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! and ?
Storage as tensorial strength

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We continue our study of the negation-free structure of multiplicative linear logic, as represented by the structure of weakly distributive categories, to consider the "exponentials" ! and ? in the weakly distributive context. In addition to the usual triple and cotriple structure that one would expect on each of the two operators, there must be some connection between them, to replace the de Morgan relationship found in the linear logic context: that turns out to be the notion of tensorial strength. We analyze coherence for this situation using a modification of the usual nets due to Danos, a form suitable for linear logic with exponentials but without negation.

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0. Introduction

The main goal of this paper is to study the categorical structure of the modalities of linear logic (Girard 1987). We choose as the framework for this study the *weakly distributive categories* of (Cockett and Seely 1991). We begin by giving the definition of weakly distributive categories equipped with modal operators, corresponding to the "exponential" fragment of linear logic, and then proceed to derive a coherence theorem for such categories.

Weakly distributive categories correspond to the negation-free fragment of multiplicative linear logic. Thus, we have two monoidal structures which correspond to the TENSOR and PAR of this fragment. A weak distributivity can be thought of as a linearization of the traditional distributivity of distributive categories. This distributive law may also be viewed as the requirement that one tensor be strong with respect to the other. As strength is one of the themes running through this work, let us see what this means. A functor F is strong (with respect to a tensor product $\otimes$) if there is a natural transformation $F(X) \otimes Y \longrightarrow F(X \otimes Y)$ satisfying certain coherence conditions. If we take F to be the functor $F(X) = A \oplus X$, then this gives the weak distributive law $(A \oplus X) \otimes B \longrightarrow A \oplus (X \otimes B)$. Notice that this map can also be read as the costrength, with respect to the "par"† $\oplus$, of the functor $G(X) = X \otimes B$. Fundamental to the linkage between "tensor" and "par" in a weakly distributive category is the requirement that this weak distribution be *simultaneously* a strength and a costrength.

The structure of $*$-autonomous categories is a conservative enrichment of the structure of weakly distributive categories. This was established in (Blute *et al* 1992). One of the points of studying weakly distributive categories (from the perspective of linear logic) is that they allow one to analyze the structure of PAR *vis à vis* TENSOR without the de Morgan duality given by linear NEGATION obscuring the issue. We shall pursue that viewpoint further in this paper. For the moment, let us merely point out that saying PAR is de Morgan dual to TENSOR is rather unsatisfactory if it does not allow one to proceed further in the analysis. We believe the role of the distributivities, for example, was obscured by the actions of NEGATION, and similarly, the role of tensorial strength has not been sufficiently recognized before for the same reason.

Adding the modalities ! and ? is reasonably straightforward, from one point of view—we just add the appropriate two-sided versions of the sequents from linear logic to the sequent calculus for weakly distributive categories (*i.e.* the tensor-par fragment of linear logic). It is straightforward to translate thinning, contraction, and dereliction into categorical terms, but the storage rule presents a richer story—as is the case in traditional linear logic, see (Seely 1989). Here we find that the storage rule breaks into several components, the only non-local one being functoriality. One component we believe has not been noticed before, and is the only part of the formulation that expresses the connection between ! and ?; this is the condition we have called "relative strength" (and dually "relative costrength"). It is a generalization of the usual notion of tensorial strength,

† As with the previous papers in this series, we use the symbol $\oplus$ for the cotensor or "par", rather than Girard's "par" symbol $⅋$.

and is very similar therefore to the weak distributivities discussed above. It would seem that this strength is the " shadow" of de Morgan duality in the negation-free case. As was the case with weakly distributive categories, simply adding negation is sufficient to deliver the full structure of linear logic: that is a $*$-autonomous category with a cotriple $!$, satisfying the usual properties, in which $?$ is given by the de Morgan dual. This conservativity result is an obvious generalization of the result in (Blute *et al* 1992).

We could at this point derive the standard Fox theorems (Fox 1976) which in our context would assert that in the category of algebras for $?$, the cotensor ("par") is a coproduct, and in the category of coalgebras for $!$, the tensor product is a Cartesian product. We shall in fact leave these results to the reader (beyond pointing them out here in the introduction), as the proofs are more or less straightforward, following the familiar pattern as originally done in (Fox 1976), and as they have no role to play in the present paper. The analogous situation for intuitionistic linear logic has been the object of careful study (*e.g.* (Benton *et al* 1992; Bierman 1995)), making our job here much simpler. However, we think that an explicit outline of what coherence conditions are necessary is useful, and so have tried to make the list given here complete, apart from naturality diagrams. Of course, one of the points that we are making is that the net approach to coherence equivalently presents these seemingly endless pages of commutative diagrams as a small number of simple net rewrites.

Our approach to solving the coherence problem follows the methods established by the first author in (Blute 1993). There the relationship between proof nets and morphisms in monoidal categories was exploited. For, when categorical morphisms can be represented as proof nets, the strongly normalizing reduction systems of these nets translate into categorical coherence theorems. Lambek's technique of representing morphisms as proofs of a sequent calculus (Lambek 1969) was intended to accomplish the same goal. While proof nets are a representation of the same proofs, they use a natural deduction style. For monoidal categories, natural deduction systems have a significant advantage: they capture the basic tensor coherences very succinctly and naturally. This, in turn, allows a succinct and particularly natural expression of coherence results in these systems.

In (Blute 1993) the nets defined in (Danos and Regnier 1989) were used. These make substantial use of the involutive negation and thus were not well-suited for expressing coherence results for weakly distributive settings. In (Blute *et al* 1992) these nets were modified to provide a two sided version, appropriate for sequents with premises and conclusions. In these nets Girard's cut links are replaced by grafting of trees, and cut-elimination is replaced by Prawitz-style normalization (Prawitz 1965). The modified nets were then used to provide coherence results for both commutative and noncommutative weakly distributive categories.

A central feature of the coherence results in (Blute *et al* 1992) was the fact that the units were also correctly handled. Units introduce a complication into the description of proof net equivalence as there is considerable freedom over where they may be introduced or eliminated while maintaining proof equivalence. In the proof nets a tensor unit elimination and a "par" unit introduction are marked by a thinning link. A key realization, due to (Trimble 1994), was that one should be allowed to *rewire* these thinning links within the empire of the unit. It was this feature which allowed the coherence theorems to handle the units for the two connectives. However, it also meant that the reductions of these nets was only confluent modulo the permuting conversions represented by these rewirings. The necessity of this more complex view of coherence is supported by results which show that the addition of unit rules to the multiplicative system greatly adds to the computational complexity of provability in the system (Lincoln and Winkler 1994). In this paper, we will need the rewiring system again to handle the weakening rules of the exponential fragment as well as the units themselves.

It would be a mistake to imagine that the proof nets (or circuits) we draw in this paper are merely pictures. There is now a fairly deep formalism underlying them which does not simply rely on the geometry of tensor calculus (Joyal and Street 1991). In (Blute *et al* 1992) we introduced for our circuits a term calculus, called "circuit expressions", which gives considerable precision not only to the thinning links, but also more generally to the use of the pictures in this paper. Our contention is that these nets are technically very useful: they are good exactly for those tricky detailed manipulations which continually rely on coherence. In this paper we shall just use the graphical presentation of circuits, but the reader ought to keep in mind that a calculus of circuit expressions (a term calculus) is available (Blute *et al* 1992).

The present context requires a new element: storage boxes, to handle the storage rules for $!$ and $?$. Again we borrow from the work of V. Danos (Danos 1990). Again, we modify his nets to obtain a two sided system; we also introduce additional rewrites. We have rewirings corresponding to the two types of weakening rules, and a system of *box-expansions*, which corresponds to delaying the application of the storage rule as much as possible. We obtain a coherence theorem which states that two morphisms are equal if and only if they are assigned the same nets modulo the rewiring relation.

We have not used any of the systems without storage boxes because we have not found that they retain enough information about sequent derivations to enable the coherence results to go through. From our point of view, attempts to remove boxes from the nets seem ill-advised. The necessity of keeping the functoriality of $!$ and $?$ as part of the system suggests that boxes are an essential reflection of the non-local component of the structure of the modalities. It will be clear from our presentation that we could have used "functor boxes" instead of storage boxes, but the reduction and expansion process for storage boxes is quite well known and very well suited to our purposes, providing a simpler rewrite system than would have been possible with functor boxes, so we were happy to retain them. In fact, one may go further, and say that specifically for the nets (circuits) used in this paper, and more generally for the nets used in the series of papers we have written on this subject, that these circuits have shown clearly that they capture the essence of the proofs they represent and the essence of morphisms in categories of the appropriate doctrines. Coherence results for (monoidal) categories represent an essential part of the proof theory of (linear) logic, and the results in these papers would not have been possible without these circuits.

Finally, a few remarks about logic: in developing the categorical structure suitable for $!$ and $?$, we have given an alternate presentation of the sequent rules for these operators, one in which the only rules needed are the functoriality of the operators, the rest of the structure being given by axioms. Of course, this system will not have a cut-elimination theorem, but it does explicate some of the structure, even at the naive level of the logic. Of course, our point is that this presentation, and its relative simplicity, extends to the proof theory as well, in its categorical version. In connection with cut-elimination, it may be worth mentioning that the system presented here (in Table 1) does admit cut-elimination, and needs no term assignment system (as in (Benton *et al* 1992), for example) to facilitate this. This is because we have no negation, nor implication. We have also studied systems with implication, particularly Lambek's bilinear logic and Hyland and de Paiva's full intuitionistic linear logic, and this will be the object of a sequel to this paper (Cockett and Seely 1995).

1. Logic for Weakly Distributive Categories with ! and ?

In this section, we discuss the logical framework for studying weakly distributive categories with ! and ?. We will first present a sequent calculus, which will of course be a fragment of linear logic. Then, a proof net system for this fragment will be presented. This system is a straightforward modification of the simply-typed nets of Danos. The normalization rules for this system are presented, and the equivalence of nets is discussed.

1.1. *Sequent Calculus*

This system is of course a subsystem of linear logic, but since there is no negation, we are forced to write two-sided sequents. Thus, we require twice the number of usual rules. The sequent rules for !, ? are given in (Seely 1989), and are summarized in Table 1, which gives the complete syntax used here.

Under the usual Lambek equivalence between morphisms in a category and deductions in a deductive system, we must impose equations (essentially those given in Section 2) to generate an equivalence relation on deductions in this fragment. The effect as usual will be to make the term model a category with the appropriate structure, given by the categorical semantics of Section 2. Before then, however, we give the description of proof structures for this calculus, together with the criterion for proof nets (*i.e.* proof structures that represent correct deductions).

1.2. *Proof Nets*

The proof nets we define here are a straightforward generalization of the simply-typed nets of Danos. However, we define nets in the style of the two-sided nets of (Blute *et al* 1992). These were nets with premises and conclusions. Here `cut` is a derived rule, and cut-elimination is replaced by normalization. Nets will be built inductively from the basic links given below.

