ii) η satisfies the linear rule [dC.3] and the chain rule [dC.4];

(iii) η satisfies the linear rule [dC.3], the alternative chain rule [dC.4'] and the monoidal rule [dC.m].

Part (iii) is the definition of Fiore's creation map [10]: this shows that the original definition of a codereliction is equivalent to Fiore's creation map.

Now we explore deriving transformations of additive linear categories. The compatibility between a deriving transformation and the monoidal structure is described by the monoidal rule [10]—this is the strength rule which was the subject of Fiore's addendum:

[d.m] Monoidal Rule: $(1 \otimes \mathbf{d})m_{\otimes} = (\Delta \otimes 1 \otimes 1)(1 \otimes \varepsilon \otimes 1 \otimes 1)(1 \otimes \sigma \otimes 1)(m_{\otimes} \otimes 1 \otimes 1)\mathbf{d}$



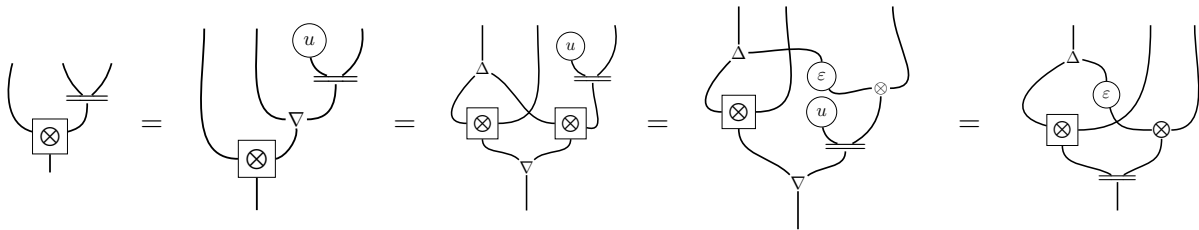
Fiore's creation operator [10] was defined to satisfy the linear rule [d.3], the chain rule [d.4], the ∇ -rule [d.∇], and the monoidal rule [d.m] – in his addendum, he pointed out the latter was redundant. It turns out that when a natural transformation satisfies both the linear rule [d.3] and the chain rule [d.4], then the monoidal rule is equivalent to both the ∇ -rule and the Leibniz rule:

Proposition 5.10 *For the induced additive bialgebra modality of an additive linear category, the following are equivalent for a natural transformation $\mathbf{d}_A : !A \otimes A \rightarrow A$ which satisfies the linear rule [d.3] and the chain rule [d.4]:*

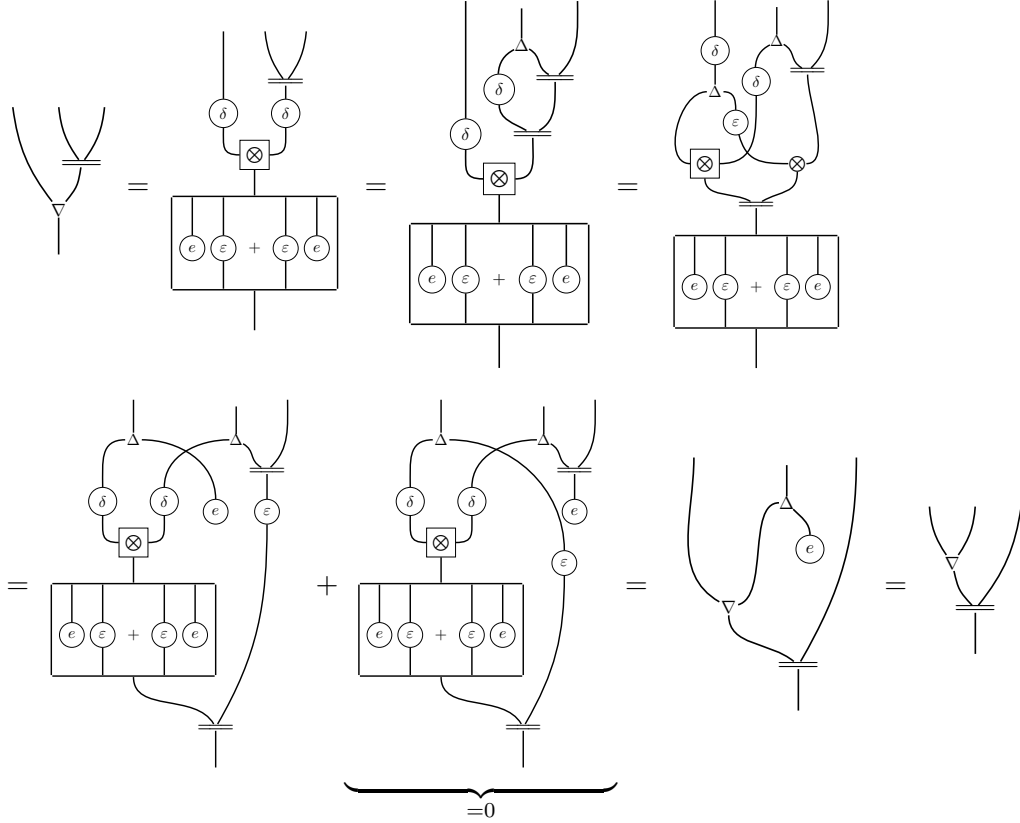
- (i) \mathbf{d} satisfies the Leibniz rule [d.2];
- (ii) \mathbf{d} satisfies the ∇ -rule [d.∇];
- (iii) \mathbf{d} satisfies the monoidal rule [d.m]

PROOF: Since this is an extension of Proposition 4.5, it suffices to show that the ∇ -rule [d.∇] and the monoidal rule [d.m] are equivalent.

[d.∇] \Rightarrow [d.m]: It is easy to see that since \mathbf{d} satisfies the linear rule [d.3] and the chain rule [d.4], $(u \otimes 1)\mathbf{d}$ satisfies the codereliction linear rule [dC.3] and chain rule [dC.4], and therefore by Corollary 5.9 is a codereliction and which by Proposition 5.7 satisfies the codereliction monoidal rule [dC.m]. Therefore, by the compatibility relations of Lemma 5.6, we have:



$[\mathbf{d.m}] \Rightarrow [\mathbf{d.\nabla}]$: By expanding ∇ using m_\otimes , we obtain the following equality:



□

Corollary 5.11 *For the induced additive bialgebra modality of an additive linear category, the following are equivalent for a natural transformation $\mathbf{d}_A : !A \otimes A \rightarrow A$:*

- (i) \mathbf{d} is a deriving transformation;
- (ii) \mathbf{d} satisfies the product rule $[\mathbf{d.2}]$, the linear rule $[\mathbf{d.3}]$, and the chain rule $[\mathbf{d.4}]$;
- (iii) \mathbf{d} satisfies the linear rule $[\mathbf{d.3}]$, the chain rule $[\mathbf{d.4}]$, and the ∇ -rule $[\mathbf{d.\nabla}]$;
- (iv) \mathbf{d} satisfies the linear rule $[\mathbf{d.3}]$, the chain rule $[\mathbf{d.4}]$, and the monoidal rule $[\mathbf{d.m}]$.

Finally, this gives the following theorem:

Theorem 5.12 *For the induced additive bialgebra modality of an additive linear category, all deriving transformations satisfy the monoidal rule $[\mathbf{d.m}]$ and are induced by a codereliction.*

6 Non-Additive Bialgebra Modalities

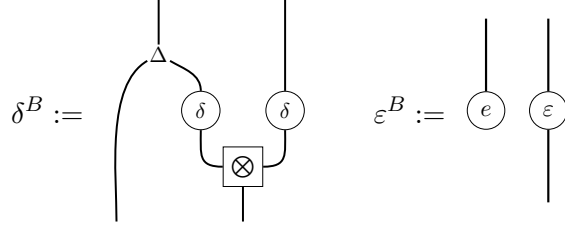
In this section we give a construction of non-additive bialgebra modalities induced by an additive algebra modalities. Given an additive bialgebra modality $(!, \delta, \varepsilon, \Delta, e, \nabla, u)$ on an additive symmetric monoidal category \mathbb{X} , for each object B consider the functor $!^B : \mathbb{X} \rightarrow \mathbb{X}$ defined on objects as

$!^B A = !B \otimes !A$, and on a map $f : A \rightarrow A'$ as $!^B(f) = 1 \otimes !(f) : !B \otimes !A \rightarrow !B \otimes !A'$. Consider the natural transformations $\delta^B : !^B(A) \rightarrow !^B(!^B(A))$ and $\varepsilon^B : !^B(A) \rightarrow A$ defined as follows:

$$\delta^B := !B \otimes !A \xrightarrow{\Delta \otimes 1} !B \otimes !B \otimes !A \xrightarrow{1 \otimes \delta \otimes \delta} !B \otimes !!B \otimes !!A \xrightarrow{1 \otimes m_\otimes} !B \otimes !(!B \otimes !A)$$

$$\varepsilon^B := !B \otimes !A \xrightarrow{e \otimes \varepsilon} A$$

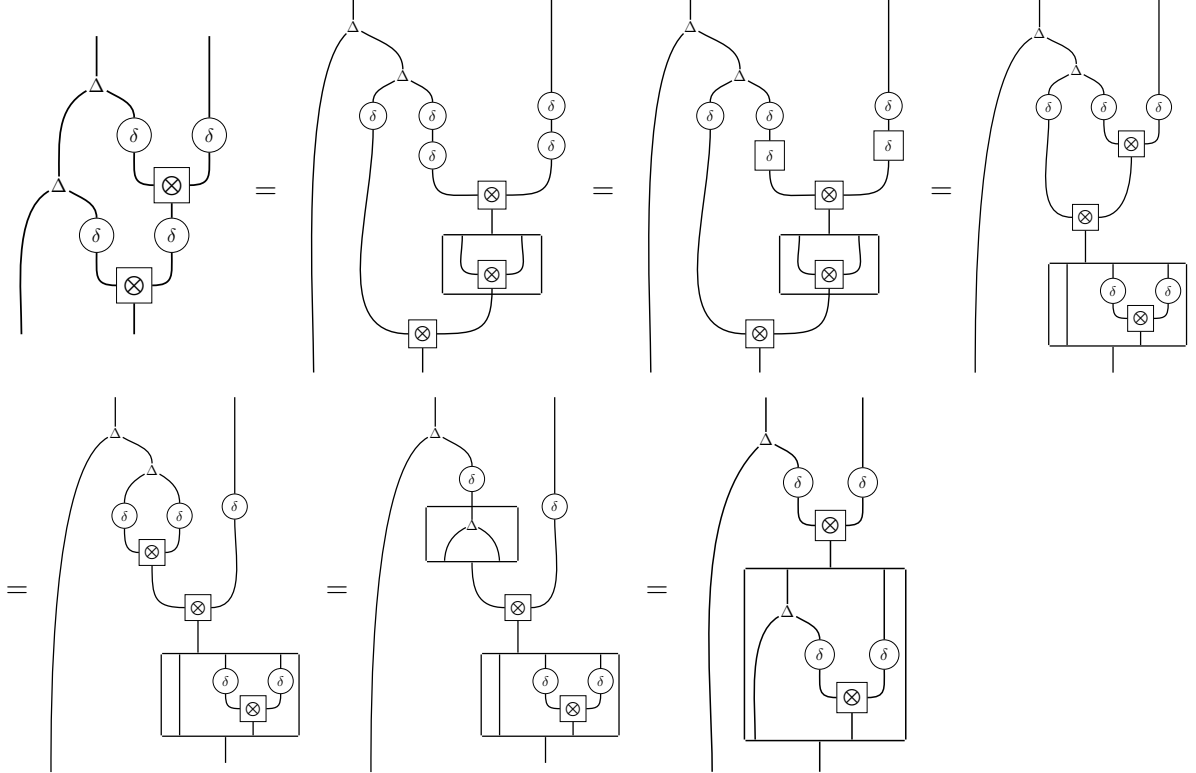
which expressed in the graphical calculus gives:



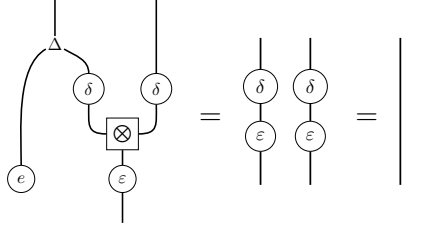
Lemma 6.1 $(!^B, \delta^B, \varepsilon^B)$ is a comonad.

PROOF: We must show the following three identities:

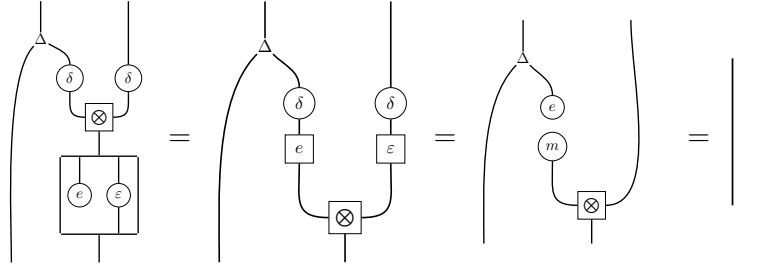
1. $\delta^B \delta^B = \delta^B !^B(\delta^B)$: Here we use that δ is a monoidal transformation, the naturality of m_\otimes , the co-associativity of Δ , the associativity of m_\otimes , the co-associativity of the comonad, and that Δ is a $!$ -coalgebra morphism:



2. $\delta^B \varepsilon^B = 1$: Here we use the counit law of the comultiplication, that ε is a monoidal transformation, and the triangle identities of the comonad:



3. $\delta^B !^B(\varepsilon^B) = 1$: Here we use the naturality of m_\otimes , that e is a monoidal transformation, the comonad triangle identities, the unit law of m_\otimes , and the counit law for the comultiplication:



□

The bialgebra structure of $!^B A$ is given by the standard tensor product of bialgebras, that is, the comultiplication Δ^B and multiplication ∇^B are defined respectively as:

$$\Delta^B := !B \otimes !A \xrightarrow{\Delta \otimes \Delta} !B \otimes !B \otimes !A \otimes !A \xrightarrow{1 \otimes \sigma \otimes 1} !B \otimes !A \otimes !B \otimes !A$$

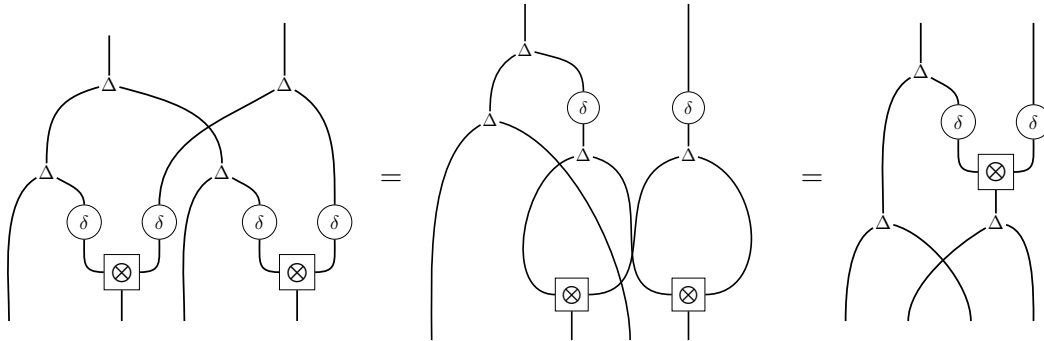
$$\nabla^B := !B \otimes !A \otimes !B \otimes !A \xrightarrow{1 \otimes \sigma \otimes 1} !B \otimes !B \otimes !A \otimes !A \xrightarrow{\nabla \otimes \nabla} !B \otimes !A$$

while the counit e^B and the unit u^B are:

$$e^B := !B \otimes !A \xrightarrow{e \otimes e} K \quad u^B := K \xrightarrow{u \otimes u} !B \otimes !A$$

Proposition 6.2 $(!^B, \delta^B, \varepsilon^B, \Delta^B, e^B, \nabla^B, u^B)$ is a bialgebra modality.