A word about the notation we shall use in this paper. Working together we have found that it is convenient to use a variant of the proof net notation which represents formula occurrences by edges of a graph, and proof rules (or "links") by nodes of the graph. We call this presentation of proof nets "circuit diagrams" to distinguish these from the standard proof nets, but the reader should be reassured that these are really just the

Table 1. *Sequent rules for* WDC + **!** **?**

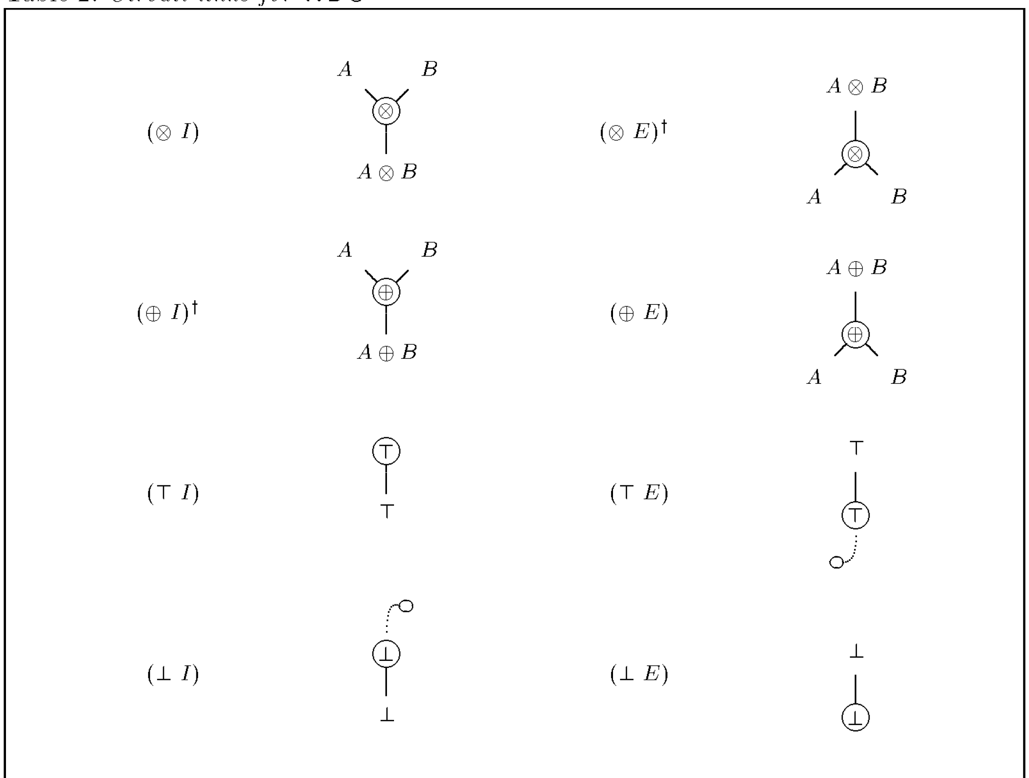
(id)	$A \vdash A$	$(exch\ R)$	$\dfrac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}$
(cut)	$\dfrac{\Gamma \vdash \Delta, A \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$	$(exch\ L)$	$\dfrac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}$
$(\otimes\ L)$	$\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}$	$(\otimes\ R)$	$\dfrac{\Gamma \vdash \Delta, A \qquad \Gamma' \vdash \Delta', B}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \otimes B}$
$(\oplus\ L)$	$\dfrac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \oplus B \vdash \Delta, \Delta'}$	$(\oplus\ R)$	$\dfrac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \oplus B}$
$(\top\ L)$	$\dfrac{\Gamma \vdash \Delta}{\Gamma, \top \vdash \Delta}$	$(\top\ R)$	$\vdash \top$
$(\bot\ L)$	$\bot \vdash$	$(\bot\ R)$	$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot}$
$(thin\ L)$	$\dfrac{\Gamma \vdash \Delta}{\Gamma, \mathbf{!}A \vdash \Delta}$	$(thin\ R)$	$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \mathbf{?}B}$
$(der\ L)$	$\dfrac{\Gamma, A \vdash \Delta,}{\Gamma, \mathbf{!}A \vdash \Delta}$	$(der\ R)$	$\dfrac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, \mathbf{?}B}$
$(contr\ L)$	$\dfrac{\Gamma, \mathbf{!}A, \mathbf{!}A \vdash \Delta}{\Gamma, \mathbf{!}A \vdash \Delta}$	$(contr\ R)$	$\dfrac{\Gamma \vdash \Delta, \mathbf{?}B, \mathbf{?}B}{\Gamma \vdash \Delta, \mathbf{?}B}$
$(stor\ L)$	$\dfrac{\mathbf{!}\Gamma, A \vdash \mathbf{?}\Delta}{\mathbf{!}\Gamma, \mathbf{?}A \vdash \mathbf{?}\Delta}$	$(stor\ R)$	$\dfrac{\mathbf{!}\Gamma \vdash \mathbf{?}\Delta, B}{\mathbf{!}\Gamma \vdash \mathbf{?}\Delta, \mathbf{!}B}$

familiar proof nets, with the variations we need for the weakly distributive context. The edges (called "wires") of the graph represent the formulas of the derivation encoded by the circuit; the nodes of the graph represent the links or deduction rules used in constructing the derivation.

We list in Tables 2, 3 the basic links for the circuit diagrams—or proof nets—that correspond to the sequent rules of Table 1: note that we need no links for the cut or exchange rules, nor for the axiom links. In the tables we label the wires with the appropriate formulas, but in general we shall only label the initial and terminal points of our circuits. Internal wires can be unambiguously labeled by these rules. The following links are switchable, in the usual sense: $(\otimes\ E), (\oplus\ I), (contr\ I), (contr\ E)$.

Some **remarks** about Tables 2, 3 are perhaps in order. First, note that the links $(\top\ I), (\bot\ E)$ have no labelled wire on one side—the wire on the other side is labelled with the corresponding unit. These links come from (respectively, go to) "nothing", as given by the sequent axioms $(\top\ R), (\bot\ L)$. Next, the thinning links $(\top\ E), (\bot\ I), (thin\ I), (thin\ E)$ all must be attached to some other wire by a thinning edge, represented here by a dotted wire—the loop at the end is attached to some other wire, just as the unit thinning links

Table 2. *Circuit links for* WDC



†Switchable links

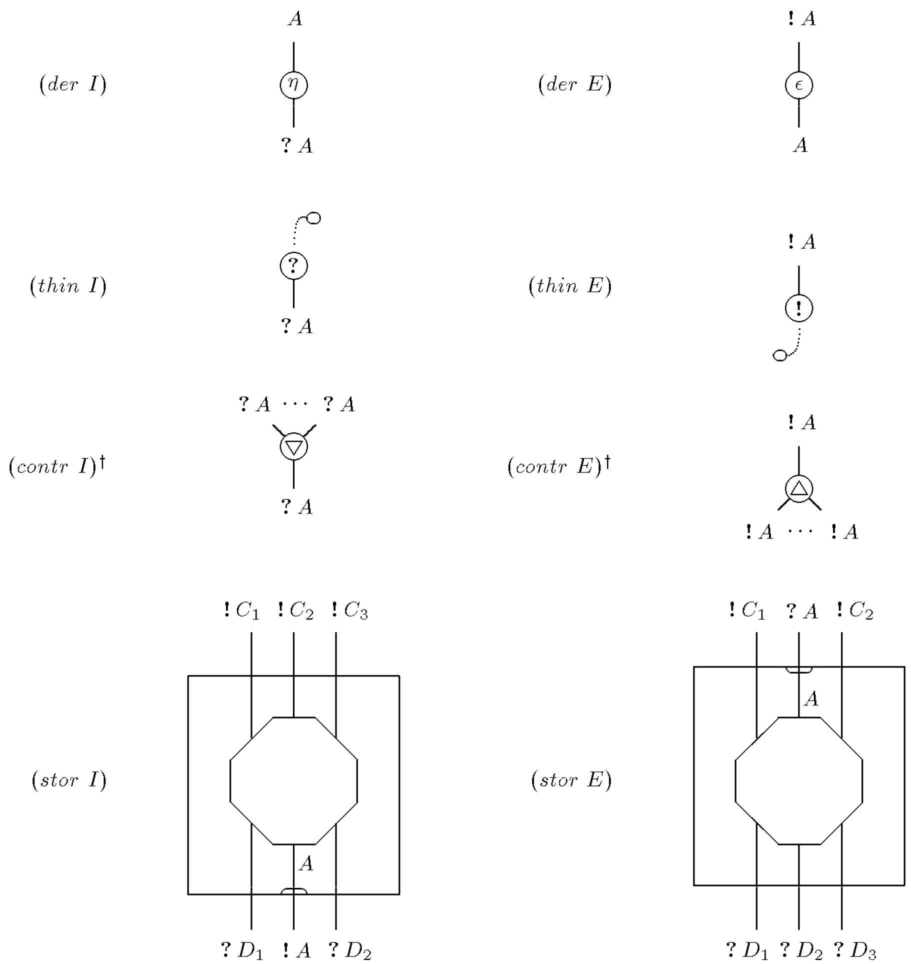
were treated in (Blute *et al* 1992). Note that the dotted edge is not labelled—it does not correspond to any formula.

The contraction links have a couple of unusual features. First they are given in the generalized n-ary form, rather than the usual binary form, in which many instances of the formula are "contracted to" (or "replicated from") one. This has relevance only for confluence of normalization; otherwise the reader may ignore the "$\cdots$" and pretend these rules are binary. We shall usually follow this advice in this paper, but the reader should keep in mind that often there ought to be "$\cdots$" added to a given circuit involving the contraction links. The context ought to make this clear.

Furthermore, these links must be understood as having their n auxiliary links *unordered*. This means that if one thinks of these links in the usual graphical sense, then one can reorder the connections these links make with the rest of the net without altering the identity of the net. This is indicated in Table 6, where we use the equality as a "rewrite" in this context (such a rewrite being unnecessary strictly speaking). Categorically, this assumption of unorderedness corresponds to the (co)commutativity of the (co)monoidal structure on the free (co)algebras, described in Definition 2.1.

Finally, the storage links are context dependent: a storage box is applied to a valid

Table 3. *Circuit links for* **!** *and* **?**



†Switchable links

net (represented by the octagons in Table 3), all of whose input wires are labelled by ! 'ed formulas and all of whose output wires are labelled by ? 'ed formulas, other than the principal formula A. That formula receives a ? or ! as indicated, and the other formulas do not change. The principal formula is indicated in the box by the half-oval located where its wire enters (or leaves) the box. We often refer to this as the "principal port" of the box; the other formulas/ports are auxiliary.

With these two-sided circuits (nets), cut-elimination is replaced by normalization, as given by reductions and expansions, as well as by some rewrites that may best be thought of as "permuting reductions", as they are not "directed" in any natural way. The reductions and expansions for the multiplicative fragment of our logic are given in (Blute *et al* 1992) and are shown in Table 4. The reductions and expansions for the exponentials ! and ? are given in Tables 5, 6, and 7. Table 7 gives the usual cut-elimination steps, whereas Table 6 gives the extra cut-elimination steps for the "unordered n-ary" treatment we give for the contraction links. Table 5 lists additional rewrites for storage boxes needed for the categorical semantics.

In (Blute *et al* 1992) a considerable effort was spent in making the rewiring of thinning links as "local" as possible (see the discussion there of rules of surgery). We shall simplify that discussion here by adopting the more "global" viewpoint of the Empire Rewiring Theorem (Proposition 3.3, (Blute *et al* 1992)): that thinning links may be rewired to any other formula in the empire of the formula at the base of the thinning link (which here may be either a unit or a formula of the form $!A$, $?A$). It is easy to show that the Empire Rewiring Theorem is valid in this context: such rewirings preserve Lambek equivalence. This means that the rewrites in Tables 5, 6, and 7 ought to be extended to also allow rewiring thinning links attached to various elements illustrated by the rewrites. To add this information is more or less straightforward if one keeps the Empire Rewiring Theorem in mind; for example, in the last rewrite in Table 7, if a thinning link were attached to the wire that leaves the box $\mathcal{N}_0$ by the principal port, then after the rewrite, that thinning link may be attached to any one of the ports entering or leaving the newly enlarged box $\mathcal{N}_1$. This indeterminacy of choice is inevitable, and is the essence of the Empire Rewiring Theorem. The only other rewrite that might cause one to pause is the cut-elimination step that "pushes" a duplication node (*contr* E) above a storage box: for example, a thinning link attached to one of the input wires of the storage box $\mathcal{N}$ could be reattached after the rewrite to the wire leading into the corresponding duplication node above the box. Other cases are handled similarly, and left to the reader.

2. Categorical semantics for ! and ?

We base the notion of categorical model of the above syntax on the now-standard idea that ! is a cotriple on a $*$-autonomous category carrying the structure of a free coalgebra over the identity functor (Seely 1989; Benton *et al* 1992). (This is of course very vague— the details follow.)

But before we begin, we must provide an alternate sequent calculus description of the

Table 4. *Reductions and Expansions for* WDC

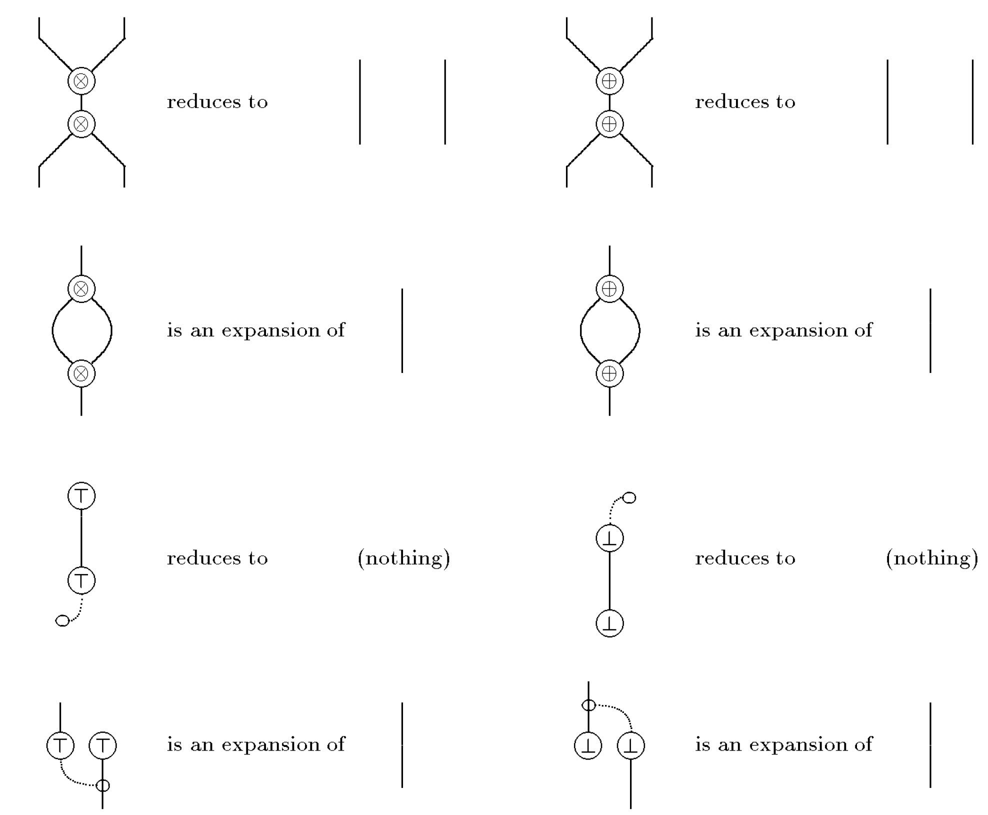
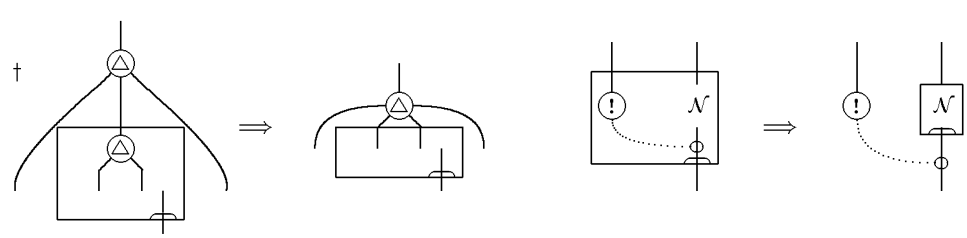


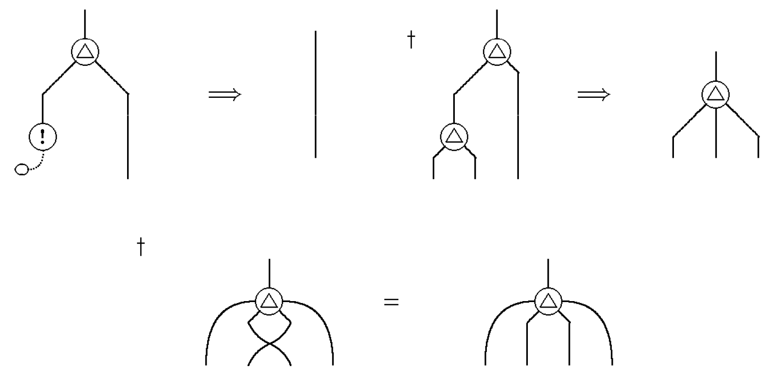
Table 5. *Box rewrite rules*

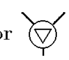


and dual rewrites for ?

†These contraction nodes are in general n-ary.

Table 6. *Cut elimination rules for duplication and contraction*



and dual rewrites for 

†These contraction nodes are in general n-ary.

storage rules

$$\frac{!\,\Gamma, A \vdash \,?\,\Delta}{!\,\Gamma, \,?\,A \vdash \,?\,\Delta} \qquad\qquad \frac{!\,\Gamma \vdash \,?\,\Delta, B}{!\,\Gamma \vdash \,?\,\Delta, \,!\,B}$$

which will show us how the tensorial strength enters the story.

It is a fairly routine exercise to show that in the presence of the cut rule the following axioms and rules are equivalent to the rules above.

$$\begin{array}{rcl} ?\,\bot & \vdash & \bot \\ ?\,A \otimes \,!\,B & \vdash & ?\,(A \otimes \,!\,B) \\ ??\,A & \vdash & ?\,A \\ ?\,(A \oplus B) & \vdash & ?\,A \oplus \,?\,B \end{array}$$

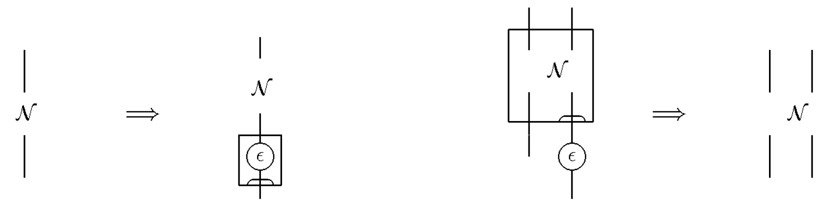
$$\begin{array}{rcl} \top & \vdash & !\,\top \\ !\,(A \oplus \,?\,B) & \vdash & !\,A \oplus \,?\,B \\ !\,A & \vdash & !!\,A \\ !\,A \otimes \,!\,B & \vdash & !\,(A \otimes B) \end{array}$$

$$\frac{A \vdash B}{?\,A \vdash \,?\,B} \qquad\qquad \frac{A \vdash B}{!\,A \vdash \,!\,B}$$

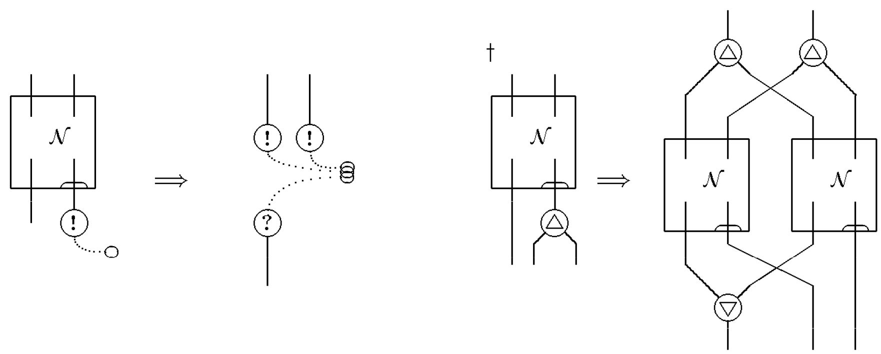
In addition, the other **! ?** rules (dereliction, thinning, and contraction) are equivalent to the following sequents (or axioms):

$$A \;\vdash\; ?\,A$$

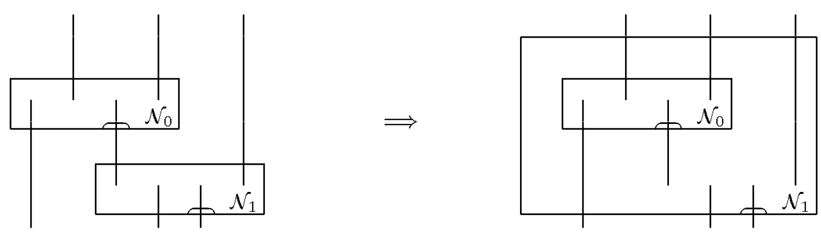
Table 7. *Cut elimination rules for* **!** *and* **?**



and dual rewrites for **?**



and dual rewrites for **?**



and three dual rewrites for other boxes

†These contraction nodes are in general n-ary.