PROOF: We first show that δ^B preserves the comultiplication. For this we use that δ preserves the comultiplication, the coassociativity of the comultiplication, and that Δ is a monoidal transformation:



Next we show the compatibility relation between ε^B and the multiplication. Here we use the compatibility between the counit and the multiplication, and the compatibility between ε and the multiplication:

□

In general, this bialgebra modality is not additive (unless $!(B) \cong K$). In particular, for the zero map we have that:

$$!^B(0) = 1 \otimes !(0) = (1 \otimes e)(1 \otimes u) \neq (e \otimes e)(u \otimes u) = e^B u^B$$

Every codereliction η on the additive bialgebra modalities induces a codereliction $\eta^B : A \rightarrow !^B(A)$ defined as follows:

$$A \xrightarrow{u \otimes \eta} !B \otimes !A \quad \eta^B := \begin{array}{c} | \\ \circlearrowleft u \\ | \end{array} \begin{array}{c} | \\ \circlearrowleft \eta \\ | \end{array}$$

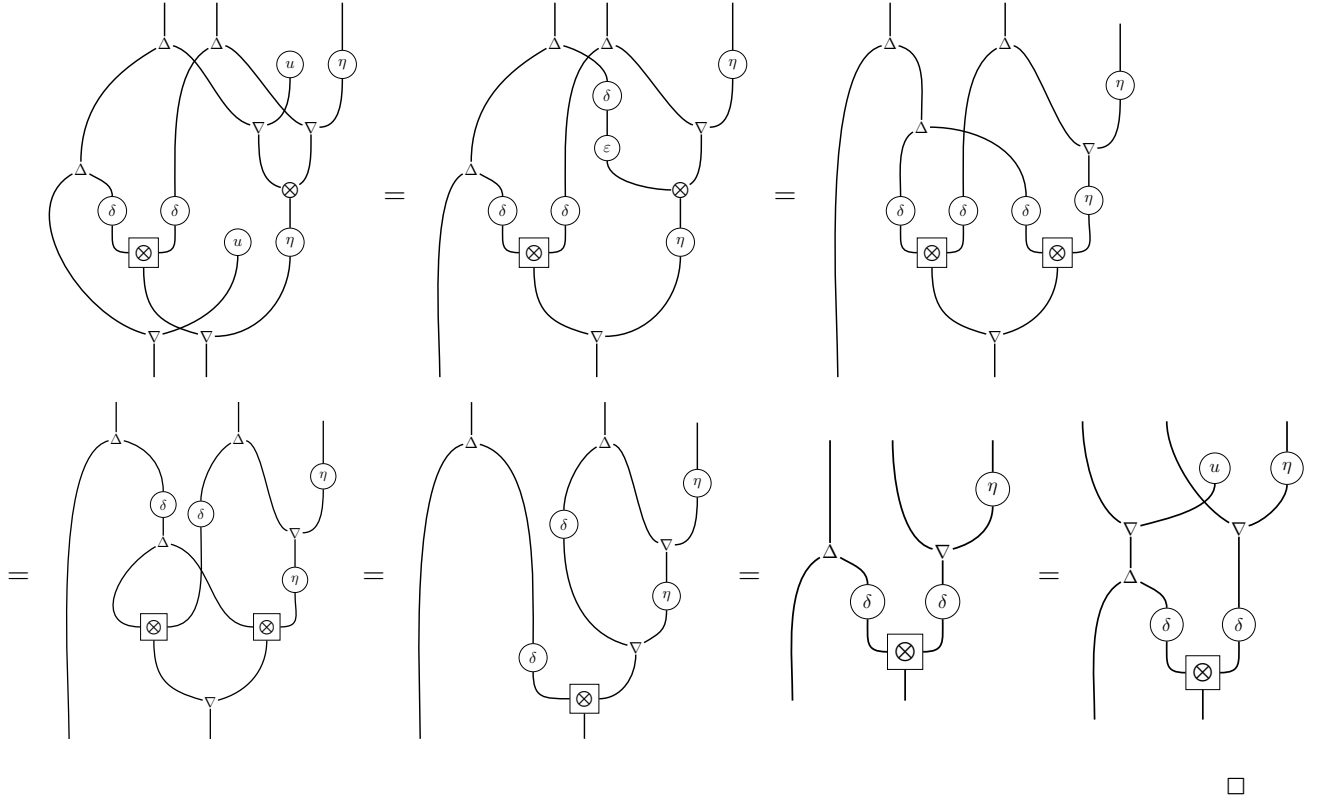
Proposition 6.3 η^B is a codereliction on $(!^B, \delta^B, \varepsilon^B, \Delta^B, e^B, \nabla^B, u^B)$.

PROOF: We must show [dC.2], [dC.3], and [dC.4]:

[dC.2]: Here we use the compatibility between the unit and the counit, and that η satisfies the product rule [dC.2]:

[dC.3]: Here we use the compatibility between the unit and the comultiplication, and that η satisfies the linear rule [dC.3]:

[dC.4]: Here we use coassociativity of the comultiplication, the third diagram of Proposition 5.6, that δ preserves the comultiplication, and that η satisfies the chain rule [dC.4] and the monoidal rule [dC.m]:



The induced deriving transformation d^B :

$$!B \otimes !A \otimes A \xrightarrow{1 \otimes d} !B \otimes !A$$

intuitively, this should be thought of as the partial derivative with respect to A .

7 Seely Isomorphisms and the Biproduct Completion

In a symmetric monoidal category with finite products \times and terminal object \mathbb{T} , a coalgebra modality has **Seely isomorphisms** [17, 1, 5] if the natural transformations χ and $\chi_{\mathbb{T}}$ defined respectively as:

$$!(A \times B) \xrightarrow{\Delta} !(A \times B) \otimes !(A \times B) \xrightarrow{!(\pi_0) \otimes !(\pi_1)} !A \otimes !B \quad !(\mathbb{T}) \xrightarrow{e} K$$

are isomorphisms, so $!(A \times B) \cong !A \otimes !B$ and $!(\mathbb{T}) \cong K$.

Definition 7.1 A **monoidal storage category** [5] is a symmetric monoidal category with finite products and a coalgebra modality which has Seely isomorphisms.

Monoidal storage categories were called **new Seely categories** in [1, 16]. As explained in [5], every coalgebra modality which satisfies the Seely isomorphisms is a monoidal coalgebra modality, where m_\otimes is

$$!A \otimes !B \xrightarrow{\chi^{-1}} !(A \times B) \xrightarrow{\delta} !!(A \times B) \xrightarrow{!(\chi)} !(!A \otimes !B) \xrightarrow{!(\varepsilon \otimes \varepsilon)} !(A \otimes B)$$

and m_K is defined as

$$K \xrightarrow{\chi_{\mathbb{T}}^{-1}} !(\mathbb{T}) \xrightarrow{\delta} !!(\mathbb{T}) \xrightarrow{!(\chi_{\mathbb{T}})} !(K)$$

Conversly, in the presence of finite products, every monoidal coalgebra modality satisfies the Seely isomorphisms [1] where the inverse of χ is

$$!A \otimes !B \xrightarrow{\delta \otimes \delta} !!A \otimes !!B \xrightarrow{m_\otimes} !(!A \otimes !B) \xrightarrow{!(\varepsilon \otimes e, e \otimes \varepsilon)} !(A \times B)$$

while the inverse of $\chi_{\mathbb{T}}$ is

$$K \xrightarrow{m_K} !(K) \xrightarrow{!(\mathfrak{t})} !(\mathbb{T})$$

where $\mathfrak{t} : K \rightarrow \mathbb{T}$ is the unique map to the terminal object. Therefore we obtain the following theorem (Theorem 3.1.6 [5]):

Theorem 7.2 *Every monoidal storage category is a linear category and conversely, every linear category with finite products is a monoidal storage category.*

This together with Proposition 7.7 below provides, as discussed in the introduction, a rather indirect verification of Theorem 5.5. We now turn our attention to monoidal storage categories with an additive structure:

Definition 7.3 An **additive monoidal storage category** is a monoidal storage category which is also an additive symmetric monoidal category.

Notice, this implies that additive monoidal storage categories have finite biproducts \times and a zero object 0 . As noted in [3], the coalgebra modality of an additive monoidal storage category is an additive bialgebra modality where the multiplication and unit are defined respectively as:

$$!A \otimes !A \xrightarrow{\chi^{-1}} !(A \times A) \xrightarrow{!(\nabla \times)} !(A) \quad K \xrightarrow{\chi_0^{-1}} !0 \xrightarrow{!(0)} !A$$

Conversly, every additive bialgebra modality satisfies the Seely isomorphisms where χ^{-1} and χ_0^{-1} are defined respectively as:

$$!A \otimes !B \xrightarrow{!(\iota_0) \otimes !(\iota_1)} !(A \times B) \otimes !(A \times B) \xrightarrow{\nabla} !(A \times B) \quad K \xrightarrow{u} !0$$

It is easy to check that this indeed gives the Seely isomorphisms, therefore we have:

Theorem 7.4 *The following are equivalent:*

- (i) *An additive monoidal storage category;*

- (ii) *An additive linear category with finite biproducts;*
- (iii) *An additive symmetric monoidal category with finite biproducts and an additive bialgebra modality.*

Every additive bialgebra modality induces an additive monoidal storage category via the biproduct completion. We first recall the biproduct completion for an additive category [15]. Let \mathbb{X} be an additive category. Define the biproduct completion of \mathbb{X} , $\mathbf{B}[\mathbb{X}]$, as the category whose objects are list of objects of \mathbb{X} : (A_1, \dots, A_n) , including the empty list $()$, and whose maps are matrices of maps of \mathbb{X} , including the empty matrix:

$$(A_1, \dots, A_n) \xrightarrow{[f_{i,j}]} (B_1, \dots, B_m)$$

where $f_{i,j} : A_i \rightarrow B_j$. The composition in $\mathbf{B}[\mathbb{X}]$ is the standard matrix multiplication:

$$[f_{i,j}][g_{l,k}] = [\sum f_{i,k}g_{k,j}]$$

while the identity is the standard identity matrix:

$$(A_1, \dots, A_n) \xrightarrow{[\delta_{i,j}]} (A_1, \dots, A_n)$$

where $\delta_{i,j} = 0$ if $i \neq j$, and $\delta_{i,i} = 1$. It is easy to see that $\mathbf{B}[\mathbb{X}]$ does in fact have biproducts:

Lemma 7.5 *$\mathbf{B}[\mathbb{X}]$ is a well-defined category with biproducts.*

If \mathbb{X} is an additive symmetric monoidal category, then so is $\mathbf{B}[\mathbb{X}]$. The monoidal unit is the same as in \mathbb{X} , the tensor product of objects is:

$$(A_1, \dots, A_n) \otimes (B_1, \dots, B_m) = (A_1 \otimes B_1, \dots, A_1 \otimes B_m, \dots, A_n \otimes B_n)$$

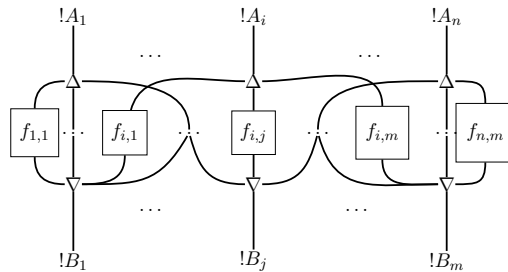
while the tensor product of maps is the standard Kronecker product of matrices.

Lemma 7.6 *If \mathbb{X} is an additive symmetric monoidal category, then so is $\mathbf{B}[\mathbb{X}]$.*

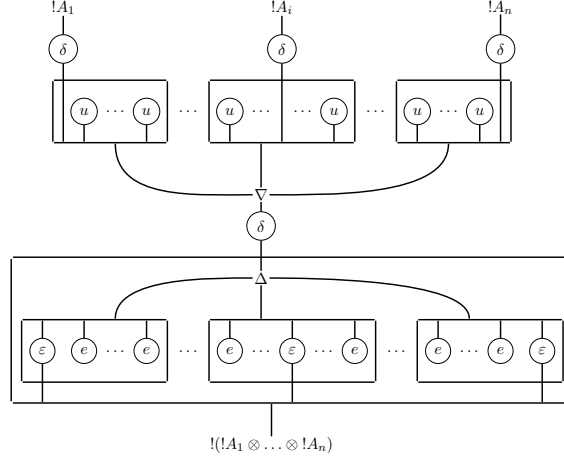
If \mathbb{X} admits an additive bialgebra modality, then $\mathbf{B}[\mathbb{X}]$ is an additive monoidal storage category where the Seelye isomorphisms are strict, i.e., equalities, and in particular it is an additive linear category. We give the additive bialgebra modality of $\mathbf{B}[\mathbb{X}]$, and leave it to the reader to check that it is in fact an additive bialgebra modality. The functor $! : \mathbf{B}[\mathbb{X}] \rightarrow \mathbf{B}[\mathbb{X}]$ is defined on objects as:

$$!(A_1, \dots, A_n) = !A_1 \otimes \dots \otimes !A_n$$

and on a map $[f_{i,j}] : (A_1, \dots, A_n) \rightarrow (B_1, \dots, B_m)$, $![f_{i,j}]$ is represented in string diagrams below:



The bialgebra structure is given by the standard tensor product of bialgebras (as discussed in the previous section), the comonad comultiplication $!(A_1, \dots, A_n) \rightarrow !(A_1, \dots, A_n)$ is represented in string diagrams as:



while the comonad counit is the following matrix:

$$\left[\varepsilon_{A_1} \otimes e \otimes \dots \otimes e, \dots, e \otimes \dots \otimes \varepsilon_{A_i} \otimes \dots \otimes e, \dots, e \otimes e \otimes \dots \otimes \varepsilon_{A_n} \right] : !A_1 \otimes \dots \otimes !A_n \longrightarrow (A_1, \dots, A_n)$$

Proposition 7.7 *If \mathbb{X} has an additive bialgebra modality, then $\mathbf{B}[\mathbb{X}]$ is an additive monoidal storage category.*

If the additive bialgebra modality of \mathbb{X} comes equipped with a codereliction η then the additive bialgebra modality of $\mathbf{B}[\mathbb{X}]$ comes equipped with a codereliction defined as follows:

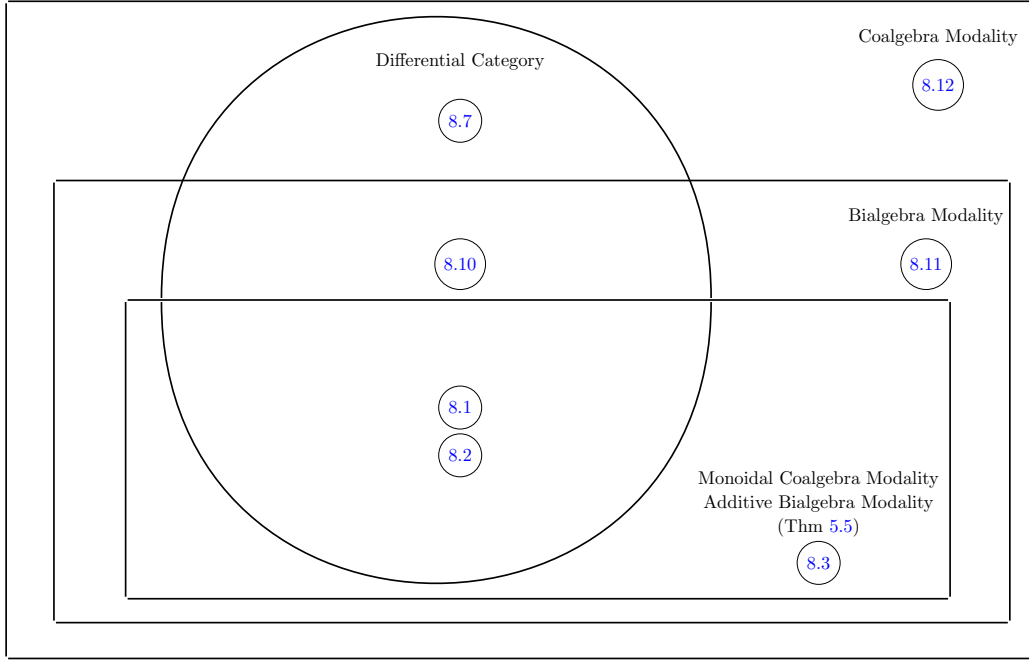
$$\left[\begin{array}{c} \eta_{A_1} \otimes u \otimes \dots \otimes u \\ \dots \\ u \otimes \dots \otimes \eta_{A_i} \otimes \dots \otimes u \\ \dots \\ u \otimes u \otimes \dots \otimes \eta_{A_n} \end{array} \right] : (A_1, \dots, A_n) \longrightarrow !A_1 \otimes \dots \otimes !A_n$$

Proposition 7.8 *If \mathbb{X} is a differential category with an additive bialgebra modality, then $\mathbf{B}[\mathbb{X}]$ is a differential category which is an additive monoidal storage category.*

8 Separating Examples

Here we present an overview of separating examples between the various structures defined throughout this paper. To help understand what the examples illustrate we present a Venn diagram which

classifies the examples we shall give below.