$$\begin{array}{rcl}
\perp & \vdash & ?\,A \\
?\,A \oplus ?\,A & \vdash & ?\,A \\
!\,A & \vdash & A \\
!\,A & \vdash & \top \\
!\,A & \vdash & !\,A \otimes !\,A
\end{array}$$

We shall see derivations of these sequents *via* proof nets, so there is no need to go into too much detail now (let us leave this as a simple exercise for the reader), but two derivations will illustrate these equivalences. First, the "costrength" sequent $!\,(A \oplus ?\,B) \vdash !\,A \oplus ?\,B$

is an easy consequence of dereliction:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad ?\,B \vdash ?\,B}{A \oplus ?\,B \vdash A,\ ?\,B}}{!\,(A \oplus ?\,B) \vdash A,\ ?\,B}}{!\,(A \oplus ?\,B) \vdash\ !\,A,\ ?\,B}}{!\,(A \oplus ?\,B) \vdash\ !\,A \oplus ?\,B}$$

And conversely, the storage rule in the case when Γ, Δ are singletons is derived as follows (the general case is much the same):

$$\dfrac{\dfrac{!\,C \vdash\ !!\,C \qquad \dfrac{\dfrac{\dfrac{!\,C \vdash A,\ ?\,D}{!\,C \vdash A \oplus ?\,D}}{!!\,C \vdash\ !\,(A \oplus ?\,D)} \qquad !\,(A \oplus ?\,D) \vdash\ !\,A \oplus ?\,D}{!!\,C \vdash\ !\,A \oplus ?\,D}}{!\,C \vdash\ !\,A \oplus ?\,D} \qquad !\,A \oplus ?\,D \vdash\ !\,A,\ ?\,D}{!\,C \vdash\ !\,A,\ ?\,D}$$

So with this sequent calculus in front of us, the following definition should seem quite natural. Of course, what is at stake here is the matter of what equivalences between proofs are "natural": we are claiming here that the "natural" equivalence relation is that given by the reductions and expansions of proof nets, as we shall see that they are equivalent to the categorical semantics described below.

Definition 2.1. (weakly distributive categories with storage) A (symmetric) weakly distributive category $\mathbf{C}$ admits storage if there is a triple $?:\mathbf{C} \longrightarrow \mathbf{C}$ and a cotriple $!:\mathbf{C} \longrightarrow \mathbf{C}$ (with the structure maps $A \xrightarrow{\eta_A} ?\,A \xleftarrow{\mu_A} ??\,A$ and $A \xleftarrow{\epsilon_A}\ !\,A \xrightarrow{\delta_A}\ !!\,A$), with the following properties:

1 the functor $?$ is comonoidal with respect to the cotensor $\oplus$, and $!$ is monoidal with respect to the tensor $\otimes$; furthermore, the natural transformations η, μ (respectively ϵ, δ) are comonoidal (respectively monoidal),

2 the functor $?$ is strong with respect to the monoidal functor $!$, and $!$ is costrong with respect to the comonoidal functor $?$; furthermore, the structure maps η, μ (respectively ϵ, δ) are strong (respectively costrong) natural transformations, and

3 each free $?$-algebra carries (naturally) the structure of a commutative $\oplus$-monoid and the algebra maps are monoid maps; each free $!$-coalgebra carries (naturally) the structure of a commutative $\otimes$-comonoid and the coalgebra maps are comonoid maps.

We shall make these clauses explicit, by displaying the required natural transformations and commutative diagrams.

First, the notion of a monoidal (respectively comonoidal) functor is fairly standard—though there may be some question as to which is which, and so we show the natural transformations we need:

$?$ is comonoidal:

$$?\,\bot \xrightarrow{n_\bot} \bot$$

$$
\begin{array}{rcl}
?\,(A \oplus B) & \xrightarrow{n_{AB}} & ?\,A \oplus ?\,B \\
\multicolumn{3}{l}{!\ \text{is monoidal:}} \\
\top & \xrightarrow{m_\top} & !\,\top \\
!\,A \otimes !\,B & \xrightarrow{m_{AB}} & !\,(A \otimes B)
\end{array}
$$

Furthermore these are subject to the following commutativity conditions.

$$
\begin{tikzcd}
?\,(\bot \oplus A) \arrow[r, "n_{\bot A}"] \arrow[d, "?\,u"'] & ?\,\bot \oplus ?\,A \arrow[d, "n_\bot \oplus ?\,A"'] \\
?\,A & \bot \oplus ?\,A \arrow[l, "u"']
\end{tikzcd}
\qquad
\begin{tikzcd}
!\,\top \otimes !\,A \arrow[r, "m_{\top A}"] & !\,(\top \otimes A) \arrow[d, "!\,u"] \\
\top \otimes !\,A \arrow[u, "m_\top \otimes !\,A"] \arrow[r, "u"] & !\,A
\end{tikzcd}
$$

where $u = u^L$ are the left-unit isomorphisms; there are similar diagrams for the right-unit isomorphisms u^R.

$$
\begin{tikzcd}
?\,((A \oplus B) \oplus C) \arrow[r, "?\,a"] \arrow[d, "n"'] & ?\,(A \oplus (B \oplus C)) \arrow[d, "n"'] \\
?\,(A \oplus B) \oplus ?\,C \arrow[d, "n \oplus ?\,C"'] & ?\,A \oplus ?\,(B \oplus C) \arrow[d, "?\,A \oplus n"'] \\
(?\,A \oplus ?\,B) \oplus ?\,C \arrow[r, "a"] & ?\,A \oplus (?\,B \oplus ?\,C)
\end{tikzcd}
\quad
\begin{tikzcd}
(!\,A \otimes !\,B) \otimes !\,C \arrow[r, "a"] \arrow[d, "m \otimes !\,C"] & !\,A \otimes (!\,B \otimes !\,C) \arrow[d, "!\,A \otimes m"] \\
!\,(A \otimes B) \otimes !\,C \arrow[d, "m"] & !\,A \otimes !\,(B \otimes C) \arrow[d, "m"] \\
!\,((A \otimes B) \otimes C) \arrow[r, "!\,a"] & !\,(A \otimes (B \otimes C))
\end{tikzcd}
$$

where a are the associativity isomorphisms.

$$
\begin{tikzcd}
?\,(A \oplus B) \arrow[r, "n"] \arrow[d, "?\,s"'] & ?\,A \oplus ?\,B \arrow[d, "s"'] \\
?\,(B \oplus A) \arrow[r, "n"] & ?\,B \oplus ?\,A
\end{tikzcd}
\qquad
\begin{tikzcd}
!\,A \otimes !\,B \arrow[r, "m"] \arrow[d, "s"] & !\,(A \otimes B) \arrow[d, "!\,s"] \\
!\,B \otimes !\,A \arrow[r, "m"] & !\,(B \otimes A)
\end{tikzcd}
$$

where s are the symmetry isomorphisms. We leave the diagrams expressing that m, n are natural transformations to the reader. Finally we have the diagrams expressing that η, μ, ϵ, δ are monoidal natural transformations:

$$\begin{tikzcd}
?(A \oplus B) \arrow[r, "n"] & ?A \oplus ?B \\
& A \oplus B \arrow[ul, "\eta"] \arrow[u, "\eta \oplus \eta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \otimes !B \arrow[r, "m"] \arrow[d, "\epsilon \otimes \epsilon"] & !(A \otimes B) \arrow[dl, "\epsilon"] \\
A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
\bot & ?\bot \arrow[l, "n"'] \\
& \bot \arrow[u, "\eta"'] \arrow[ul, equal]
\end{tikzcd}
\qquad
\begin{tikzcd}
!\top \arrow[d, "\epsilon"] & \top \arrow[l, "m"'] \arrow[dl, equal] \\
\top &
\end{tikzcd}$$

$$\begin{tikzcd}
??A \oplus ??B \arrow[r, "\mu \oplus \mu"] & ?A \oplus ?B \\
?(?A \oplus ?B) \arrow[u, "n"] & \\
??(A \oplus B) \arrow[u, "?n"] \arrow[r, "\mu"] & ?(A \oplus B) \arrow[uu, "n"]
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \otimes !B \arrow[r, "\delta \otimes \delta"] \arrow[dd, "m"] & !!A \otimes !!B \arrow[d, "m"] \\
& !(!A \otimes !B) \arrow[d, "!m"] \\
!(A \otimes B) \arrow[r, "\delta"] & !!(A \otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
?\bot \arrow[r, "n"] & \bot \\
??\bot \arrow[u, "?n"] \arrow[r, "\mu"] & ?\bot \arrow[u, "n"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\top \arrow[r, "m"] \arrow[d, "m"] & !\top \arrow[d, "!m"] \\
!\top \arrow[r, "\delta"] & !!\top
\end{tikzcd}$$

Second, the notion of relative (co)strength needs some explanation. By this we mean the following.

Definition 2.2. (strong and costrong functors) A functor $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ is strong with respect to (or relative to) a $\otimes$-monoidal functor $G\colon \mathbf{C} \longrightarrow \mathbf{D}$ if there is a natural transformation

$$\theta\colon F(A) \otimes G(B) \longrightarrow F(A \otimes GB)$$

subject to the following commutativity conditions. (Costrength is the dual notion, so that F is costrong with respect to a $\oplus$-comonoidal functor G if the natural transformation goes in the opposite direction.)

$$\begin{tikzcd}
F(A) \otimes \top \arrow[r, "FA \otimes m"] \arrow[d, "u"', "\cong"] & F(A) \otimes G(\top) \arrow[dd, "\theta"] \\
F(A) \arrow[d, "(Fu)^{-1}"', "\cong"] & \\
F(A \otimes \top) \arrow[r, "F(A \otimes m)"] & F(A \otimes G(\top))
\end{tikzcd}$$

$$\begin{tikzcd}
(F(A) \otimes G(B)) \otimes G(C) \arrow[r, "\theta \otimes GC"] \arrow[d, "a"'] & F(A \otimes G(B)) \otimes G(C) \arrow[r, "\theta"] & F((A \otimes G(B)) \otimes G(C)) \arrow[d, "Fa"] \\
F(A) \otimes (G(B) \otimes G(C)) \arrow[d, "FA \otimes m"'] & & F(A \otimes (G(B) \otimes G(C))) \arrow[d, "F(A \otimes m)"] \\
F(A) \otimes G(B \otimes C) \arrow[rr, "\theta"] & & F(A \otimes G(B \otimes C))
\end{tikzcd}$$

where m refers to the monoidal structure of G, as *e.g.* with **!** above. (The diagrams for a costrength are dual.)