A well-known example of a differential category (whose coalgebra modality also happens to be monoidal) comes from the free symmetric algebra construction which actually gives a co-differential category. We briefly recall this example (see [3] for more details):

Example 8.1 Let R be a commutative ring. For an R -module M , define $\text{Sym}(M)$, called the free symmetric algebra over M , as following (see Section 8, Chapter XVI in [14] for more details):

$$\text{Sym}(M) = \bigoplus_{n=0}^{\infty} \text{Sym}^n(M) = R \oplus M \oplus \text{Sym}^2(M) \oplus \dots$$

where $\text{Sym}^n(M)$ is simply the quotient of $M^{\otimes n}$ by the tensor symmetry equalities:

$$a_1 \otimes \dots \otimes a_i \otimes \dots \otimes a_n = a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(i)} \otimes \dots \otimes a_{\sigma(n)}$$

$\text{Sym}(M)$ is a commutative algebra where the multiplication $\nabla : \text{Sym}(M) \otimes \text{Sym}(M) \rightarrow \text{Sym}(M)$ is the concatenation of words $\nabla(v_1 \otimes \dots \otimes v_n, w_1 \otimes \dots \otimes w_m) = v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m$ which we then extend by linearity, and the unit $u : R \rightarrow \text{Sym}(M)$ is the injection map of R into $\text{Sym}(M)$. Furthermore, $\text{Sym}(M)$ is the free commutative algebra over M , that is, we obtain an adjunction:

$$\text{MOD}_R \begin{array}{c} \xrightarrow{\text{Sym}} \\ \xleftarrow{\perp} \\ \xleftarrow{U} \end{array} \text{CALG}_R$$

The unit $\eta : M \rightarrow \text{Sym}(M)$ is the injection map of M into $\text{Sym}(M)$ and for an algebra A , the counit $\epsilon : \text{Sym}(A) \rightarrow A$ is defined on pure tensors as $\epsilon(a_1 \otimes \dots \otimes a_n) = a_1 \dots a_n$, which we then extend by linearity. The induced monad (Sym, μ, η) is an algebra modality (the dual of a coalgebra modality) the multiplication of the monad is an algebra morphism (as it's a map in the category of algebras). Furthermore, this algebra modality satisfies the Seely isomorphism [14], that is:

$$\text{Sym}(M \times N) \cong \text{Sym}(M) \otimes \text{Sym}(N) \quad \text{Sym}(0) \cong R$$

which implies that the free symmetric algebra adjunction induces an additive linear category (or equivalently an additive Seely category) structure on MOD_R . Furthermore, it comes equipped with a deriving transformation, making MOD_R into a co-differential category. The deriving transformation $d : \text{Sym}(M) \rightarrow \text{Sym}(M) \otimes M$ on pure tensors is defined as follows:

$$d(a_1 \otimes \dots \otimes a_n) = \sum_{i=1}^n (a_1 \otimes \dots \otimes a_{i-1} \otimes a_{i+1} \otimes \dots \otimes a_n) \otimes a_i$$

which we then extend by linearity (if this map looks backwards, recall that MOD_R is a co-differential category).

It is important to note that this differential category structure on MOD_R can be generalized to the category of modules over any ring R . In fact, this example can be generalized further. Indeed, the free symmetric algebra construction on appropriate additive symmetric monoidal categories induces a differential category structure, such as on the category of sets and relations (see [3] for more details).

Example 8.2 Convenient vector spaces provides an another example (given by R. Blute, T. Ehrhard and C. Tasson) of a co-differential category with a monoidal algebra modality [6].

Interestingly, the free differential algebra construction – which one might suppose would give rise rather naturally to a modality with a differential – gives an example of an additive bialgebra modality which *does not* admit a deriving transformation:

Example 8.3 Let R be a commutative ring. A (commutative) **differential algebra** (of weight 0) over R (see [12]) is a pair (A, D) consisting of a commutative R -algebra A and a linear map $D : A \rightarrow A$ such that D satisfies the Leibniz rule:

$$D(ab) = D(a)b + aD(b) \quad \forall a, b \in A$$

where the multiplication of the R -algebra A has been written as juxtaposition. A **map of differential algebras** $f : (A, D) \rightarrow (C, D')$ is an R -algebra morphism $f : A \rightarrow C$ such that $fD' = Df$.

The forgetful functor from the category of differential algebras over R , CDA_R to modules over R has a left adjoint:

$$\text{MOD}_R \begin{array}{c} \xrightarrow{\text{DIFF}} \\ \perp \\ \xleftarrow{U} \end{array} \text{CDA}_R$$

which induces an algebra modality, Diff , on the category of modules: we shall now give an explicit description of this modality and, furthermore, show that it is an additive bialgebra modality which *does not* admit a deriving transformation.

Let M be an R -module, then the free commutative differential R -algebra $\text{Diff}(M)$ is defined as follows:

$$\text{Diff}(M) = \text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right)$$

where the unit and multiplication are just that of the symmetric algebra, $u : R \rightarrow \text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right)$ and $\nabla : \text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right) \otimes \text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right) \rightarrow \text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right)$. The differential is obtained by “shifting” the infinite sum (on which the symmetric algebras is built) up one

$$\phi := \langle \iota_{n+1} \rangle_{n=0}^{\infty} : \bigoplus_{n=0}^{\infty} M \rightarrow \bigoplus_{n=0}^{\infty} M$$

and defining the map $D : \text{Diff}(M) \rightarrow \text{Diff}(M)$ as:

$$\text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right) \xrightarrow{d} \text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right) \otimes \bigoplus_{n=0}^{\infty} M \xrightarrow{1 \otimes \phi} \text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right) \otimes \bigoplus_{n=0}^{\infty} M \xrightarrow{d^\circ} \text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right)$$

where d is the deriving transformation of the free symmetric algebra modality and $d^\circ := (1 \otimes \eta)\nabla$ (where η is the unit of the free symmetric algebra monad). By the associativity and unit laws of the multiplication, we note the following identities d° satisfies:

$$\begin{aligned} (u \otimes 1)d^\circ &= \eta \\ (\nabla \otimes 1)d^\circ &= (1 \otimes 1 \otimes d^\circ)\nabla \end{aligned}$$

We then have:

$$\begin{aligned} \nabla D &= \nabla d(1 \otimes \phi)d^\circ \\ &= (1 \otimes d)(\nabla \otimes 1)(1 \otimes \phi)d^\circ + (d \otimes 1)(1 \otimes \sigma)(\nabla \otimes 1)(1 \otimes \phi)d^\circ \\ &= (1 \otimes d)(1 \otimes 1 \otimes \phi)(\nabla \otimes 1)d^\circ + (d \otimes 1)(1 \otimes \phi \otimes 1)(1 \otimes \sigma)(\nabla \otimes 1)d^\circ \\ &= (1 \otimes d)(1 \otimes 1 \otimes \phi)(1 \otimes 1 \otimes d^\circ)\nabla + (d \otimes 1)(1 \otimes \phi \otimes 1)(d^\circ \otimes 1)\nabla \\ &= (1 \otimes D)\nabla + (D \otimes 1)\nabla \end{aligned}$$

showing that it is a differential on the algebra.

To show it is a monad set the unit to be the natural transformation $\alpha : M \rightarrow \text{Diff}(M)$ defined as

$$\alpha := M \xrightarrow{\iota_0} \bigoplus_{n=0}^{\infty} M \xrightarrow{\eta} \text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right)$$

where η is the unit of the free symmetric algebra monad, and define the multiplication as the natural transformation $\nu : \text{Diff}(\text{Diff}(M)) \rightarrow \text{Diff}(M)$ to be

$$\text{Sym}\left(\bigoplus_{n=0}^{\infty} \left(\text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right)\right)\right) \xrightarrow{\text{Sym}(\psi)} \text{Sym}\left(\text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right)\right) \xrightarrow{\mu} \text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right)$$

where μ is the multiplication of the free symmetric algebra monad and $\psi := \langle D^n \rangle_{n=0}^{\infty} : \bigoplus_{n=0}^{\infty} A \rightarrow A$ is the map with $\iota_k \psi = D^k$ where A is a differential algebra: notice that ψ is natural for differential algebra maps. We then have:

Lemma 8.4 $(\text{Diff}, \nu, \alpha)$ is a monad on MOD_R .

PROOF: We must verify the three monad identities:

$\text{Diff}(\nu)\nu = \nu\nu$: Here we use that ψ is natural with respect to differential algebra morphisms, that ν is a differential algebra morphism, the monad associativity of μ , and the naturality of μ :

$$\begin{aligned} \text{Diff}(\nu)\nu &= \text{Sym}\left(\bigoplus_{n=0}^{\infty} \nu\right)\nu = \text{Sym}\left(\bigoplus_{n=0}^{\infty} \nu\right)\text{Sym}(\psi)\mu = \text{Sym}\left(\left(\bigoplus_{n=0}^{\infty} \nu\right)\psi\right)\mu \\ &= \text{Sym}(\psi\nu)\mu = \text{Sym}(\psi)\text{Sym}(\nu)\mu = \text{Sym}(\psi)\text{Sym}(\text{Sym}(\psi)\mu)\mu \\ &= \text{Sym}(\psi)\text{Sym}(\text{Sym}(\psi))\mu\mu = \text{Sym}(\psi)\mu\text{Sym}(\psi)\mu = \nu\nu \end{aligned}$$

$\alpha\nu = 1$: Here we use the naturality of η , the definition of ψ , and the monad triangle identity of μ and η :

$$\alpha\nu = \iota_0\eta\text{Sym}(\psi)\mu = \iota_0\psi\eta\mu = 1.$$

$\text{Diff}(\alpha)\nu = 1$: Here we have:

$$\begin{aligned} \text{Diff}(\alpha)\nu &= \text{Sym}\left(\bigoplus_{n=0}^{\infty} \alpha\right)\nu = \text{Sym}\left(\bigoplus_{n=0}^{\infty} \alpha\right)\text{Sym}(\psi)\mu \\ &= \text{Sym}\left(\left(\bigoplus_{n=0}^{\infty} \alpha\right)\psi\right)\mu = \text{Sym}(\eta)\mu = 1 \end{aligned}$$

where the equality $\left(\bigoplus_{n=0}^{\infty} \alpha\right)\psi = \eta$ is used in the penultimate step which we must now establish.

Notice, using the linear rule [d.3], the following identity holds:

$$\eta\mathbf{D} = \eta\mathbf{d}(1 \otimes \phi)\mathbf{d}^\circ = (\mathbf{u} \otimes 1)(1 \otimes \phi)\mathbf{d}^\circ = \phi(\mathbf{u} \otimes 1)\mathbf{d}^\circ = \phi\eta$$

This allows us to observe that

$$\iota_k\left(\bigoplus_{n=0}^{\infty} \alpha\right)\psi = \alpha\iota_k\psi = \alpha\mathbf{D}^k = \iota_0\eta\mathbf{D}^k = \iota_0\phi^k\eta = \iota_k\eta$$

so that $\left(\bigoplus_{n=0}^{\infty} \alpha\right)\psi = \eta$ as desired.

□

Proposition 8.5 $(\text{Diff}, \nu, \alpha, \nabla, \mathbf{u}, \Delta, \mathbf{e})$ is an additive bialgebra modality.

PROOF: First observe that ν is an algebra morphism as both $\text{Sym}(\psi)$ and μ are algebra morphisms.

To establish that the modality is an additive bialgebra modality we exhibit the Seelye isomorphisms. Since Sym has Seelye isomorphisms it follows that:

$$\text{Diff}(0) = \text{Sym}\left(\bigoplus_{n=0}^{\infty} 0\right) = \text{Sym}(0) \cong R$$

$$\mathrm{Sym}\left(\bigoplus_{n=0}^{\infty} (M \times N)\right) \cong \mathrm{Sym}\left(\bigoplus_{n=0}^{\infty} M \times \bigoplus_{n=0}^{\infty} N\right) \cong \mathrm{Sym}\left(\bigoplus_{n=0}^{\infty} M\right) \otimes \mathrm{Sym}\left(\bigoplus_{n=0}^{\infty} N\right)$$

□

We now set about proving that this modality does not admit a deriving transformation. This is accomplished by proving that if there is a deriving transformation, then the ring R over which the modules are taken must be trivial: that is in R we must have $1 = 0$. More precisely we prove that $1_{M \otimes M} = -\sigma : M \otimes M \rightarrow M \otimes M$, however, by substituting $R + R$ for M this gives the matrix equality

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

but this then immediately gives.

$$\begin{aligned} 0 &= \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = 1 \end{aligned}$$

Theorem 8.6 *For any category of modules over a non-trivial commutative ring the free differential algebra modality, Diff , does not admit a deriving transformation.*

PROOF: Suppose then that there is a natural transformation $\mathbf{b} : \mathrm{Diff}(M) \rightarrow \mathrm{Diff}(M) \otimes M$ which is a deriving transformation. Then for each R -module M , the Leibniz rule implies that \mathbf{b} is a $\mathrm{Sym}\left(\bigoplus_{n=0}^{\infty} M\right)$ -derivation, and therefore by the universality of the deriving transformation of $\mathrm{Sym}\left(\bigoplus_{n=0}^{\infty} M\right)$

(see [2, 7] for more details), there exists a unique $\mathrm{Sym}\left(\bigoplus_{n=0}^{\infty} M\right)$ -module homomorphism f^\sharp making

$$\begin{array}{ccc} \mathrm{Sym}\left(\bigoplus_{n=0}^{\infty} M\right) & \xrightarrow{\mathbf{d}} & \mathrm{Sym}\left(\bigoplus_{n=0}^{\infty} M\right) \otimes \bigoplus_{n=0}^{\infty} M \\ & \searrow \mathbf{b} & \downarrow f^\sharp \\ & & \mathrm{Sym}\left(\bigoplus_{n=0}^{\infty} M\right) \otimes M \end{array}$$

commute. However, as f^\sharp is a morphism between free modules it is determined by a map

$$f : \bigoplus_{n=0}^{\infty} M \rightarrow \mathrm{Sym}\left(\bigoplus_{n=0}^{\infty} M\right) \otimes M \quad \text{where} \quad f^\sharp = (1 \otimes f)(\nabla \otimes 1).$$

Now both \mathbf{d} and \mathbf{b} are determined by derelictions, respectively $\widehat{\mathbf{d}} := \mathbf{d}(e \otimes 1)$ and $\widehat{\mathbf{b}} = \mathbf{b}(e \otimes 1)$, as they are on additive algebra modalities. Thus, setting $f = f(e \otimes 1)$ we have:

$$\widehat{\mathbf{b}} = \mathbf{b}(e \otimes 1) = \mathbf{d}f^\sharp(e \otimes 1) = \mathbf{d}(1 \otimes f)(\nabla \otimes 1)(e \otimes 1) = \mathbf{d}(e \otimes f(e \otimes 1)) = \widehat{\mathbf{d}}f(e \otimes 1) = \widehat{\mathbf{d}}f.$$

But then $f^\sharp = 1 \otimes f$ as $\mathbf{d}(1 \otimes f) = \Delta(1 \otimes (\widehat{\mathbf{d}} f)) = \Delta(1 \otimes \widehat{\mathbf{b}}) = \mathbf{b}$. Thus we have now shown that

$$\begin{array}{ccc} \text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right) & \xrightarrow{\mathbf{d}} & \text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right) \otimes \bigoplus_{n=0}^{\infty} M \\ & \searrow \mathbf{b} & \downarrow 1 \otimes f \\ & & \text{Sym}\left(\bigoplus_{n=0}^{\infty} M\right) \otimes M \end{array}$$

commutes. Furthermore, f is a natural transformation as if $g : M \rightarrow N$ then, using that \mathbf{d} is natural and \mathbf{b} is assumed to be a natural, both $\bigoplus_{n=0}^{\infty} gf$ and fg provide the unique mediating maps between \mathbf{d} and $\text{Diff}(g)\mathbf{b}$.