Definition 2.3. (strong and costrong natural transformations) A natural transformation $\alpha: F \longrightarrow F'$ between strong functors (with respect to a monoidal functor G) is strong if the following diagram commutes. (And dually for costrength.)

$$\begin{tikzcd}
F(A) \otimes G(B) \arrow[r, "\alpha \otimes GB"] \arrow[d, "\theta"'] & F'(A) \otimes G(B) \arrow[d, "\theta'"'] \\
F(A \otimes G(B)) \arrow[r, "\alpha"] & F'(A \otimes G(B))
\end{tikzcd}$$

where $\theta,\ \theta'$ are the strength natural transformations for $F,\ F'$ respectively.

So the second condition of Definition 2.1 means that we have natural transformations

? is strong w.r.t. **!** :

$$\mathbf{?}\,A \otimes \mathbf{!}\,B \xrightarrow{\theta_{AB}} \mathbf{?}\,(A \otimes \mathbf{!}\,B)$$

! is costrong w.r.t. **?** :

$$\mathbf{!}\,(A \oplus \mathbf{?}\,B) \xrightarrow{\phi_{AB}} \mathbf{!}\,A \oplus \mathbf{?}\,B$$

satisfying the appropriate commutativity conditions. We shall leave the reader the exercise of showing that if F is strong, so is FF, and that the identity functor is strong, so as to make sense of the requirement that $\eta,\ \mu$ (respectively $\epsilon,\ \delta$) are strong (respectively costrong) natural transformations.

This then gives the following diagrams:

$$
\begin{tikzcd}
?A \otimes \top \arrow[r, "?A \otimes m"] \arrow[d, "u"', "\cong"] & ?A \otimes !\top \arrow[dd, "\theta"] \\
?A \arrow[d, "?u^{-1}"', "\cong"] & \\
?(A \otimes \top) \arrow[r, "?(A \otimes m)"] & ?(A \otimes !\top)
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \oplus ?\bot \arrow[r, "!A \oplus n"] & !A \oplus \bot \\
& !A \arrow[u, "u^{-1}", "\cong"'] \\
!(A \oplus ?\bot) \arrow[uu, "\phi"'] \arrow[r, "!(A \oplus n)"] & !(A \oplus \bot) \arrow[u, "!u", "\cong"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
(?A \otimes !B) \otimes !C \arrow[rr, "\theta \otimes !C"] \arrow[d, "a"] & & ?(A \otimes !B) \otimes !C \arrow[d, "\theta"'] \\
?A \otimes (!B \otimes !C) \arrow[d, "?A \otimes m"] & & ?((A \otimes !B) \otimes !C) \arrow[d, "?a"'] \\
?A \otimes !(B \otimes C) \arrow[dr, "\theta"'] & & ?(A \otimes (!B \otimes !C)) \arrow[dl, "?(A \otimes m)"] \\
& ?(A \otimes !(B \otimes C)) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
!(A \oplus ?B) \oplus ?C \arrow[rr, "\phi \oplus ?C"] & & (!A \oplus ?B) \oplus ?C \\
!((A \oplus ?B) \oplus ?C) \arrow[u, "\phi"'] & & !A \oplus (?B \oplus ?C) \arrow[u, "a^{-1}"] \\
!(A \oplus (?B \oplus ?C)) \arrow[u, "!a^{-1}"'] & & !A \oplus ?(B \oplus C) \arrow[u, "!A \oplus n"] \\
& !(A \oplus ?(B \oplus C)) \arrow[ul, "?(A \oplus n)"] \arrow[ur, "\phi"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
?A \otimes !B \arrow[r, "\theta"] & ?(A \otimes !B) \\
& A \otimes !B \arrow[ul, "\eta \otimes !B"] \arrow[u, "\eta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
!(A \oplus ?B) \arrow[r, "\phi"] \arrow[d, "\epsilon"] & !A \oplus ?B \arrow[dl, "\epsilon \oplus ?B"] \\
A \oplus ?B &
\end{tikzcd}
$$

$$
\begin{tikzcd}
??A \otimes !B \arrow[r, "\mu \otimes !B"] \arrow[d, "\theta"'] & ?A \otimes !B \arrow[dd, "\theta"'] \\
?(?A \otimes !B) \arrow[d, "?\theta"'] & \\
??(A \otimes !B) \arrow[r, "\mu"] & ?(A \otimes !B)
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \oplus ?B \arrow[r, "\delta \oplus ?B"] & !!A \oplus ?B \\
& !(!A \oplus ?B) \arrow[u, "\phi"] \\
!(A \oplus ?B) \arrow[uu, "\phi"] \arrow[r, "\delta"] & !!(A \oplus ?B) \arrow[u, "!\phi"]
\end{tikzcd}
$$

Also, $!$ and $?$ must be associatively bistrong, meaning that the next diagrams must commute (here ∂, ∂' are the appropriate weak distributivities that reassociate $\otimes$ and $\oplus$).

$$
\begin{tikzcd}
?A \otimes !B \arrow[r, "\theta"] \arrow[d, "?A \otimes \delta"'] & ?(A \otimes !B) \arrow[d, "?(A \otimes \delta)"'] \\
?A \otimes !!B \arrow[r, "\theta"] & ?(A \otimes !!B)
\end{tikzcd}
\qquad
\begin{tikzcd}
!(A \oplus ?B) \arrow[r, "\phi"] & !A \oplus ?B \\
!(A \oplus ??B) \arrow[r, "\phi"] \arrow[u, "!(A \oplus \mu)"'] & !A \oplus ??B \arrow[u, "!A \oplus \mu"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
?A \otimes !(B \oplus ?C) \arrow[r, "?A \otimes \phi"] \arrow[d, "\theta"'] & ?A \otimes (!B \oplus ?C) \arrow[dd, "\partial"'] \\
?(A \otimes !(B \oplus ?C)) \arrow[d, "?(A \otimes \phi)"'] & \\
?(A \otimes (!B \oplus ?C)) \arrow[d, "?\partial"'] & (?A \otimes !B) \oplus ?C \arrow[dd, "\theta \oplus ?C"'] \\
?((A \otimes !B) \oplus ?C) \arrow[d, "n"'] & \\
?(A \otimes !B) \oplus ??C \arrow[r, "id \oplus \mu"] & ?(A \otimes !B) \oplus ?C
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \oplus (?B \otimes !C) \arrow[r, "!A \oplus \theta"] & !A \oplus ?(B \otimes !C) \\
& !(A \oplus ?(B \otimes !C)) \arrow[u, "\phi"'] \\
(!A \oplus ?B) \otimes !C \arrow[uu, "\partial"'] & !(A \oplus (?B \otimes !C)) \arrow[u, "!(A \oplus \theta)"'] \\
& !((A \oplus ?B) \otimes !C) \arrow[u, "!\partial"'] \\
!(A \oplus ?B) \otimes !C \arrow[r, "id \otimes \delta"] \arrow[uu, "\phi \otimes !C"'] & !(A \oplus ?B) \otimes !!C \arrow[u, "m"']
\end{tikzcd}
$$

Third, that the free $?$-algebras carry commutative $\oplus$-monoid structure (respectively, free $!$-coalgebras carry commutative $\otimes$-comonoid structure) means that we must have monoidal natural transformations

$$
\begin{array}{rcl}
\bot & \xrightarrow{i_A} & ?A \\
?A \oplus ?A & \xrightarrow{c_A} & ?A
\end{array}
\qquad\qquad
\begin{array}{rcl}
!A & \xrightarrow{e_A} & \top \\
!A & \xrightarrow{d_A} & !A \otimes !A
\end{array}
$$

subject to the usual commutativity conditions (the "Mac Lane pentagons", the triangles for the units, and the (co)associativity and (co)commutativity of the (co)multiplication). We shall leave these diagrams to the reader. Furthermore, the (co)monoid structure maps are to be (co)algebra maps. This means the following must commute.

$$
\begin{tikzcd}
\bot \arrow[rr, equal] \arrow[dr, "i"] & & \bot \\
& ?\bot \arrow[ur, "n"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\top \arrow[rr, equal] \arrow[dr, "m"] & & \top \\
& !\top \arrow[ur, "e"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bot \oplus \bot \arrow[r, "u"] & \bot \\
{?}\bot \oplus {?}\bot \arrow[u, "n \oplus n"] \arrow[r, "c"] & {?}\bot \arrow[u, "n"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\top \arrow[r, "u^{-1}"] \arrow[d, "m"] & \top \otimes \top \arrow[d, "m \otimes m"] \\
{!}\top \arrow[r, "d"] & {!}\top \otimes {!}\top
\end{tikzcd}
$$

$$
\begin{tikzcd}
{?}A \oplus {?}B & \bot \oplus \bot \arrow[l, "i \oplus i"'] \arrow[d, "u"] \\
{?}(A \oplus B) \arrow[u, "n"] & \bot \arrow[l, "i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
{!}A \otimes {!}B \arrow[r, "e \otimes e"] \arrow[d, "m"] & \top \otimes \top \arrow[d, "u"] \\
{!}(A \otimes B) \arrow[r, "e"] & \top
\end{tikzcd}
$$

$$
\begin{tikzcd}
({?}A \oplus {?}A) \oplus ({?}B \oplus {?}B) \arrow[r, "c \oplus c"] & {?}A \oplus {?}B \\
({?}A \oplus {?}B) \oplus ({?}A \oplus {?}B) \arrow[u, "s"] & \\
{?}(A \oplus B) \oplus {?}(A \oplus B) \arrow[u, "n \oplus n"] \arrow[r, "c"] & {?}(A \oplus B) \arrow[uu, "n"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
{!}A \otimes {!}B \arrow[r, "d \otimes d"] \arrow[dd, "m"] & ({!}A \otimes {!}A) \otimes ({!}B \otimes {!}B) \arrow[d, "s"] \\
& ({!}A \otimes {!}B) \otimes ({!}A \otimes {!}B) \arrow[d, "m \otimes m"] \\
{!}(A \otimes B) \arrow[r, "d"] & {!}(A \otimes B) \otimes {!}(A \otimes B)
\end{tikzcd}
$$

where s (in each case) is the canonical isomorphism derived from symmetry and associativity.

$$
\begin{tikzcd}
\bot \arrow[r, "i"] & {?}A \\
{?}\bot \arrow[u, "n"] \arrow[r, "{?}i"] & {??}A \arrow[u, "\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
{!}A \arrow[r, "e"] \arrow[d, "\delta"] & \top \arrow[d, "m"] \\
{!!}A \arrow[r, "{!}e"] & {!}\top
\end{tikzcd}
$$

$$
\begin{tikzcd}
?A \oplus ?A \arrow[r, "c"] & ?A \\
??A \oplus ??A \arrow[u, "\mu \oplus \mu"] & \\
?(?A \oplus ?A) \arrow[u, "n"] \arrow[r, "?c"] & ??A \arrow[uu, "\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "d"] \arrow[dd, "\delta"] & !A \otimes !A \arrow[d, "\delta \otimes \delta"] \\
& !!A \otimes !!A \arrow[d, "m"] \\
!!A \arrow[r, "!d"] & !(!A \otimes !A)
\end{tikzcd}
$$

The monoid natural transformations i, c must be strong with respect to $!$, and the comonoid natural transformations e, d must be costrong with respect to $?$.

$$
\begin{tikzcd}
\bot \arrow[r, "i"] & ?(A \otimes !B) \\
\bot \otimes \top \arrow[u, "u"] & \\
\bot \otimes !B \arrow[u, "\bot \otimes e"] \arrow[r, "i \otimes !B"] & ?A \otimes !B \arrow[uu, "\theta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
!(A \oplus ?B) \arrow[r, "e"] \arrow[dd, "\phi"] & \top \arrow[d, "u^{-1}"] \\
& \top \oplus \bot \arrow[d, "\top \oplus i"] \\
!A \oplus ?B \arrow[r, "e \oplus ?B"] & \top \oplus ?B
\end{tikzcd}
$$

$$
\begin{tikzcd}
?(A \otimes !B) \oplus ?(A \otimes !B) \arrow[r, "c"] & ?(A \otimes !B) \\
(?A \otimes !B) \oplus (?A \otimes !B) \arrow[u, "\theta \oplus \theta"] & \\
((?A \otimes !B) \oplus ?A) \otimes !B \arrow[u, "\partial"] & \\
((?A \oplus ?A) \otimes !B) \otimes !B \arrow[u, "\partial' \otimes !B"] & \\
(?A \oplus ?A) \otimes (!B \otimes !B) \arrow[u, "a^{-1}"] & \\
(?A \oplus ?A) \otimes !B \arrow[u, "id \otimes d"] \arrow[r, "c \otimes !B"] & ?A \otimes !B \arrow[uuuuu, "\theta"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
!(A \oplus ?B) \arrow[r, "d"] \arrow[dddddd, "\phi"] & !(A \oplus ?B) \otimes !(A \oplus ?B) \arrow[d, "\phi \otimes \phi"] \\
& (!A \oplus ?B) \otimes (!A \oplus ?B) \arrow[d, "\partial"] \\
& ((!A \oplus ?B) \otimes !A) \oplus ?B \arrow[d, "\partial' \oplus ?B"] \\
& ((!A \otimes !A) \oplus ?B) \oplus ?B \arrow[d, "a"] \\
& (!A \otimes !A) \oplus (?B \oplus ?B) \arrow[d, "id \oplus c"] \\
& \\
!A \oplus ?B \arrow[r, "d \oplus ?B"] & (!A \otimes !A) \oplus ?B
\end{tikzcd}
$$

Finally, for the (co)algebra maps to be (co)monoid maps, we require the following to commute.

$$
\begin{tikzcd}
?A \oplus ?A \arrow[r, "c"] & ?A \\
??A \oplus ??A \arrow[u, "\mu \oplus \mu"] \arrow[r, "c"] & ??A \arrow[u, "\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "d"] \arrow[d, "\delta"'] & !A \otimes !A \arrow[d, "\delta \otimes \delta"] \\
!!A \arrow[r, "d"] & !!A \otimes !!A
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bot \arrow[rr, "i"] \arrow[dr, "i"'] & & ?A \\
& ??A \arrow[ur, "\mu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[rr, "e"] \arrow[dr, "\delta"'] & & \top \\
& !!A \arrow[ur, "e"'] &
\end{tikzcd}
$$

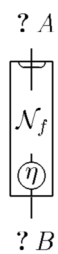
3. Equivalences

We must now show that the categorical semantics is complete and sound for WDC+ **! ?** — that in effect we have the right categorical notion. In one direction, we shall show that the structure of formulas and equivalence classes of derivations forms a polycategory (Cockett and Seely 1991) whose category part is a weakly distributive category admitting storage. Here two derivations of the same sequent are equivalent if they induce proof nets that may be linked by a sequence of reductions and expansions of nets. For the converse, we must show that all the reductions and expansions do correspond to commutative diagrams in weakly distributive categories with storage. Of course this amounts to lots of very boring calculations: we shall illustrate sufficient to (we hope) convince the reader of the truth of these assertions. We are helped in this task by knowing that much is completely analogous to the situation for intuitionistic linear logic, where similar matters have been the object of much study (*e.g.* (Benton *et al* 1992; Bierman 1995)), and so we concentrate on those aspects peculiar to the current context.

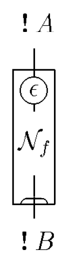
3.1. *Nets form a weakly distributive category admitting storage*

To show that the polycategory of formulas and equivalence classes of derivations induces a weakly distributive category admitting storage, we must display rewrites of various nets corresponding to the required commutative diagrams. We do this in a series of Figures, each labelled with the appropriate commutativity condition it verifies.

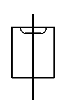
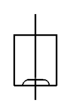
However, before we begin, we must show what nets correspond to the basic morphisms of our definitions. We follow the notation of Definition 2.1. First, we need to know the effect of **!** and **?** on morphisms: given a net $\mathcal{N}_f$ corresponding to a morphism $A \xrightarrow{f} B$, we have the following nets corresponding to (respectively) $?A \xrightarrow{?f} ?B$ and $!A \xrightarrow{!f} !B$.



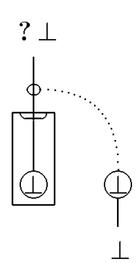
and

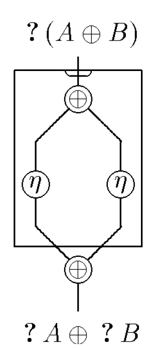


Of course, η and ϵ are given by the links of those names. μ and δ are given by the appropriate storage boxes applied to identity links:

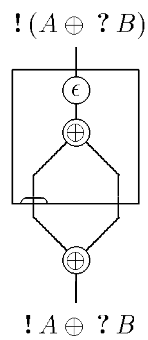
$\mu =$  and $\delta =$ 

The comonoidal structure of **?** is given by these nets: (the monoidal structure of **!** is dual—just turn the page upside-down!)

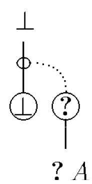
$n_\perp =$ 

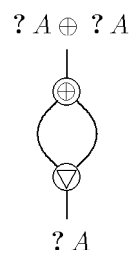
and $n_{AB} =$ 

The cotensorial costrength of **!** is given by this net (the tensorial strength of **?** is dual):

$\phi_{AB} =$ 

The monoid structure on $?A$ is given by

$i_A =$ 

and $c_A =$ 

(and duals for **!**.)

With these maps we then have to check lots of commutative diagrams: Figures 1 to 7b on the following pages should suffice as an illustration of the evident technique of using the reductions and expansions to derive common rewrites of the appropriate nets.