Consider the map $\pi_1 : \bigoplus_{n=0}^{\infty} A_n \rightarrow A_1$ defined as $\pi_1 = \langle \delta_{k,1} \rangle$, that is, the unique map which makes the following diagram commute for each injection map:

$$\begin{array}{ccc} A_k & \xrightarrow{\iota_k} & \bigoplus_{n=0}^{\infty} A_k \\ & \searrow \delta_{k,1} & \downarrow \pi_1 \\ & & A_1 \end{array}$$

where $\delta_{1,1} = 1$ and $\delta_{k,1} = 0$ for $k \neq 1$. Because \mathbf{b} satisfied the chain rule, $\nu\mathbf{b} = \mathbf{b}(\nu \otimes \mathbf{b})(\nabla \otimes 1)$, the following equality of natural transformations, obtained by sandwiching the chain rule between the same maps, must also hold:

$$(\alpha \otimes \alpha)\nabla\iota_1\eta\nu\mathbf{b}((\varepsilon\pi_1) \otimes 1) = (\alpha \otimes \alpha)\nabla\iota_1\eta\mathbf{b}(\nu \otimes \mathbf{b})(\nabla \otimes 1)(\varepsilon \otimes 1)(\pi_1 \otimes 1)$$

However, we will show that this forces $1 = -\sigma : M \otimes M \rightarrow M \otimes M$ which forces the module category to be trivial. Preliminary to this we note the following useful identities:

$$\begin{aligned} \alpha &= \iota_0\eta \\ \alpha D &= \iota_0\eta D = \iota_0\phi\eta = \iota_1\eta \\ \iota_0 f &= \iota_0\eta\widehat{\mathbf{d}}f = \alpha\widehat{\mathbf{b}} = 1 \\ (\eta \otimes \eta)\nabla\mathbf{d} &= (\eta \otimes \eta)(1 \otimes \mathbf{d})(\nabla \otimes 1) + (\eta \otimes \eta)(\mathbf{d} \otimes 1)(1 \otimes \sigma)(\nabla \otimes 1) \\ &= (\eta \otimes \mathbf{u} \otimes 1)(\nabla \otimes 1) + (\mathbf{u} \otimes 1 \otimes \eta)(1 \otimes \sigma)(\nabla \otimes 1) \\ &= \eta \otimes 1 + \sigma(\eta \otimes 1) \end{aligned}$$

Now explicitly calculating out the first map above we have:

$$(\alpha \otimes \alpha)\nabla\iota_1\eta\nu\mathbf{b}(\varepsilon\pi_1 \otimes 1)$$

$$\begin{aligned}
&= (\alpha \otimes \alpha) \nabla_{\iota_1 \eta} \text{Sym}(\psi) \mu d(\varepsilon \pi_1 \otimes f) \\
&= (\alpha \otimes \alpha) \nabla_{\iota_1 \psi \eta} \mu d(\varepsilon \pi_1 \otimes f) \\
&= (\alpha \otimes \alpha) \nabla D d(\varepsilon \pi_1 \otimes f) \\
&= ((\alpha \otimes \alpha D) \nabla + (\alpha D \otimes \alpha) \nabla) d(\varepsilon \pi_1 \otimes f) \\
&= ((\iota_0 \eta \otimes \iota_1 \eta) \nabla + (\iota_1 \eta \otimes \iota_0 \eta) \nabla) d(\varepsilon \pi_1 \otimes f) \\
&= (\iota_0 \otimes \iota_1 + \iota_1 \otimes \iota_0) (\eta \otimes \eta) \nabla d(\varepsilon \pi_1 \otimes f) \\
&= (\iota_0 \otimes \iota_1 + \iota_1 \otimes \iota_0) ((\eta \otimes 1) + \sigma(\eta \otimes 1)) (\varepsilon \pi_1 \otimes f) \\
&= (\iota_0 \otimes \iota_1 + \iota_1 \otimes \iota_0) ((\pi_1 \otimes f) + \sigma(\pi_1 \otimes f)) \\
&= (\iota_0 \otimes \iota_1) (\pi_1 \otimes f) + (\iota_1 \otimes \iota_0) (\pi_1 \otimes f) \\
&\quad + \sigma(\iota_1 \otimes \iota_0) (\pi_1 \otimes f) + \sigma(\iota_0 \otimes \iota_1) (\pi_1 \otimes f) \\
&= 0 + 1 \otimes 1 + \sigma + 0 \\
&= 1 \otimes 1 + \sigma
\end{aligned}$$

while for the second map:

$$\begin{aligned}
&(\alpha \otimes \alpha) \nabla_{\iota_1 \eta} \mathbf{b}(\nu \otimes \mathbf{b})(\nabla \otimes 1)(\varepsilon \pi_1 \otimes 1) \\
&= (\alpha \otimes \alpha) \nabla_{\iota_1 \eta} d(\nu \otimes \mathbf{b})(\nabla \otimes 1)(\varepsilon \pi_1 \otimes 1) \\
&= (\alpha \otimes \alpha) \nabla_{\iota_1} (u \otimes 1)(\nu \otimes \mathbf{b})(\nabla \otimes 1)(\varepsilon \pi_1 \otimes 1) \\
&= (\alpha \otimes \alpha) \nabla_{\iota_1} (u\nu \otimes \mathbf{b})(\nabla \otimes 1)(\varepsilon \pi_1 \otimes 1) \\
&= (\alpha \otimes \alpha) \nabla_{\iota_1} (u \otimes \mathbf{b})(\nabla \otimes 1)(\varepsilon \pi_1 \otimes 1) \\
&= (\alpha \otimes \alpha) \nabla_{\iota_1} \mathbf{b}((u \otimes 1) \nabla \otimes 1)(\varepsilon \pi_1 \otimes 1) \\
&= (\alpha \otimes \alpha) \nabla_{\iota_1} \mathbf{b}(\varepsilon \pi_1 \otimes 1) \\
&= (\alpha \otimes \alpha) \nabla_{\iota_1} \left(\bigoplus_{n=0}^{\infty} \mathbf{b}(\varepsilon \pi_1 \otimes 1) \right) \mathbf{f} \\
&= (\alpha \otimes \alpha) \nabla \mathbf{b}(\varepsilon \pi_1 \otimes 1) \iota_1 \mathbf{f} \\
&= (\iota_0 \otimes \iota_0) (\eta \otimes \eta) \nabla d(\varepsilon \pi_1 \otimes \mathbf{f}) \iota_1 \mathbf{f} \\
&= (\iota_0 \otimes \iota_0) (\eta \otimes 1 + \sigma(\eta \otimes 1)) (\varepsilon \pi_1 \otimes \mathbf{f}) \iota_1 \mathbf{f} \\
&= (\iota_0 \otimes \iota_0) (\pi_1 \otimes \mathbf{f}) \iota_1 \mathbf{f} + \sigma(\iota_0 \otimes \iota_0) (\pi_1 \otimes \mathbf{f}) \iota_1 \mathbf{f} \\
&= 0 + 0 = 0
\end{aligned}$$

Therefore, if \mathbf{b} satisfied the chain rule, this would imply that $1_{M \otimes M} = -\sigma$ for every R -module M which only happens when R is the trivial ring as discussed above. Therefore, Diff does not have a deriving transformation when the ring is non-trivial. \square

By way of contrast Rota-Baxter algebras – an algebraic abstraction of integration – whose algebra modality is not additive always gives a differential category:

Example 8.7 Let R be a commutative ring. A (commutative) **Rota-Baxter algebra (of weight 0)** [11] over R is a pair (A, \mathbf{P}) consisting of a commutative R -algebra A and an R -linear map $\mathbf{P} : A \rightarrow A$ such that \mathbf{P} satisfies the Rota-Baxter equation, that is, the following equality holds:

$$\mathbf{P}(a)\mathbf{P}(b) = \mathbf{P}(a\mathbf{P}(b)) + \mathbf{P}(\mathbf{P}(a)b) \quad \forall ab, \in A$$

The map P is called a **Rota-Baxter operator** (we refer the reader to [11] for more details on Rota-Baxter algebras). It turns out that there is a left adjoint to the forgetful functor between the category of Rota-Baxter algebras, $CRBA_R$, and the category of commutative algebras over R , $CALG_R$. We quickly review the construction of the free Rota-Baxter algebra over an algebra (for more details see chapter 3 of [11]): let M be an R -module and consider the **shuffle algebra**, $Sh(M)$, over M which is defined as follows: $Sh(M) = R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \dots = \bigoplus_{n \in \mathbb{N}} M^{\otimes n}$ where $M^{\otimes 0} = R$ and where the multiplication \sqcup , called the **shuffle product** [11], is defined inductively on pure tensors $w = a \otimes w'$ and $v = b \otimes v'$ as follows:

$$w \sqcup v = a \otimes (w' \sqcup v) + b \otimes (w \sqcup v')$$

which we then extend by linearity (notice that the unit for the shuffle product is 1_R). Denote the multiplication and unit maps of the shuffle algebra by $\blacktriangledown : Sh(M) \otimes Sh(M) \rightarrow Sh(M)$ and $v : R \rightarrow Sh(M)$ respectively. The free commutative Rota-Baxter over a commutative R -algebra A , $RB(A)$, is then the tensor product of shuffle algebra and A itself: $RB(A) = Sh(A) \otimes A$. The Rota-Baxter operator $P : RB(A) \rightarrow RB(A)$ is defined on pure tensors as follows:

$$w \otimes b = \begin{cases} (w \cdot b) \otimes 1_A & \text{if } w \in R \\ (w \otimes b) \otimes 1_A & \text{otherwise} \end{cases}$$

which we then extend by linearity. The induced functor $RB : CALG_R \rightarrow CRBA_R$ is the left adjoint to the forgetful functor $U : CRBA_R \rightarrow CALG_R$:

$$CALG_R \begin{array}{c} \xrightarrow{RB} \\ \perp \\ \xleftarrow{U} \end{array} CRBA_R$$

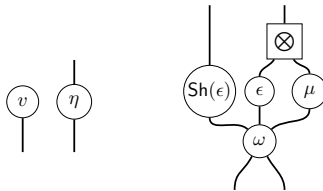
(for more details on this adjunction and monad see [19]) where for an algebra A , the unit of the adjunction is defined as $v \otimes 1 : A \rightarrow Sh(A) \otimes A$, while for a Rota-Baxter algebra (B, Q) , the counit $\omega : Sh(B) \otimes B \rightarrow B$ is defined on pure tensors as

$$\omega_B((b_1 \otimes \dots \otimes b_n) \otimes b) = Q(\dots Q(Q(b_1)b_2) \dots b_n)b$$

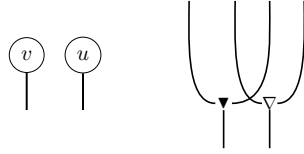
which we then extend by linearity. To obtain an algebra modality on MOD_R , we compose the free Rota-Baxter algebra adjunction and the free symmetric algebra adjunction:

$$MOD_R \begin{array}{c} \xrightarrow{Sym} \\ \perp \\ \xleftarrow{U} \end{array} CALG_R \begin{array}{c} \xrightarrow{RB} \\ \perp \\ \xleftarrow{U} \end{array} CRBA_R$$

The monad induced by the resulting adjunction between MOD_R and $CRBA_R$ is clearly an algebra modality by construction again. After some simplifications, the unit and multiplication of the monad are represented in string diagrams respectively as:



While the unit and multiplication of the algebra structure are represented in string diagrams respectively as:

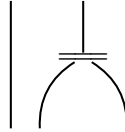


However this algebra modality, $\text{RB}(\text{Sym}(M))$, does not have the Seely isomorphism as Sh is not a strong monoidal functor (i.e. $\text{Sh}(A \otimes B) \not\cong \text{Sh}(A) \otimes \text{Sh}(B)$):

$$\begin{aligned}
 \text{RB}(\text{Sym}(M \times N)) &\cong \text{RB}(\text{Sym}(M) \otimes \text{Sym}(N)) \\
 &= \text{Sh}(\text{Sym}(M) \otimes \text{Sym}(N)) \otimes \text{Sym}(M) \otimes \text{Sym}(N) \\
 &\not\cong \text{Sh}(\text{Sym}(M)) \otimes \text{Sh}(\text{Sym}(N)) \otimes \text{Sym}(M) \otimes \text{Sym}(N) \\
 &\cong \text{Sh}(\text{Sym}(M)) \otimes \text{Sym}(M) \otimes \text{Sh}(\text{Sym}(N)) \otimes \text{Sym}(N) \\
 &= \text{RB}(\text{Sym}(M)) \otimes \text{RB}(\text{Sym}(N))
 \end{aligned}$$

Therefore, this algebra modality is not a bialgebra modality or a comonoidal algebra modality. We should mention that while it is true that the shuffle algebra is a bialgebra, its comultiplication is not cocommutative [11]. However, we may still use the free Rota-Baxter adjunction to obtain a differential category structure on MOD_R . The deriving transformation is defined as

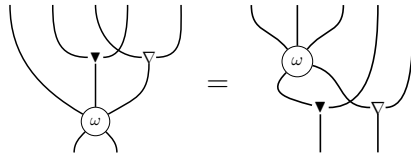
$$1 \otimes d : \text{Sh}(\text{Sym}(M)) \otimes \text{Sym}(M) \rightarrow \text{Sh}(\text{Sym}(M)) \otimes \text{Sym}(M) \otimes M$$



where recall that d is the deriving transformation of Sym (if this looks upside-down, recall that we are working in a co-differential category). It may seem trivial that this is a deriving transformation, but in fact proving the chain rule is quite non-trivial! We will need the following lemma to prove the chain rule:

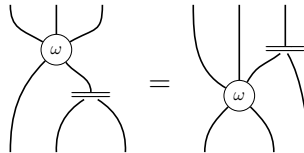
Lemma 8.8 *Let R be a commutative ring.*

- (i) *For every commutative R -algebra A , $(1 \otimes \diamond)\omega = (\omega \otimes 1 \otimes 1)\diamond$*



where \diamond is the multiplication on $\text{RB}(A)$.

- (ii) *For every R -module M , $\omega(1 \otimes d) = (1 \otimes 1 \otimes d)(\omega \otimes 1)$*



PROOF: *i*): Let A be a commutative R -algebra. It suffices to prove this equality on pure tensors. Consider the following pure tensor of $\text{Sh}(\text{Sh}(A) \otimes A) \otimes \text{Sh}(A) \otimes A$:

$$([A_1 \otimes a_1] \otimes \dots \otimes [A_n \otimes a_n]) \otimes A \otimes a \in \text{Sh}(\text{Sh}(A) \otimes A) \otimes \text{Sh}(A) \otimes A$$

where $A, A_1 \dots A_n \in \text{Sh}(A)$ and $a, a_1 \dots, a_n \in A$. By definition of ω_A , we obtain that:

$$\begin{aligned} & \omega_A(([A_1 \otimes a_1] \otimes \dots \otimes [A_n \otimes a_n]) \otimes A \otimes a) \\ &= \text{P}(\text{P}(\dots \text{P}(\text{P}(A_1 \otimes a_1) \diamond (A_2 \otimes a_2)) \dots \diamond (A_n \otimes a_n))) \diamond (A \otimes a) \\ &= (\text{P}(\dots \text{P}(\text{P}(A_1 \otimes a_1) \diamond (A_2 \otimes a_2)) \dots \diamond (A_n \otimes a_n))) \otimes 1 \diamond (A \otimes a) \\ &= (\text{P}(\dots \text{P}(\text{P}(A_1 \otimes a_1) \diamond (A_2 \otimes a_2)) \dots \diamond (A_n \otimes a_n))) \sqcup A \otimes a \end{aligned}$$

Notice that a is unaffected by ω . Now let $B \otimes b \in \text{Sh}(A) \otimes A$. Then we have the following equality by associativity of the shuffle product:

$$\begin{aligned} &= \omega_A(([A_1 \otimes a_1] \otimes \dots \otimes [A_n \otimes a_n]) \otimes ((A \otimes a) \diamond (B \otimes b))) \\ &= \omega_A(([A_1 \otimes a_1] \otimes \dots \otimes [A_n \otimes a_n]) \otimes ((A \sqcup B) \otimes (ab))) \\ &= (\text{P}(\dots \text{P}(\text{P}(A_1 \otimes a_1) \diamond (A_2 \otimes a_2)) \dots \diamond (A_n \otimes a_n))) \sqcup (A \sqcup B) \otimes (ab) \\ &= ((\text{P}(\dots \text{P}(\text{P}(A_1 \otimes a_1) \diamond (A_2 \otimes a_2)) \dots \diamond (A_n \otimes a_n))) \sqcup A) \sqcup B \otimes (ab) \\ &= ((\text{P}(\dots \text{P}(\text{P}(A_1 \otimes a_1) \diamond (A_2 \otimes a_2)) \dots \diamond (A_n \otimes a_n))) \sqcup A) \otimes a \diamond (B \otimes b) \\ &= \omega_A(([A_1 \otimes a_1] \otimes \dots \otimes [A_n \otimes a_n]) \otimes A \otimes a) \diamond (B \otimes b) \end{aligned}$$

ii): Let M be an R -module. It suffices to prove this equality on pure tensors. Consider the following pure tensor of $\text{Sh}(\text{Sh}(\text{Sym}(M)) \otimes \text{Sym}(M)) \otimes \text{Sh}(\text{Sym}(M)) \otimes \text{Sym}(M)$:

$$([W_1 \otimes w_1] \otimes \dots \otimes [W_n \otimes w_n]) \otimes W_0 \otimes w_0 \in \text{Sh}(\text{Sh}(\text{Sym}(M)) \otimes \text{Sym}(M)) \otimes \text{Sh}(\text{Sym}(M)) \otimes \text{Sym}(M)$$

where $W, W_1, \dots, W_n \in \text{Sh}(\text{Sym}(M))$ and $w, w_1 \dots, w_n \in \text{Sym}(M)$. By definition of $\omega_{\text{Sym}(M)}$, we obtain that:

$$\begin{aligned} & \omega_W(([W_1 \otimes w_1] \otimes \dots \otimes [W_n \otimes w_n]) \otimes W \otimes w) \\ &= \text{P}(\text{P}(\dots \text{P}(\text{P}(W_1 \otimes w_1) \diamond (W_2 \otimes w_2)) \dots \diamond (W_n \otimes w_n))) \diamond (W \otimes w) \\ &= (\text{P}(\dots \text{P}(\text{P}(W_1 \otimes w_1) \diamond (W_2 \otimes w_2)) \dots \diamond (W_n \otimes w_n))) \otimes 1 \diamond (W \otimes w) \\ &= (\text{P}(\dots \text{P}(\text{P}(W_1 \otimes w_1) \diamond (W_2 \otimes w_2)) \dots \diamond (W_n \otimes w_n))) \sqcup W \otimes w \end{aligned}$$

Notice that w is unaffected by ω . Let $w = m_1 \otimes \dots \otimes m_k$, $m_i \in M$, then we have that:

$$\mathbf{d}_M(w) = \sum_{i=1}^k (m_1 \otimes \dots \otimes m_{i-1} \otimes m_{i+1} \otimes \dots \otimes m_k) \otimes m_i$$

Then we obtain the following equality:

$$\begin{aligned} & (1 \otimes \mathbf{d}_M)(\omega_{\text{Sym}(M)}(([W_1 \otimes w_1] \otimes \dots \otimes [W_n \otimes w_n]) \otimes W \otimes w)) \\ &= (1 \otimes \mathbf{d}_M)((\text{P}(\dots \text{P}(\text{P}(W_1 \otimes w_1) \diamond (W_2 \otimes w_2)) \dots \diamond (W_n \otimes w_n))) \sqcup W) \otimes w) \\ &= (\text{P}(\dots \text{P}(\text{P}(W_1 \otimes w_1) \diamond (W_2 \otimes w_2)) \dots \diamond (W_n \otimes w_n))) \sqcup W \otimes \mathbf{d}_M(w) \end{aligned}$$

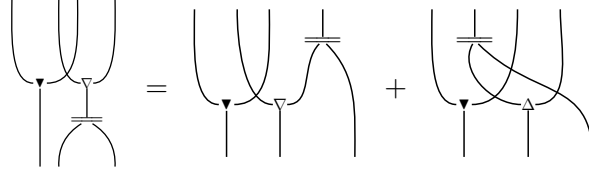
$$\begin{aligned}
&= (\mathbb{P}(\dots \mathbb{P}(\mathbb{P}(W_1 \otimes w_1) \diamond (W_2 \otimes w_2)) \dots \diamond (W_n \otimes w_n)) \sqcup W) \\
&\quad \otimes \left(\sum_{i=1}^k (m_1 \otimes \dots \otimes m_{i-1} \otimes m_{i+1} \otimes \dots \otimes m_k) \otimes m_i \right) \\
&= \sum_{i=1}^k (\mathbb{P}(\dots \mathbb{P}(\mathbb{P}(W_1 \otimes w_1) \diamond (W_2 \otimes w_2)) \dots \diamond (W_n \otimes w_n)) \sqcup W) \\
&\quad \otimes (m_1 \otimes \dots \otimes m_{i-1} \otimes m_{i+1} \otimes \dots \otimes m_k) \otimes m_i \\
&= \sum_{i=1}^k (\omega_{\text{Sym}(M)} \otimes 1) (([W_1 \otimes w_1] \otimes \dots \otimes [W_n \otimes w_n]) \otimes W) \\
&\quad \otimes (m_1 \otimes \dots \otimes m_{i-1} \otimes m_{i+1} \otimes \dots \otimes m_k) \otimes m_i \\
&= (\omega_{\text{Sym}(M)} \otimes 1) (([W_1 \otimes w_1] \otimes \dots \otimes [W_n \otimes w_n]) \otimes W) \\
&\quad \otimes \left(\sum_{i=1}^k (m_1 \otimes \dots \otimes m_{i-1} \otimes m_{i+1} \otimes \dots \otimes m_k) \otimes m_i \right) \\
&= (\omega_{\text{Sym}(M)} \otimes 1) (([W_1 \otimes w_1] \otimes \dots \otimes [W_n \otimes w_n]) \otimes W \otimes \mathbf{d}_M(w))
\end{aligned}$$

□

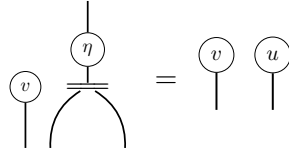
Proposition 8.9 *For the free Rota-Baxter monad, $1 \otimes \mathbf{d}$ is a deriving transformation.*

PROOF: By construction $1 \otimes \mathbf{d}$ is a natural transformation. We need to show [d.2] to [d.5] (recall that [d.1] is redundant). The linear rule [d.3], the Leibniz rule [d.2] and the interchange rule [d.5] are straightforward but the chain rule [d.4] requires some work:

[d.2]: Here we use the Leibniz rule [d.2]:

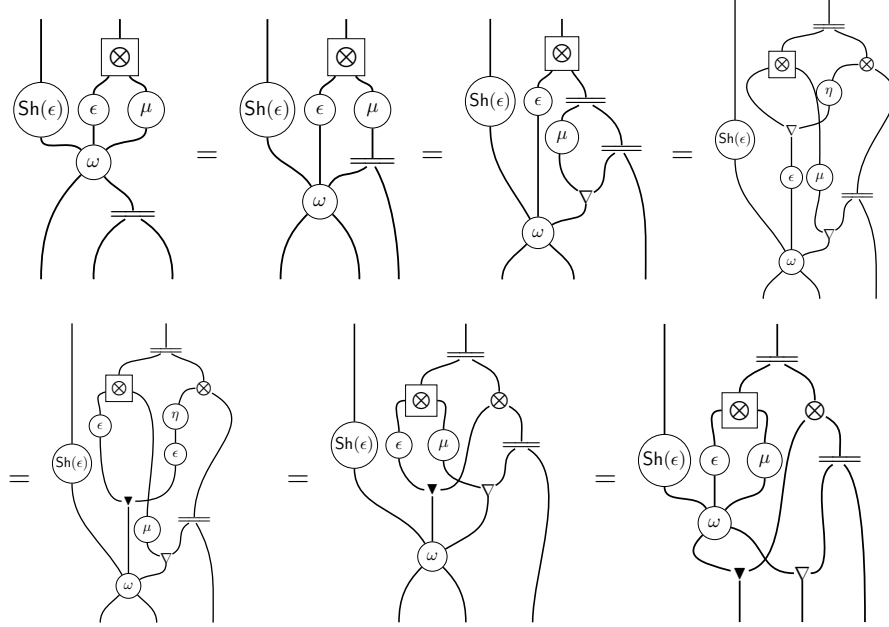


[d.3]: Here we use the linear rule [d.3]:

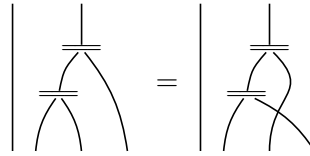


[d.4]: Here we use the counit law, Lemma 8.8, the chain rule [d.4], the monoidal rule [d.m], that

ϵ is an algebra morphism and the triangle identities of the free symmetric algebra adjunction:



[d.5]: Here we use the interchange rule [d.5]:



□

To obtain examples of non-additive bialgebra modalities, we need simply apply the construction for Section 6 to our examples of additive bialgebra modalities.

Example 8.10 The free symmetric algebra modality induces a non-additive bialgebra modality on MOD_R which has a deriving transformation.

Example 8.11 The free differential algebra modality induces a non-additive bialgebra modality on MOD_R which does not have a deriving transformation (since if it did, then DIFF would have one).

Finally to obtain a coalgebra modality which is not a bialgebra modality and does not have a deriving transformation, we look towards differential Rota-Baxter algebras.

Example 8.12 Let R be a commutative ring. A (commutative) **differential Rota-Baxter algebra (of weight 0)** [12] over R is a triple (A, D, P) consisting of a differential algebra (A, D) and a Rota-Baxter algebra (A, P) such that $PD = 1_A$. It turns out that the free Rota-Baxter algebra over a differential algebra is also its free differential Rota-Baxter algebra, and therefore inducing

the following adjunction between the category of differential algebras, CDA_R , and the category of differential Rota-Baxter algebras, $CDRBA_R$:

$$CDA_R \begin{array}{c} \xrightarrow{\text{RB}} \\ \perp \\ \xleftarrow{U} \end{array} CDRBA_R$$

The full construction can be found in [12]. Once again, to obtain an algebra modality we compose this adjunction with the free differential algebra adjunction:

$$\text{MOD}_R \begin{array}{c} \xrightarrow{\text{DIFF}} \\ \perp \\ \xleftarrow{U} \end{array} CDA_R \begin{array}{c} \xrightarrow{\text{RB}} \\ \perp \\ \xleftarrow{U} \end{array} CDRBA_R$$

This algebra modality is not comonoidal for the same reasons as the free Rota-Baxter algebra, and is not a differential category failing the chain rule like the free differential algebra.

9 Conclusion

There is a tendency to assume that the only important coalgebra modalities are those which arise through linear logic: that is those which are monoidal coalgebra modalities (or equivalently additive coalgebra modalities – in the sense of this paper). Certainly, it is true that it is the monoidal coalgebra modalities that have filled the lion’s share of the literature. Of course, this does not mean that their relatives, the mere coalgebra modalities, are not worthy of scientific attention – indeed, we believe that they have been wrongly overlooked. Thus, one objective of this paper is to reemphasize the important of these mere coalgebra modalities and to provide a stock of examples.

Of course it would have been very much simpler if all these observations had been made in [3] – the original paper. Unfortunately, they were not. Marcelo Fiore’s paper [10] made us realize that there was much more to say. In particular, the relation between deriving transformations and coderelictions had not been fully developed. Here we have revisited this relation and, in addition, filled in some of the gaps left in the original paper. That there will be yet more to say we are sure!

Significantly, it is not that the notion of differentiation varies but rather, as the setting varies, significantly different presentations of the differential become possible. Philosophically this is certainly how things should be ... and apparently it is how they are!

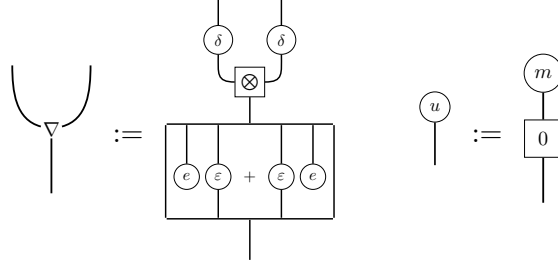
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A Monoidal Coalgebra Modalities are Additive Bialgebra Modalities

In this appendix we provide the complete proof that the monoidal coalgebra modality of an additive linear category is an additive bialgebra modality. Recall the definitions of ∇ and u in their string diagram representations:



In certain parts of the following proofs, for space and simplification, define $\phi = e \otimes \varepsilon + \varepsilon \otimes e$.

We first check that ∇ and u are indeed natural.

Lemma A.1 ∇ and u are natural transformations.

PROOF: By construction, ∇ is a natural transformation since it is the composition of natural transformations. The unit u on the other hand is not automatically a natural transformation by construction, since m_K is a map and not a natural transformation. Let $f : A \rightarrow B$, then we have the following equality:

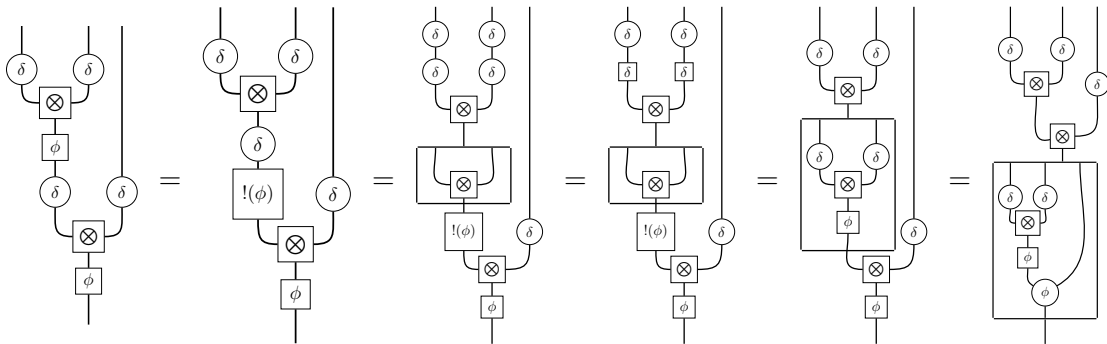
$$u_A!(f) = m_K!(0)!(f) = m_K!(0f) = m_K!(0) = u_B$$

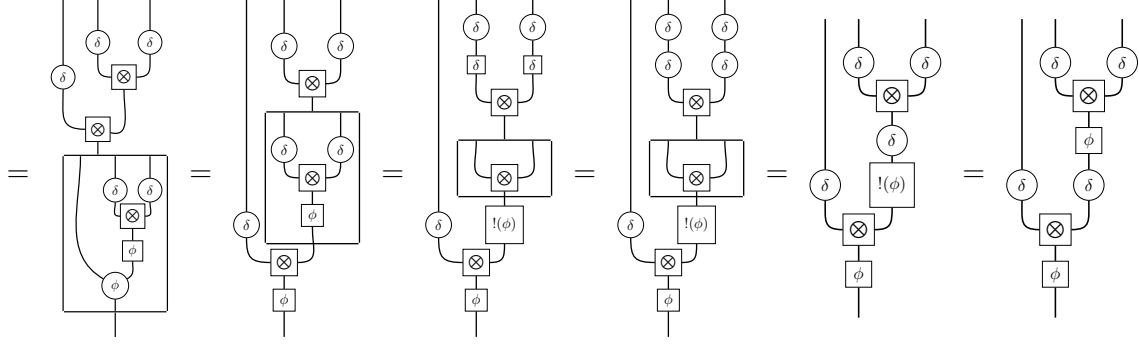
Therefore, u is also a natural transformation. □

Next we check that we have a commutative algebra.

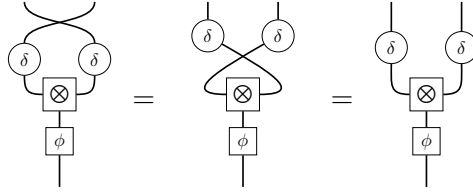
Lemma A.2 For each object A , $(!A, \nabla, u)$ is a commutative monoid.