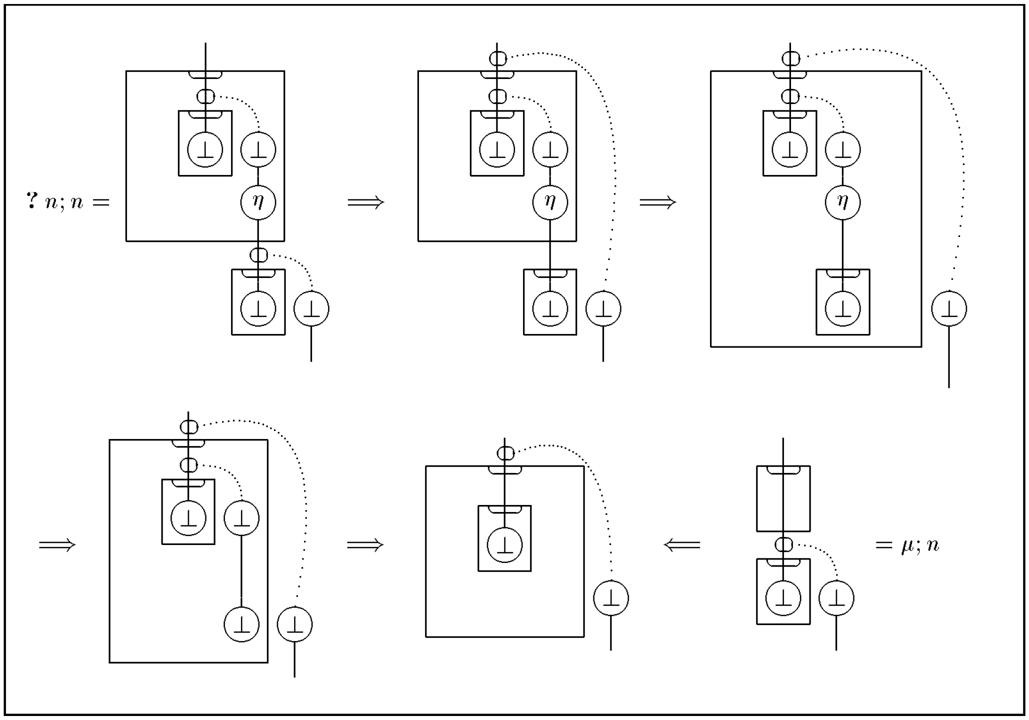


Fig. 1. $\mathbf{?}\,n; n = \mu; n$

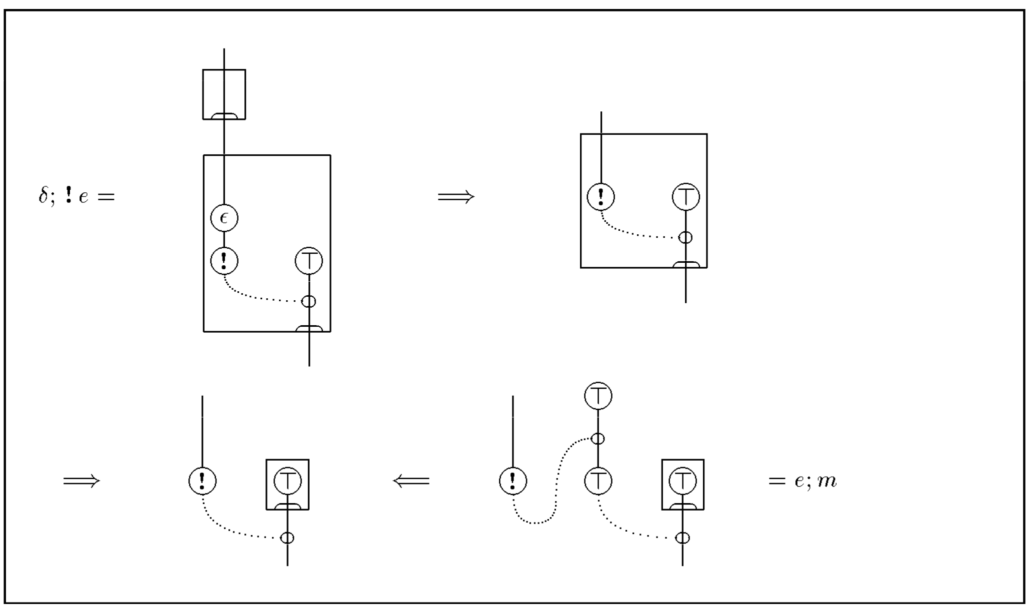


Fig. 2. $e; m = \delta; \mathbf{!}\,e$

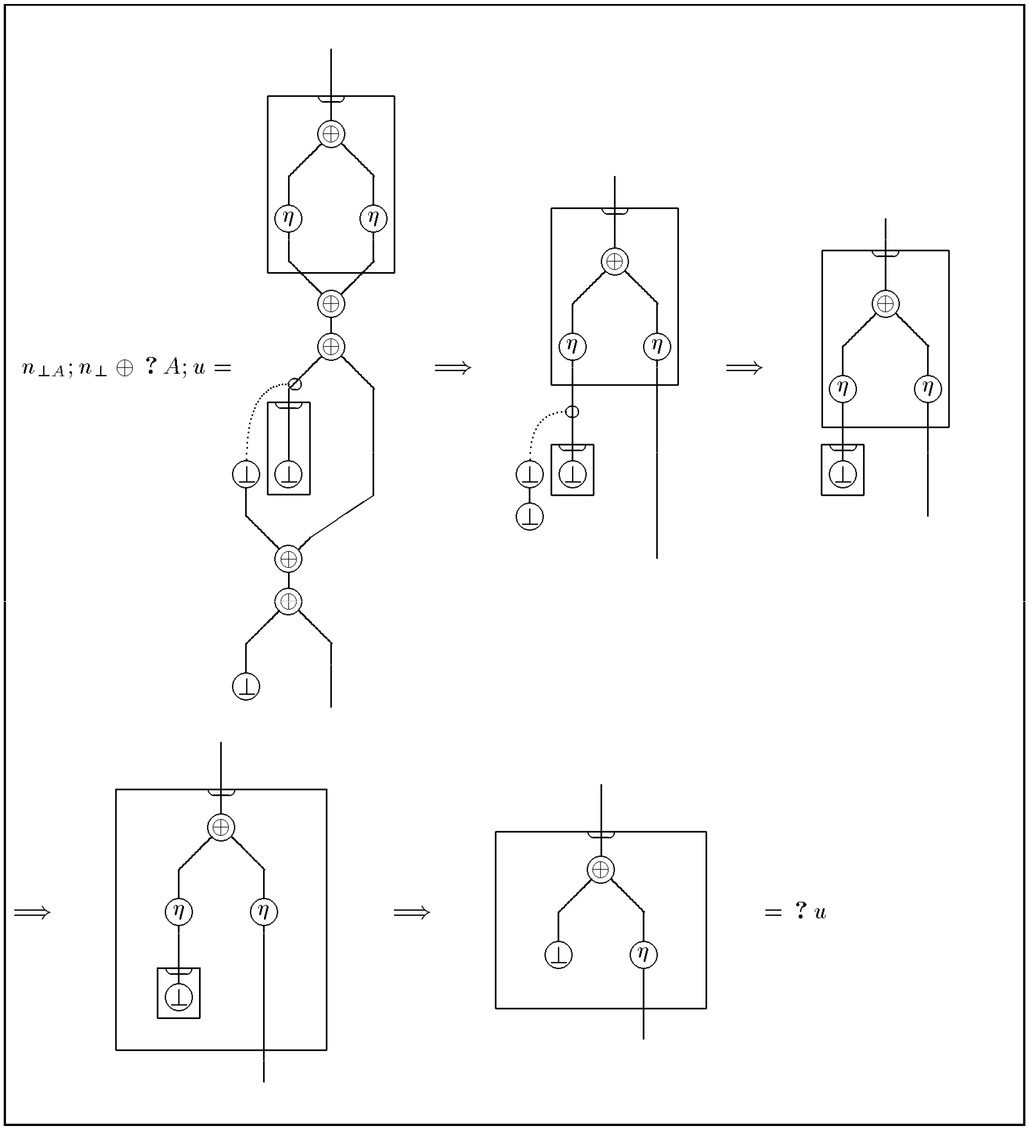


Fig. 3. $n_{\perp A}; n_\perp \oplus ?A; u = ?u$

$a; \,!A \otimes m; m =$

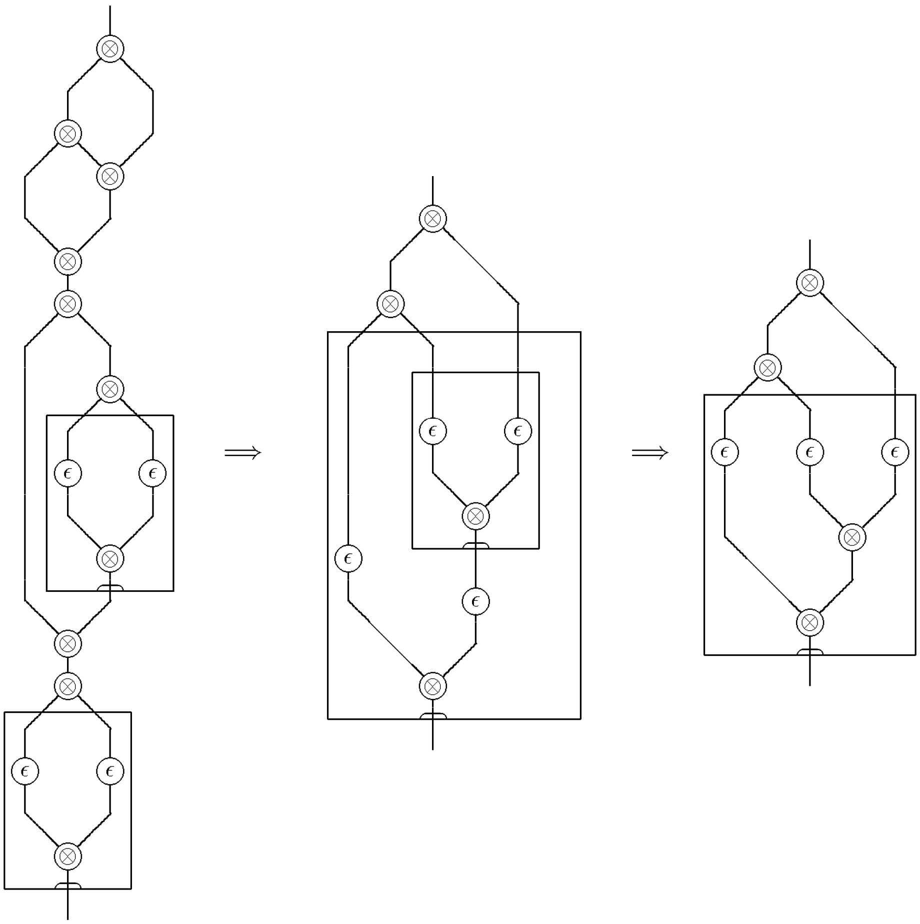


Fig. 4a. $a; \,!A \otimes m; m = m \otimes \,!C; m; \,!a$ — Expanded normal form of $a; \,!A \otimes m; m$

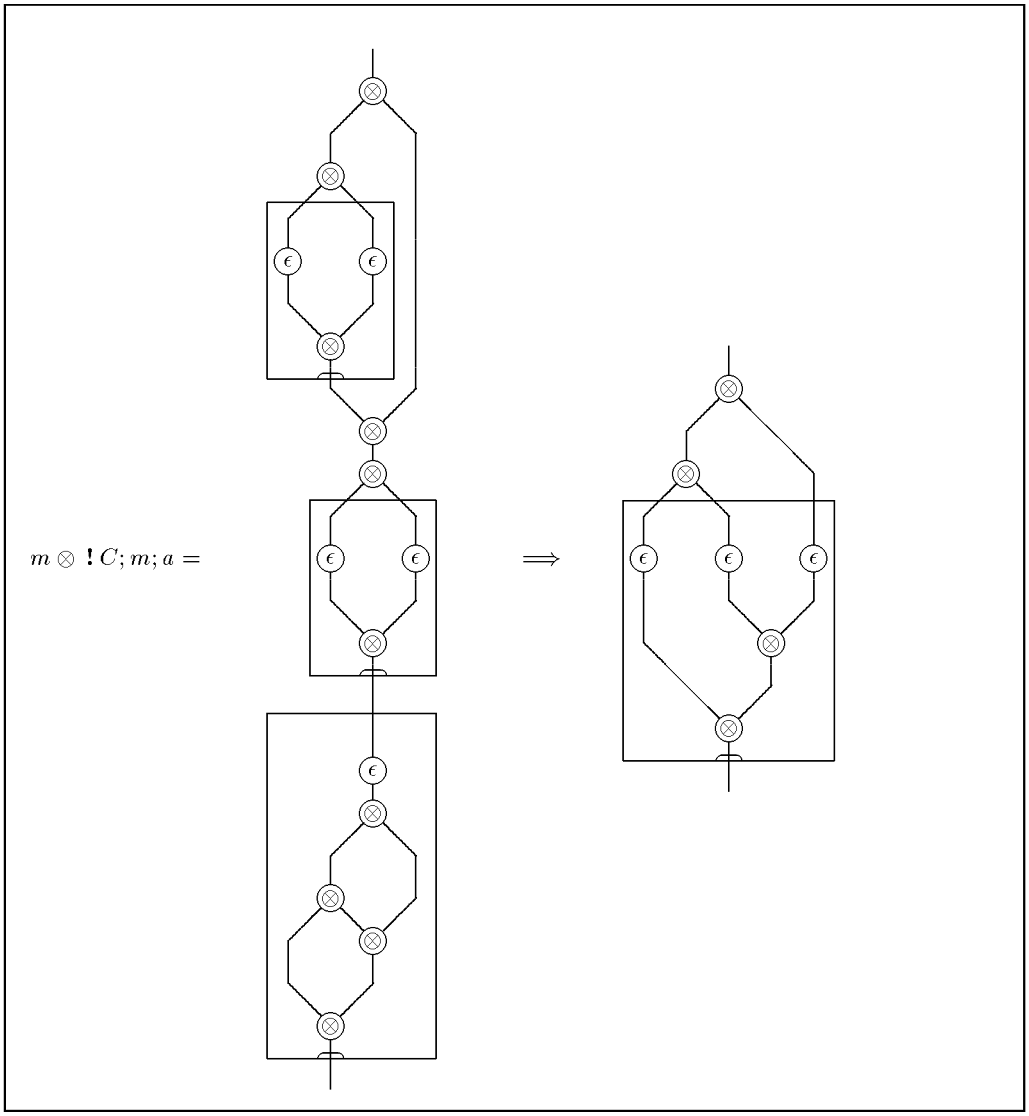
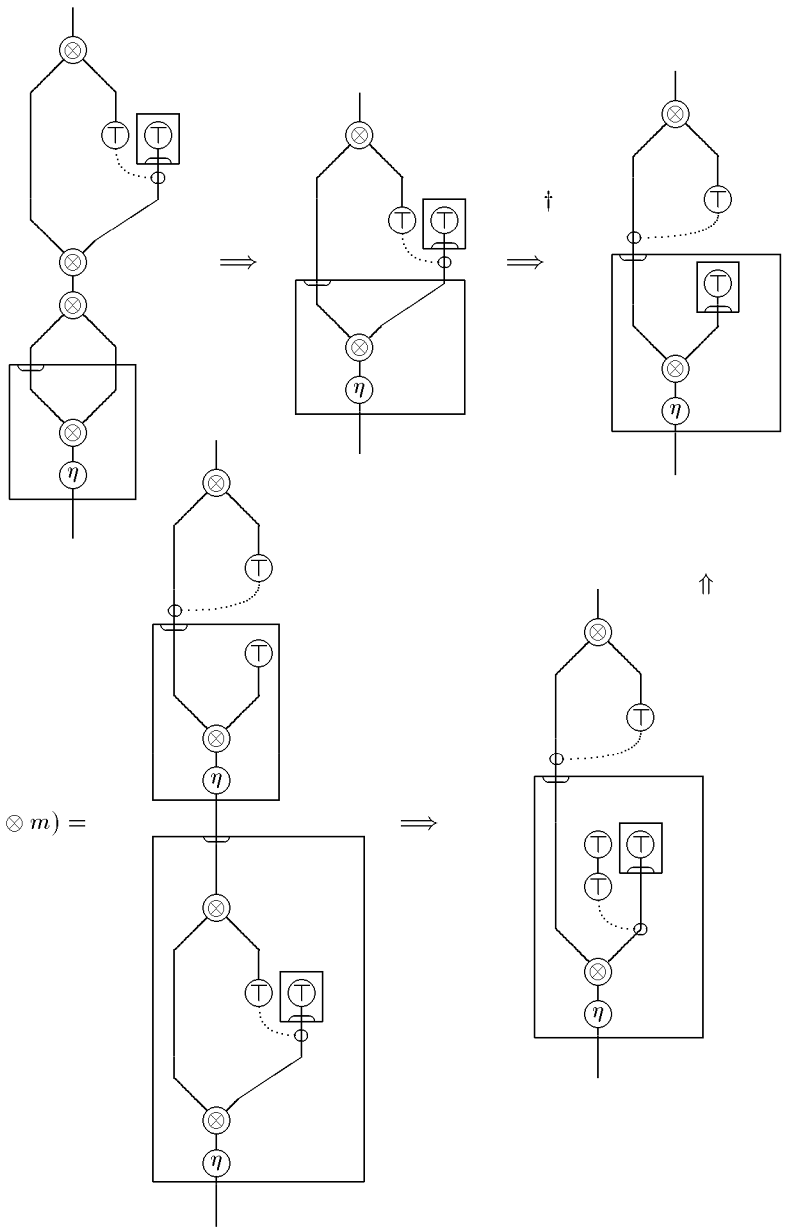