PROOF: We need to check associativity, commutativity and the unit laws. For associativity of the multiplication we have:

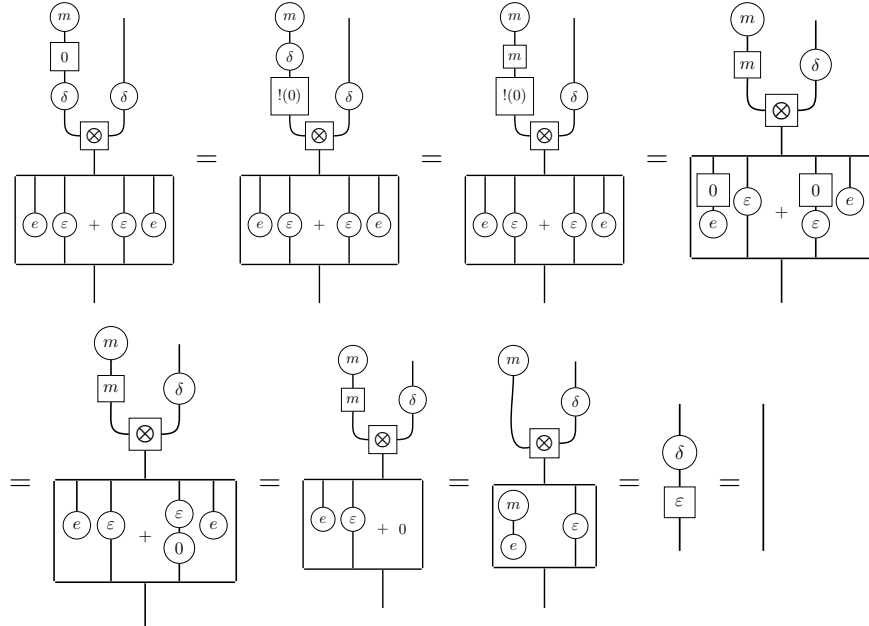




Now to show that the multiplication is commutative:



Having proved commutativity, we only need to prove the left unit law of the multiplication:



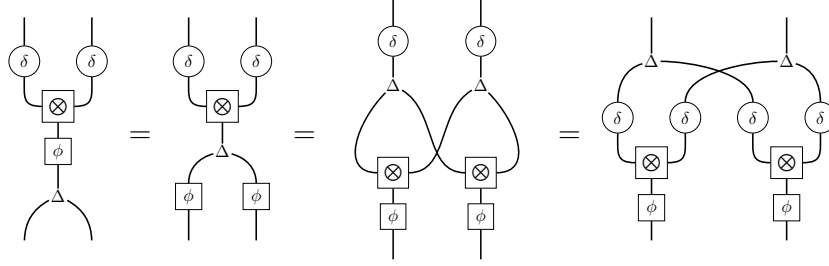
□

Next we check that we have a bialgebra:

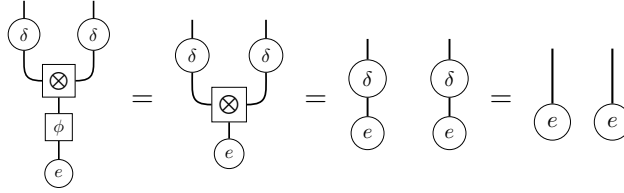
Lemma A.3 *For each object A , $(!A, \nabla, u, \Delta, e)$ is a (commutative) bialgebra.*

PROOF: We need to check the four bialgebra compatibility relations:

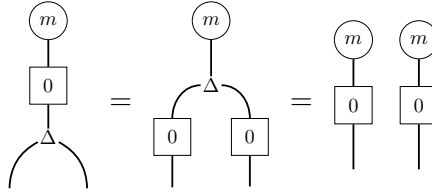
1. Multiplication and comultiplication compatibility:



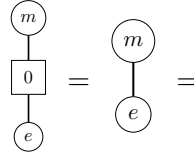
2. Multiplication and counit compatibility:



3. Comultiplication and unit compatibility:



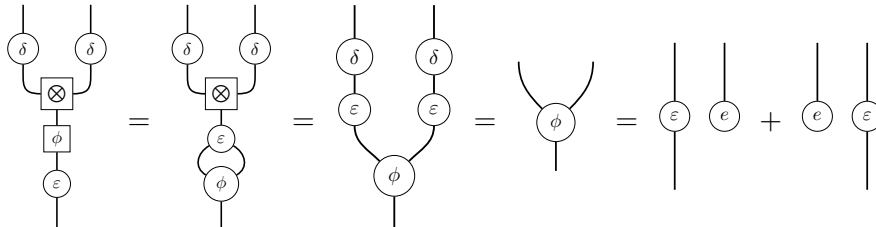
4. Counit and unit compatibility:



□

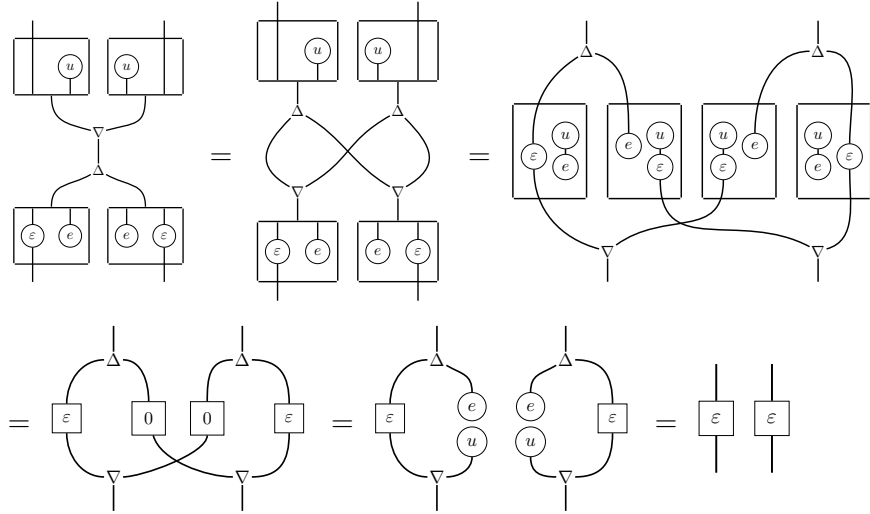
Proposition A.4 *The monoidal coalgebra modality of an additive linear category is an additive bialgebra modality.*

PROOF: We first prove that we have a bialgebra modality by proving the compatibility of ε and ∇ :



Finally, we show that the bialgebra modality is in fact additive by proving the compatibility with the additive structure:

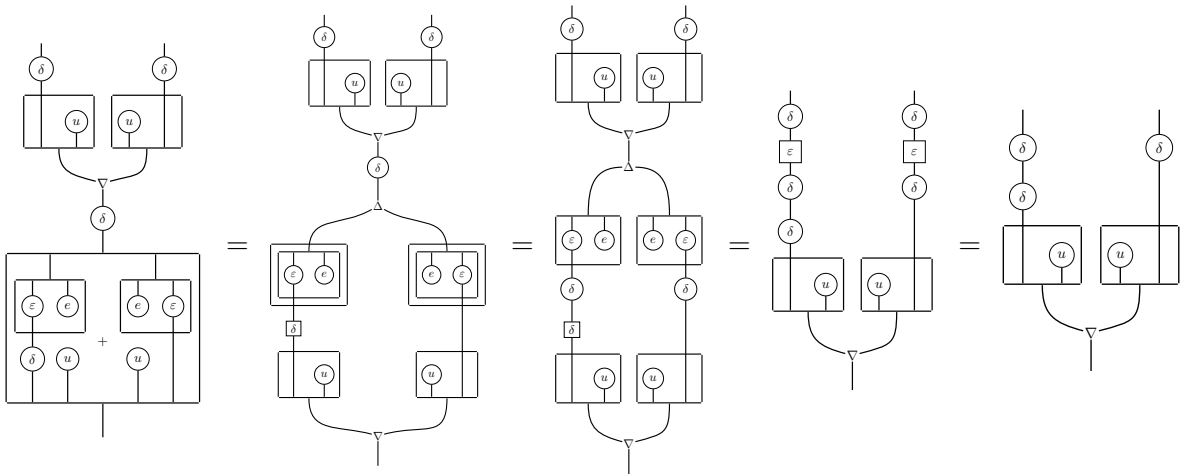
For multiple parts of the following proofs, we will need the following useful identity:



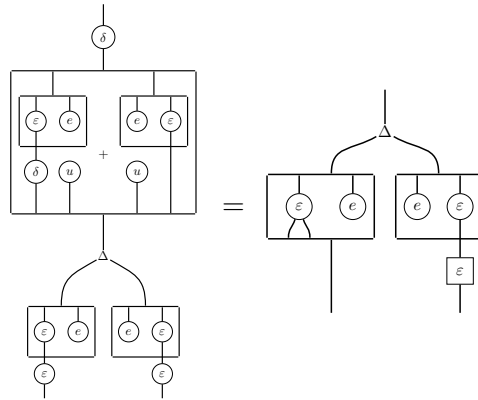
We first prove that we have a symmetric monoidal functor:

Lemma B.2 $(!, m_{\otimes}, m_K)$ is a symmetric monoidal functor.

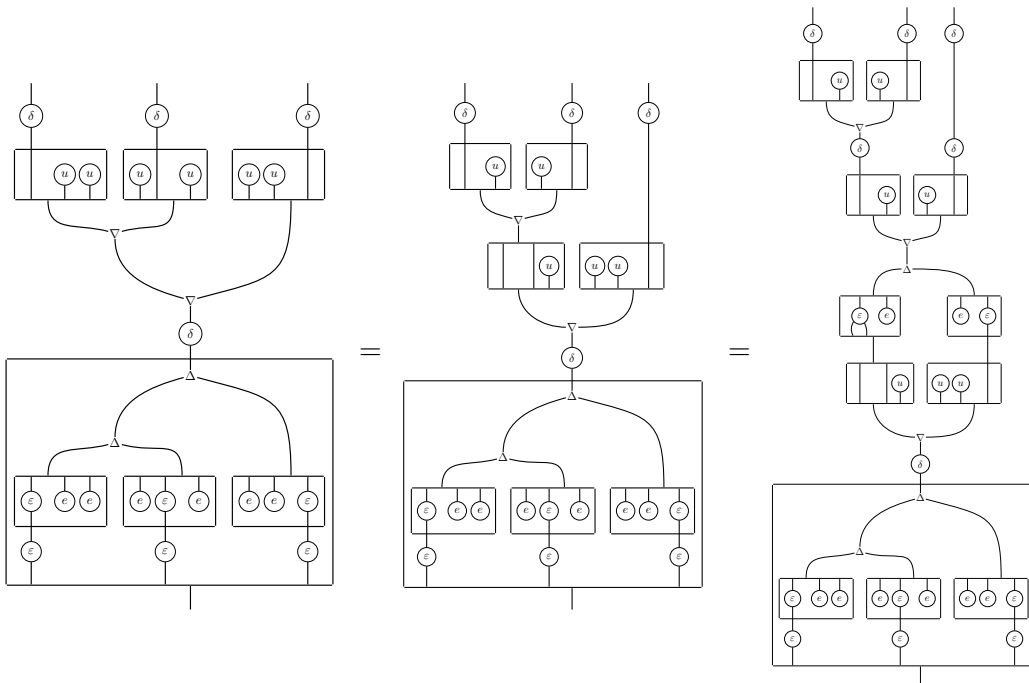
PROOF: To prove associativity of m_{\otimes} we will need the following identities (which follow from the above identity):

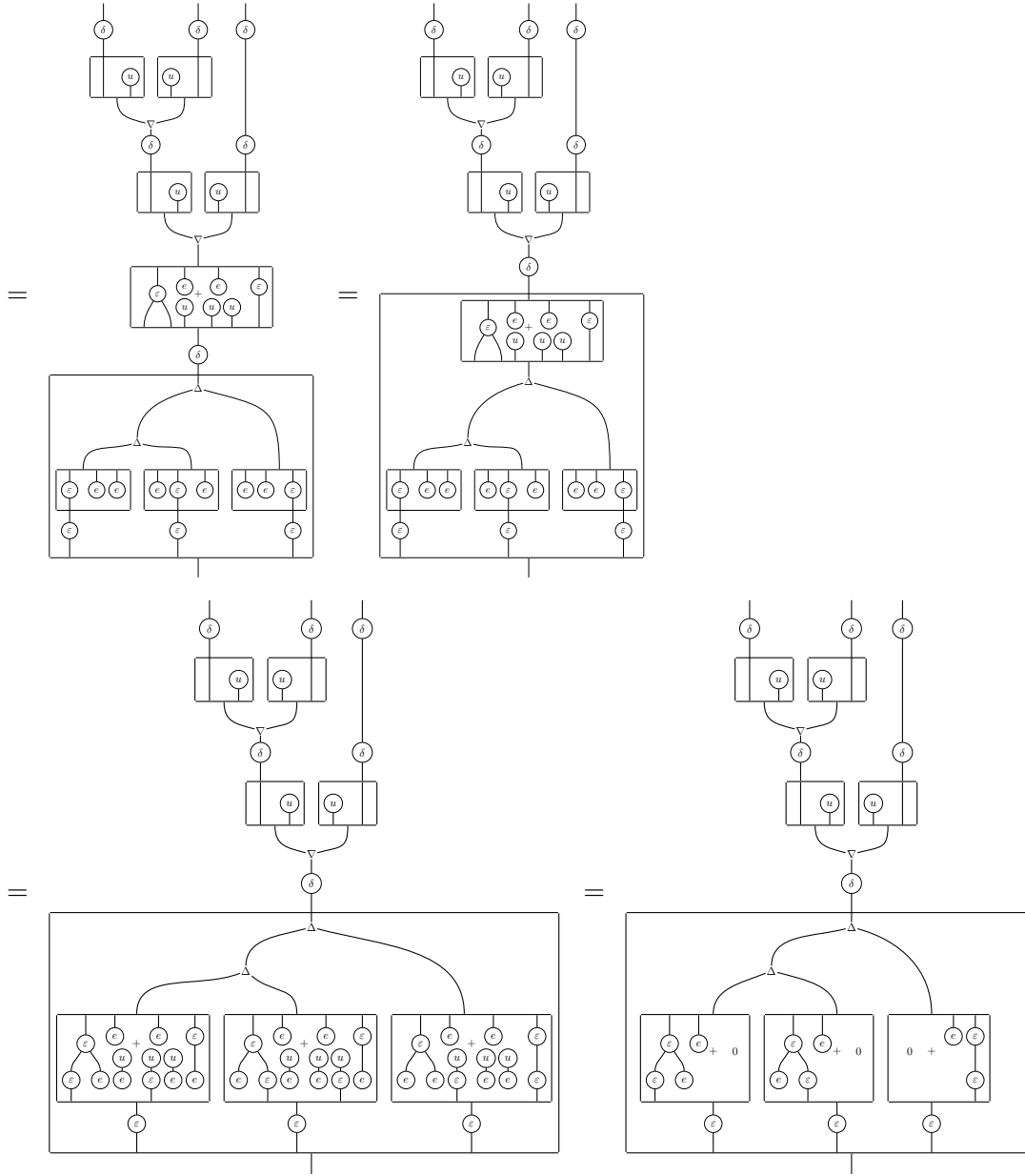


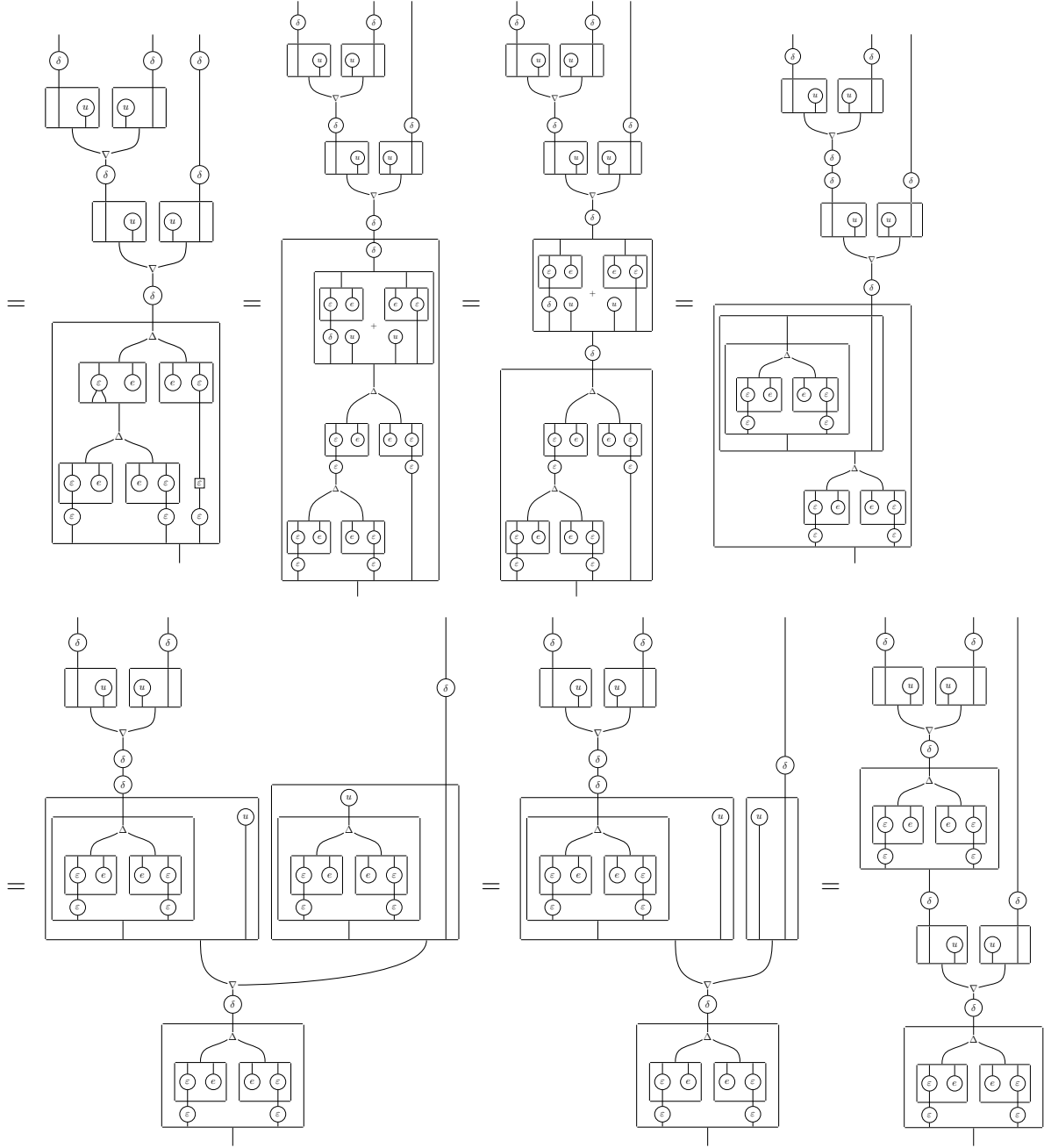
And similarly we have:



Now we can prove associativity. Consider the following series of equalities:



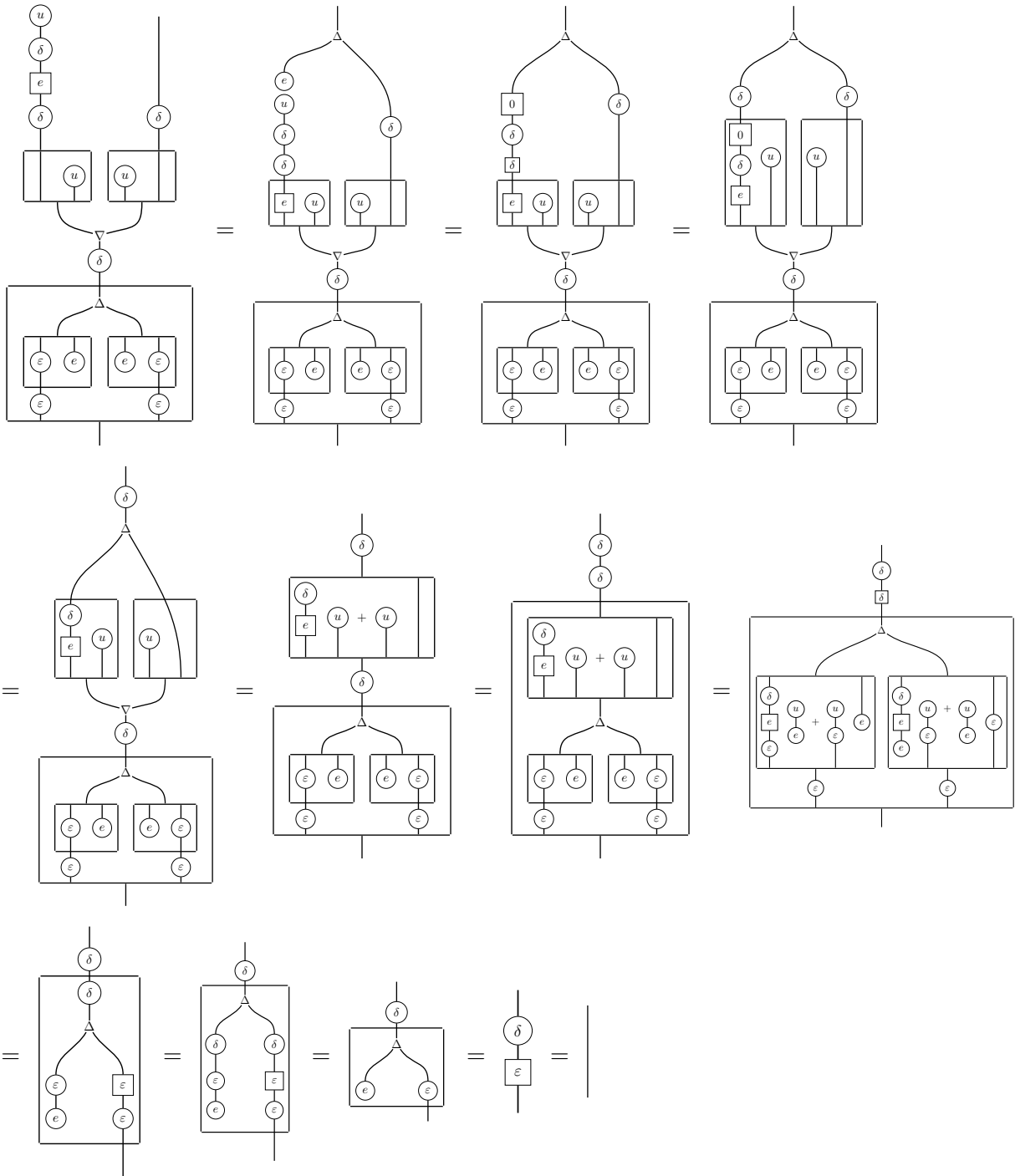




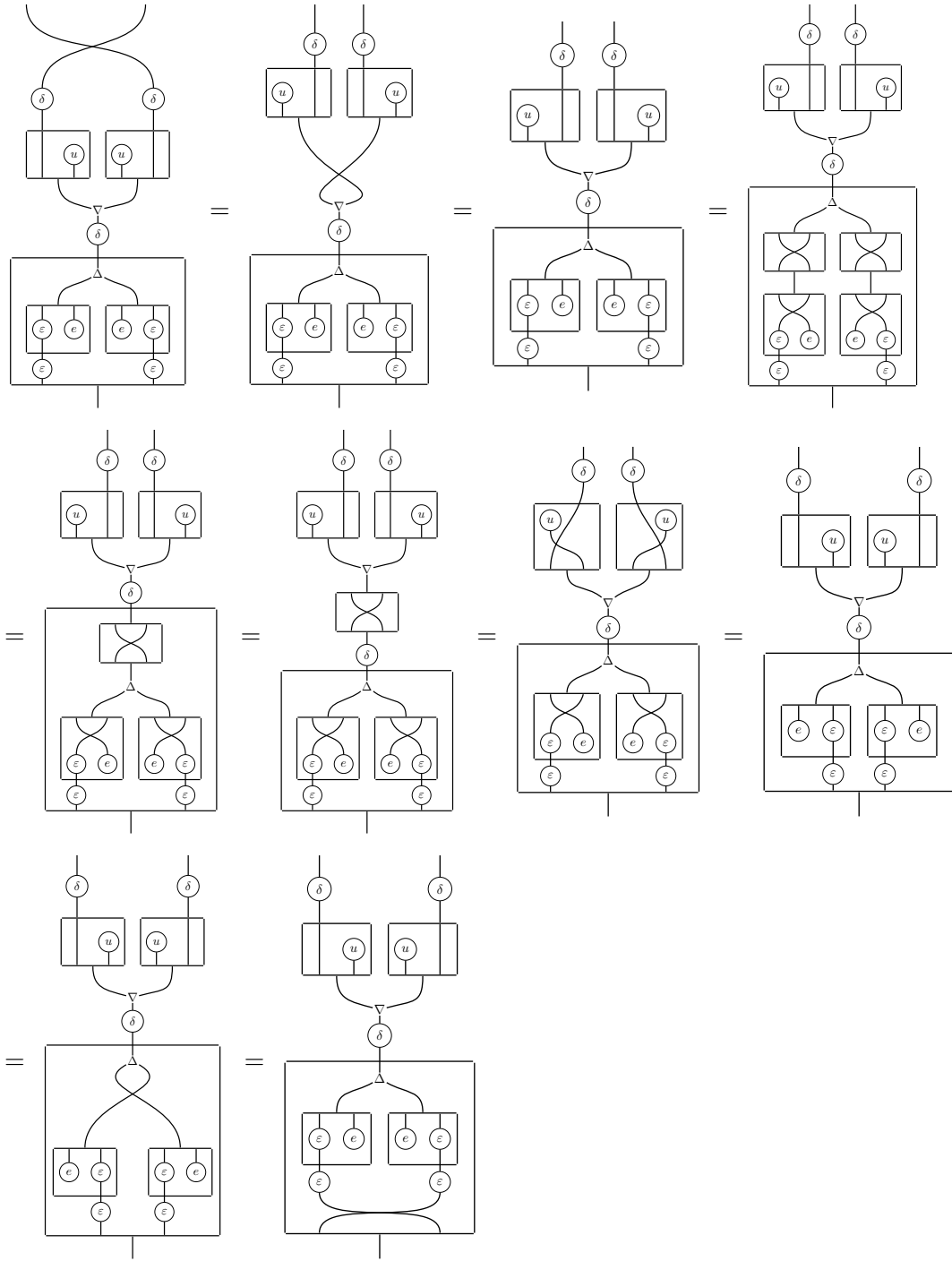
The first circuit is symmetric by associativity of the bialgebra and the last circuit is the left associativity of m_\otimes . Therefore reversing the sequence of equalities by symmetry gives the right associativity of m_\otimes . This proves the associativity coherence for m_\otimes .

Next we prove the unit laws for m_\otimes and m_K (we only prove the left unit identity, as the proof

for the right unit identity is similar):



Now we prove that m_{\otimes} preserves the symmetry:

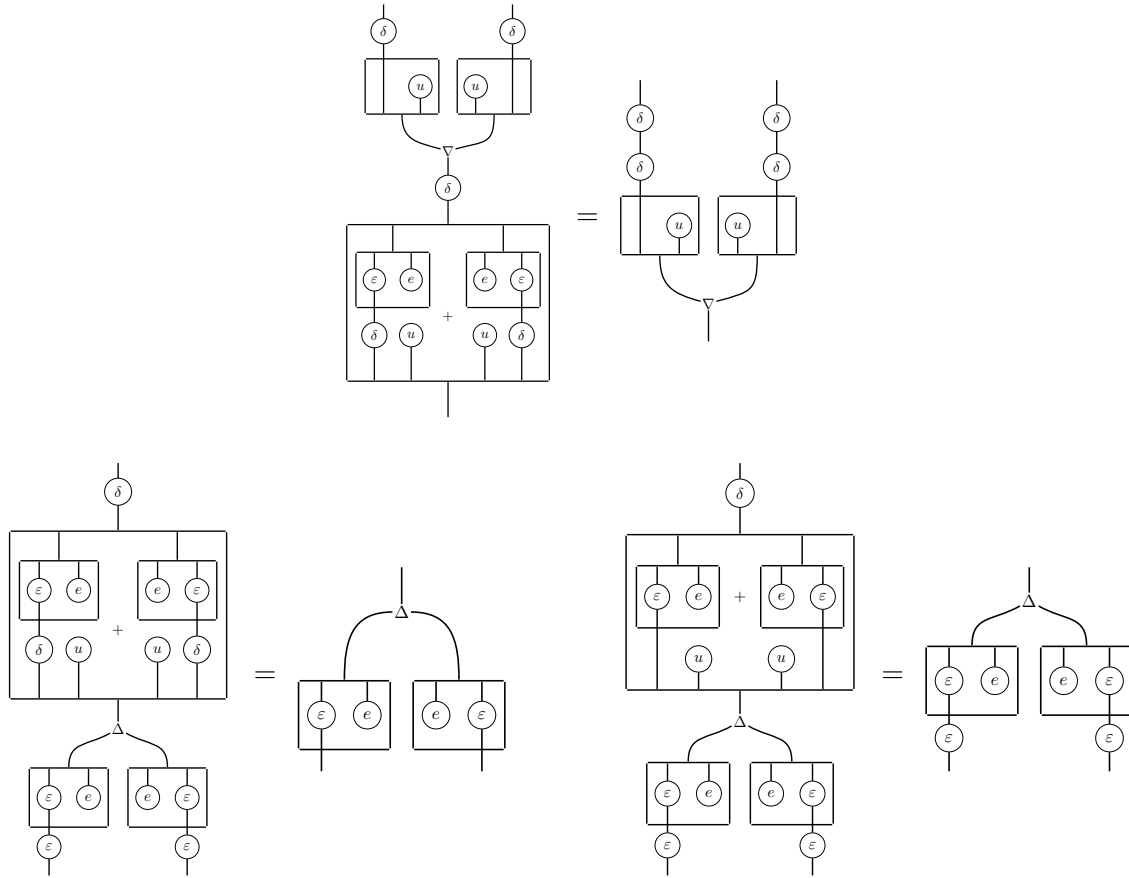


□

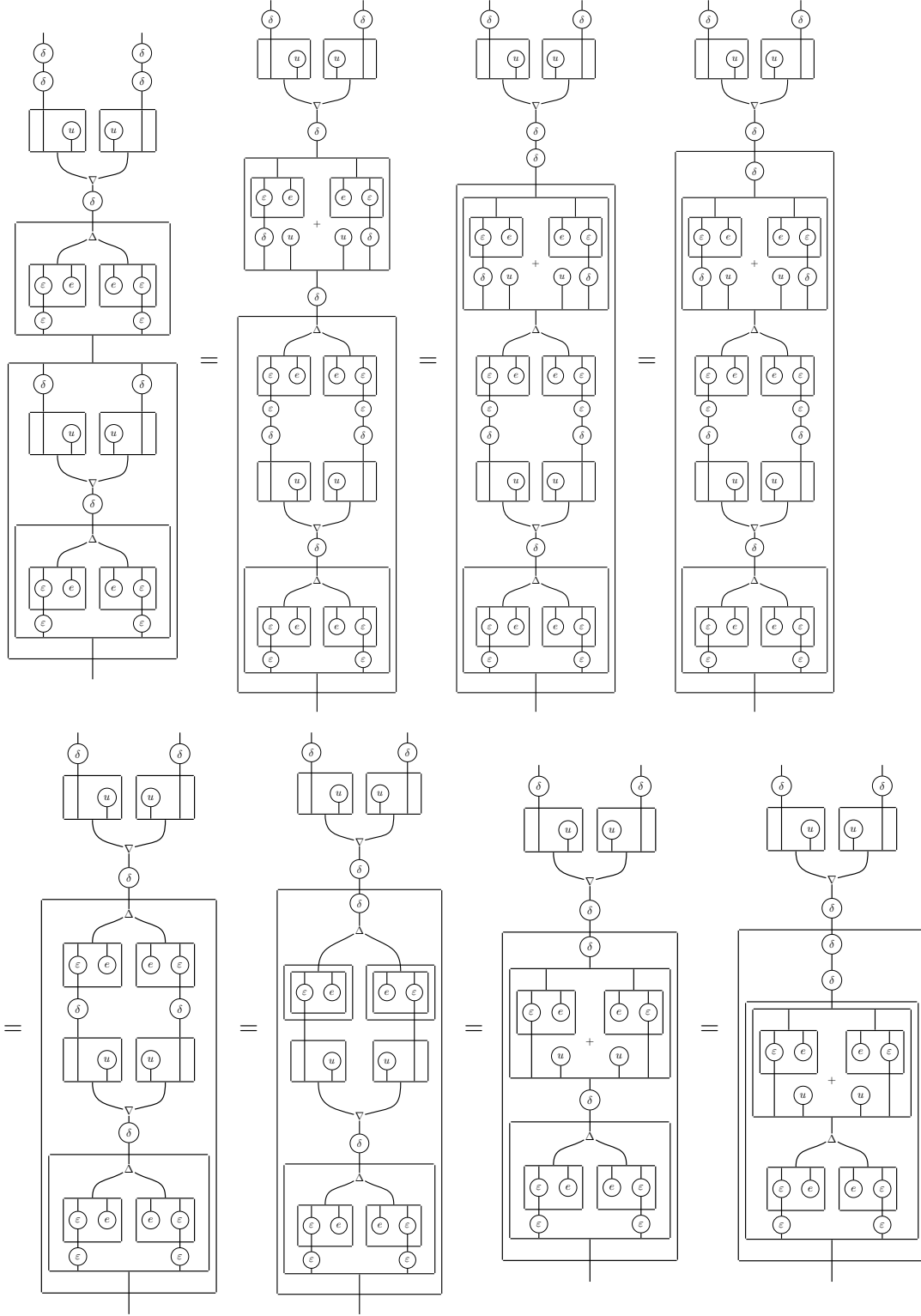
Next we prove that we have a symmetric monoidal comonad:

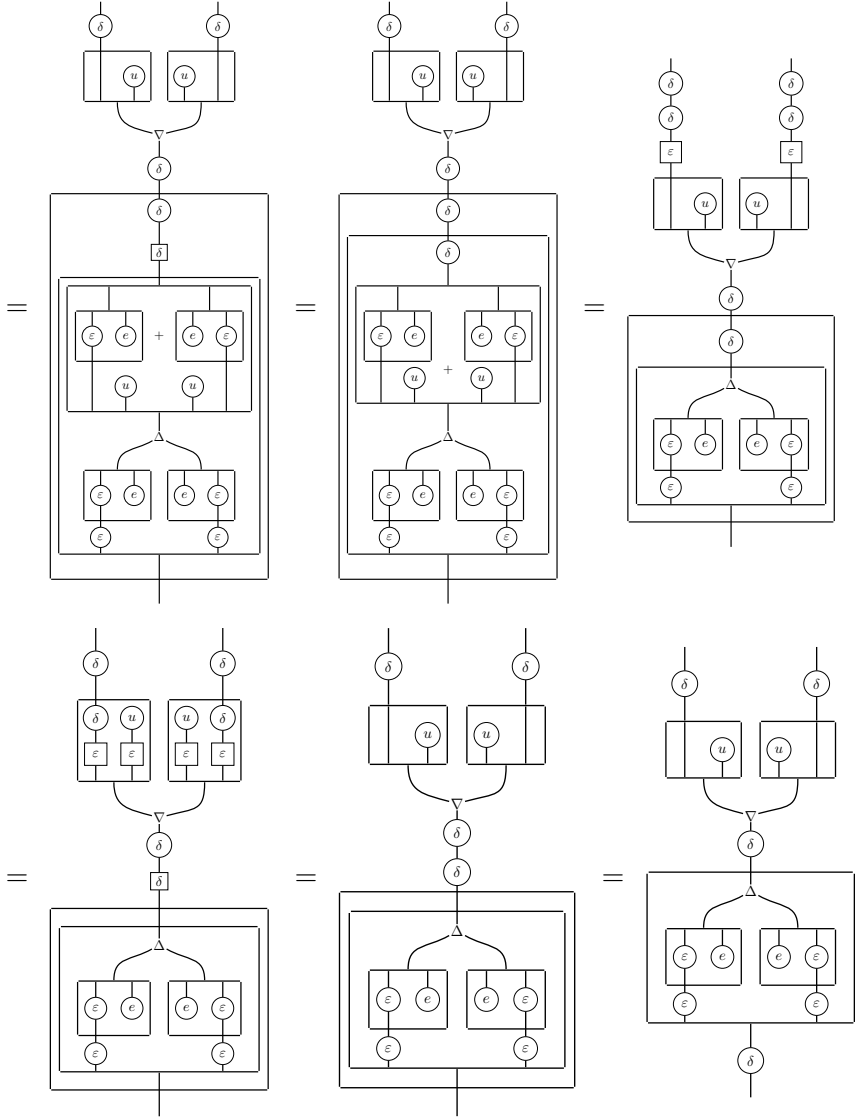
Lemma B.3 $(!, \delta, \varepsilon, m_{\otimes}, m_K)$ is a symmetric monoidal comonad.

PROOF: We need to show that δ and ε are monoidal natural transformation. To show the compatibility relations with m_{\otimes} , we will need some useful identities (for which the proofs are very similar to the identity used for the associativity coherence of m_{\otimes}):

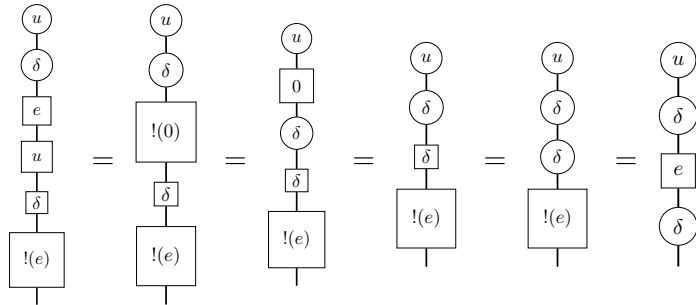


Now we show the compatibility between δ and m_{\otimes} :

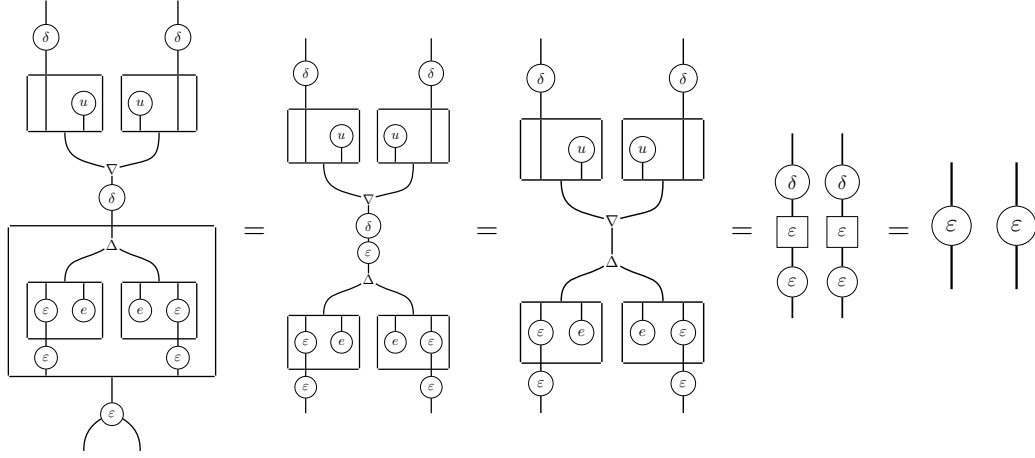




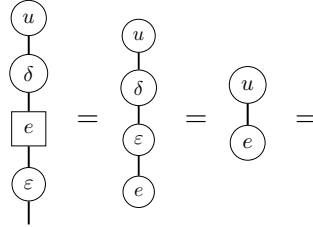
Next we show the compatibility between δ and m_K :



Now we show the compatibility between ε and m_{\otimes} :



Finally we show the compatibility between ε and m_K :

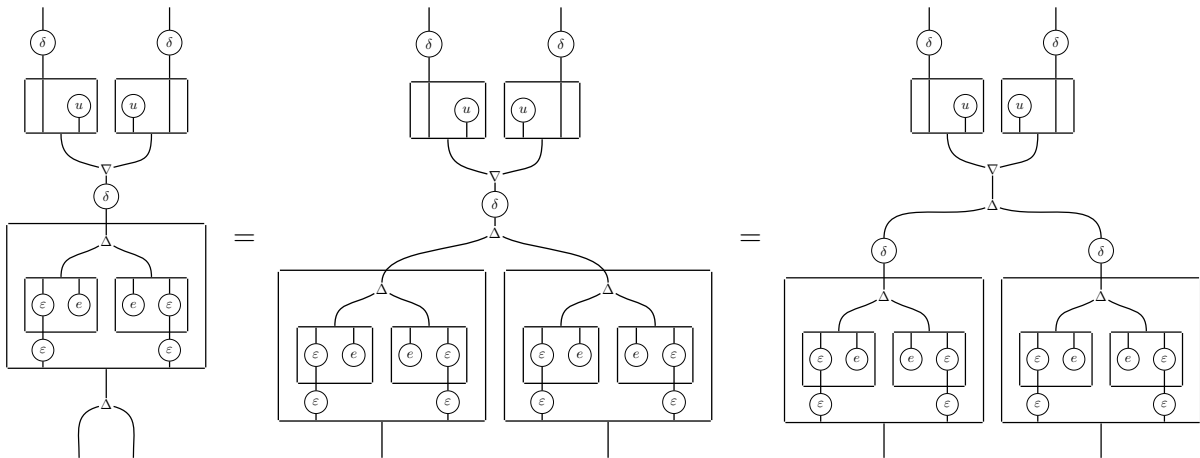


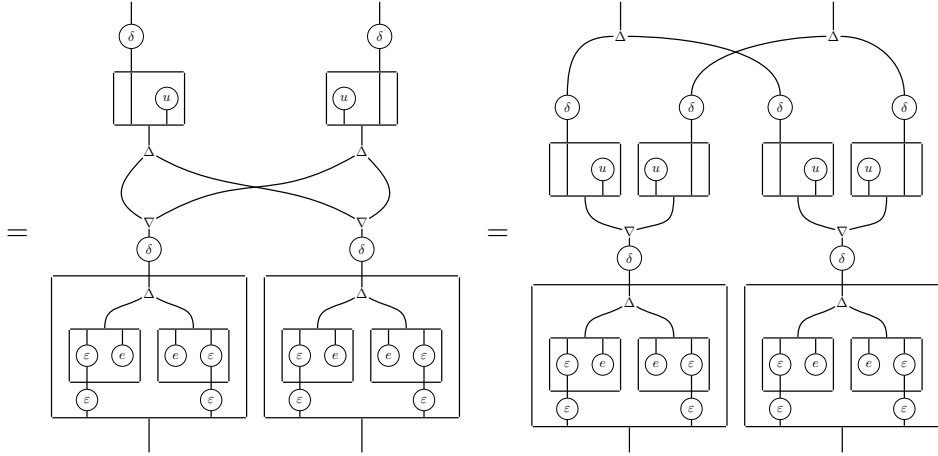
□

Finally we show that we have a monoidal coalgebra modality:

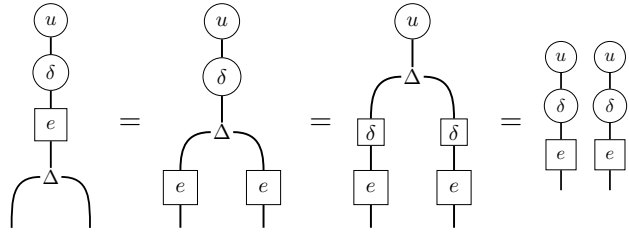
Proposition B.4 *Every additive bialgebra modality is a monoidal coalgebra modality.*

PROOF: We begin by proving that Δ is a monoidal transformation: first by proving m_{\otimes} preserves the comultiplication:

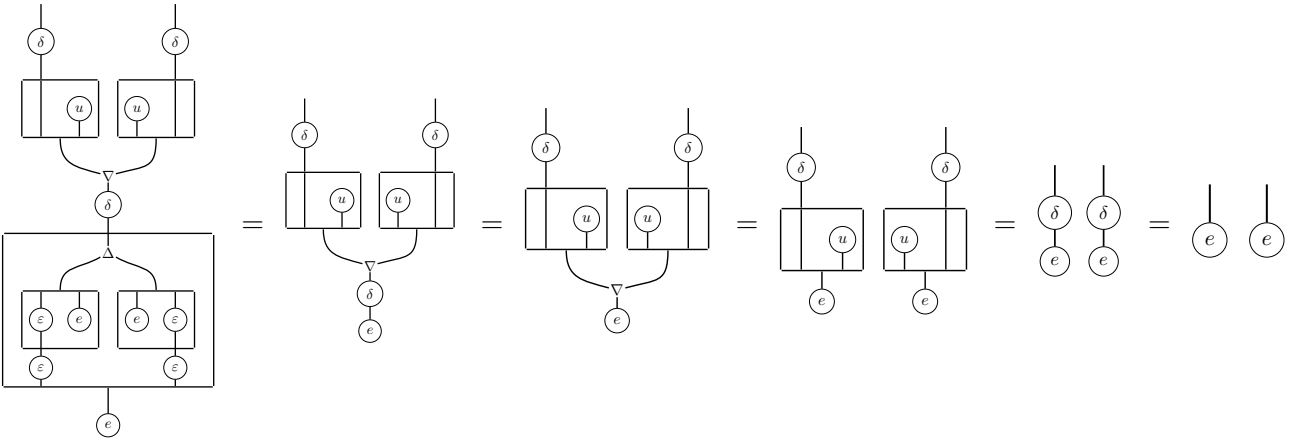




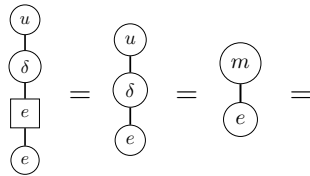
Next we show that m_K preserves the comultiplication:



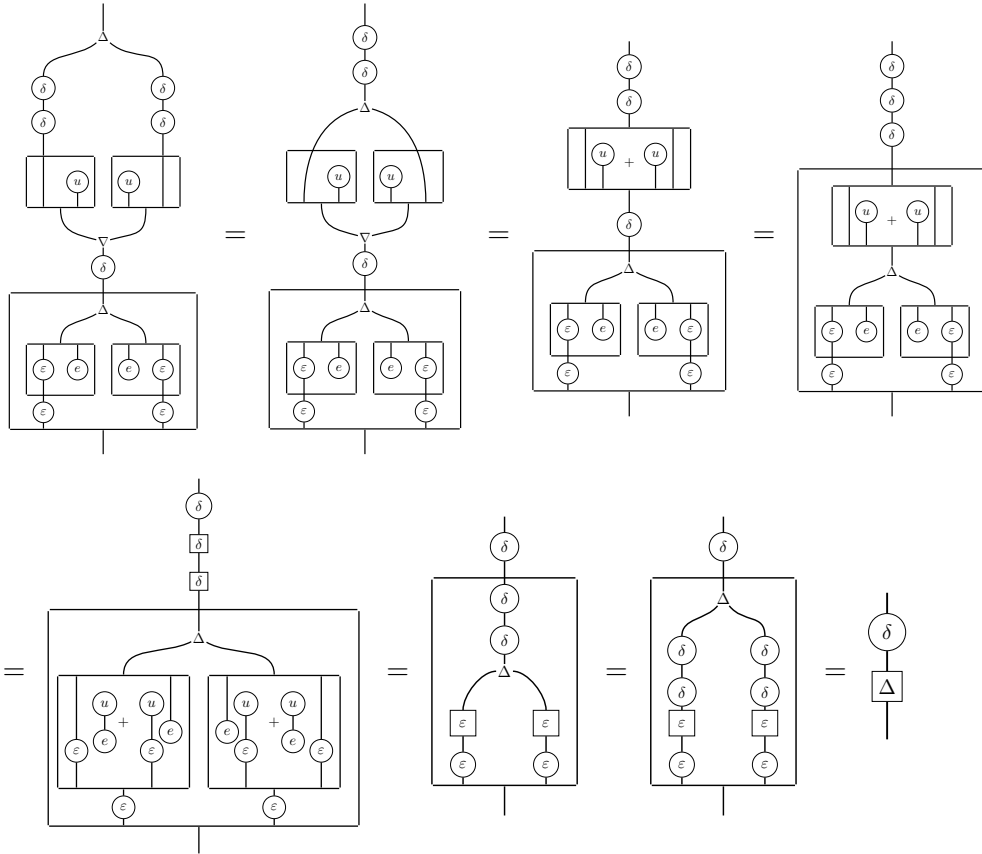
Now we we prove that e is monoidal, by first proving that m_{\otimes} preserves the counit:



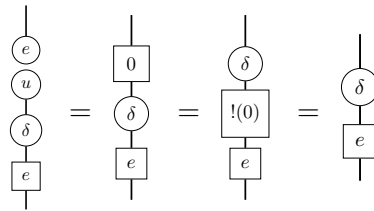
Next we show that m_K preserves the counit:



Now we prove that Δ is a $!$ -coalgebra morphism:



And finally we prove that e is a $!$ -coalgebra morphism:



□