Fig. 4b. $a;\ !A \otimes m; m = m \otimes\ !C; m;\ !a$ — Expanded normal form of $m \otimes\ !C; m;\ !a$

$? A \otimes m; \theta =$

$u; (\,?\,u)^{-1};\ ?\,(A \otimes m) =$



† Note the rewiring as the lower box expands to contain the upper one.

Fig. 5. $?\,A \otimes m; \theta = u; (\,?\,u)^{-1};\ ?\,(A \otimes m)$

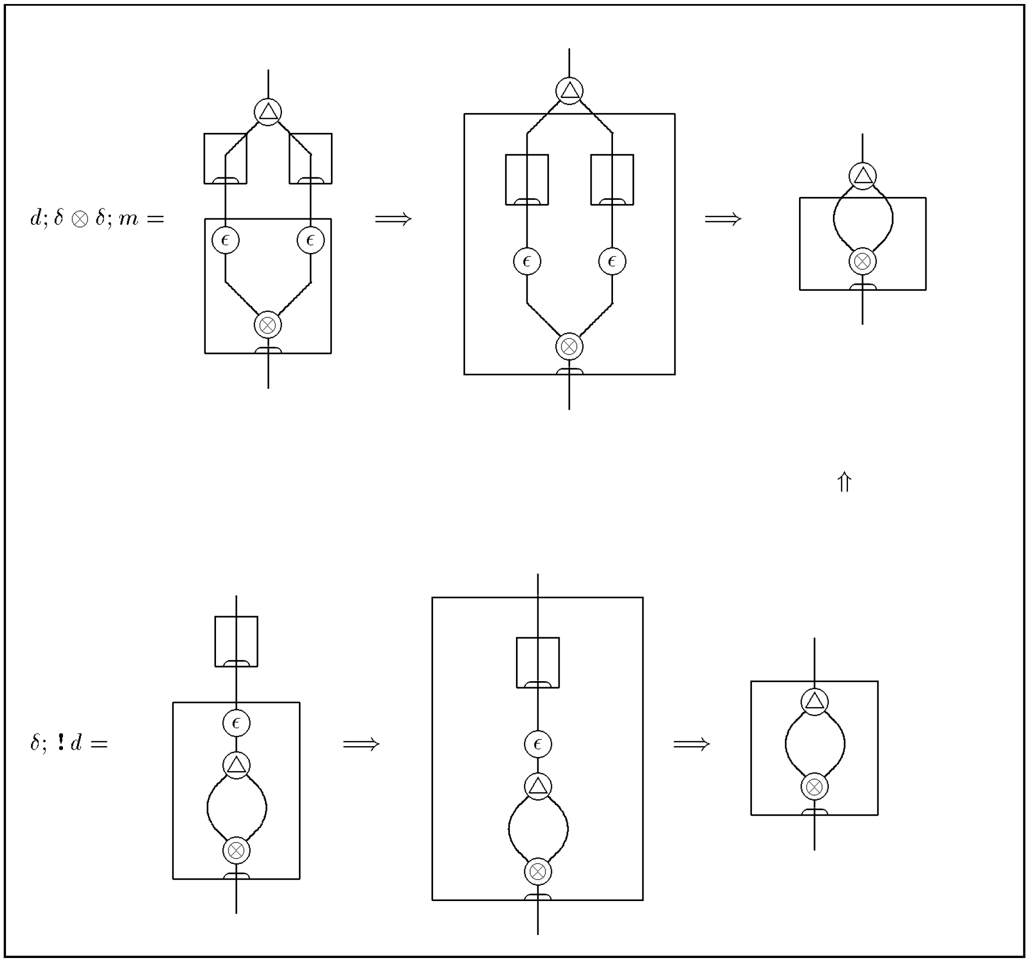


Fig. 6. $d; \delta \otimes \delta; m = \delta; \mathbf{!}\, d$

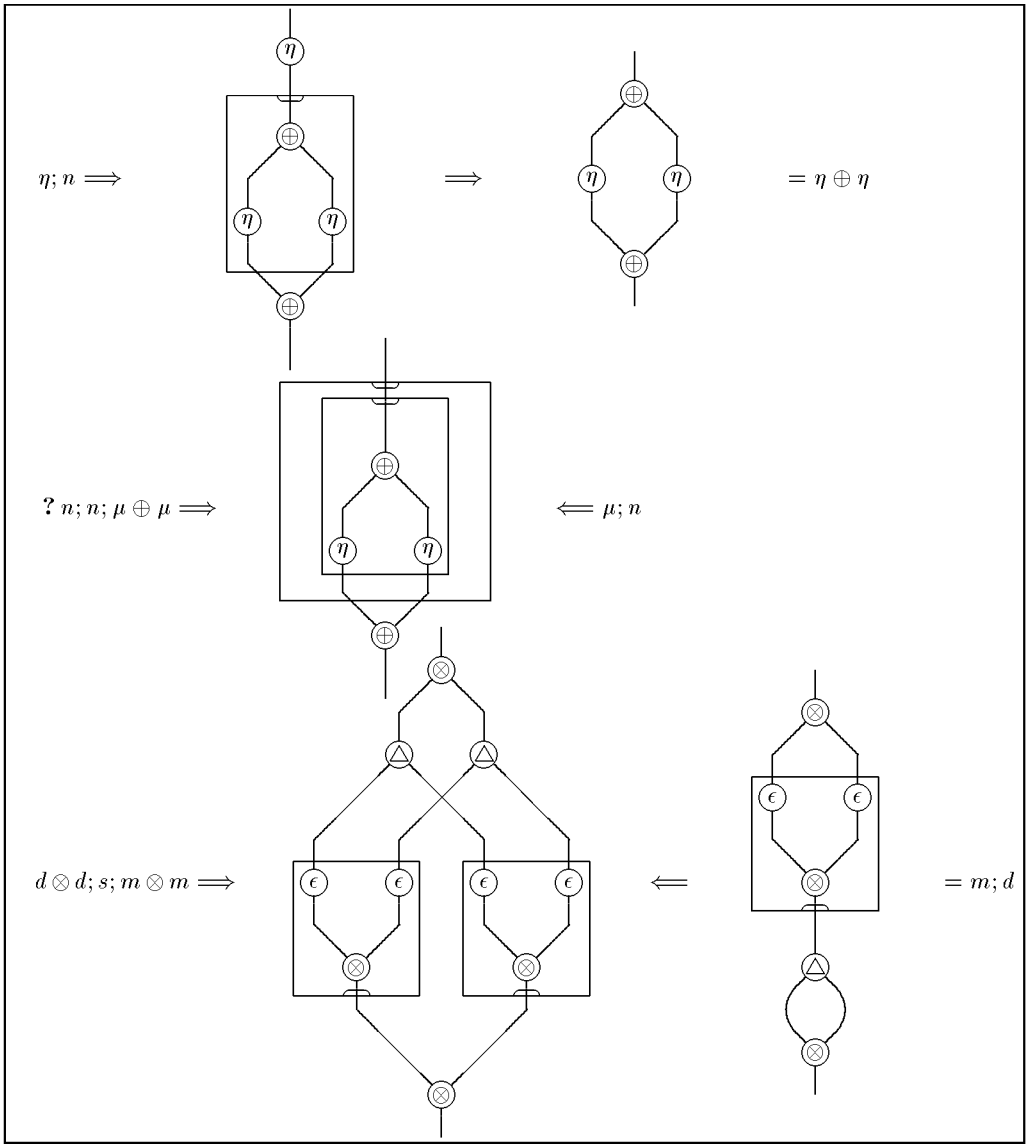


Fig. 7a. Some expanded normal forms

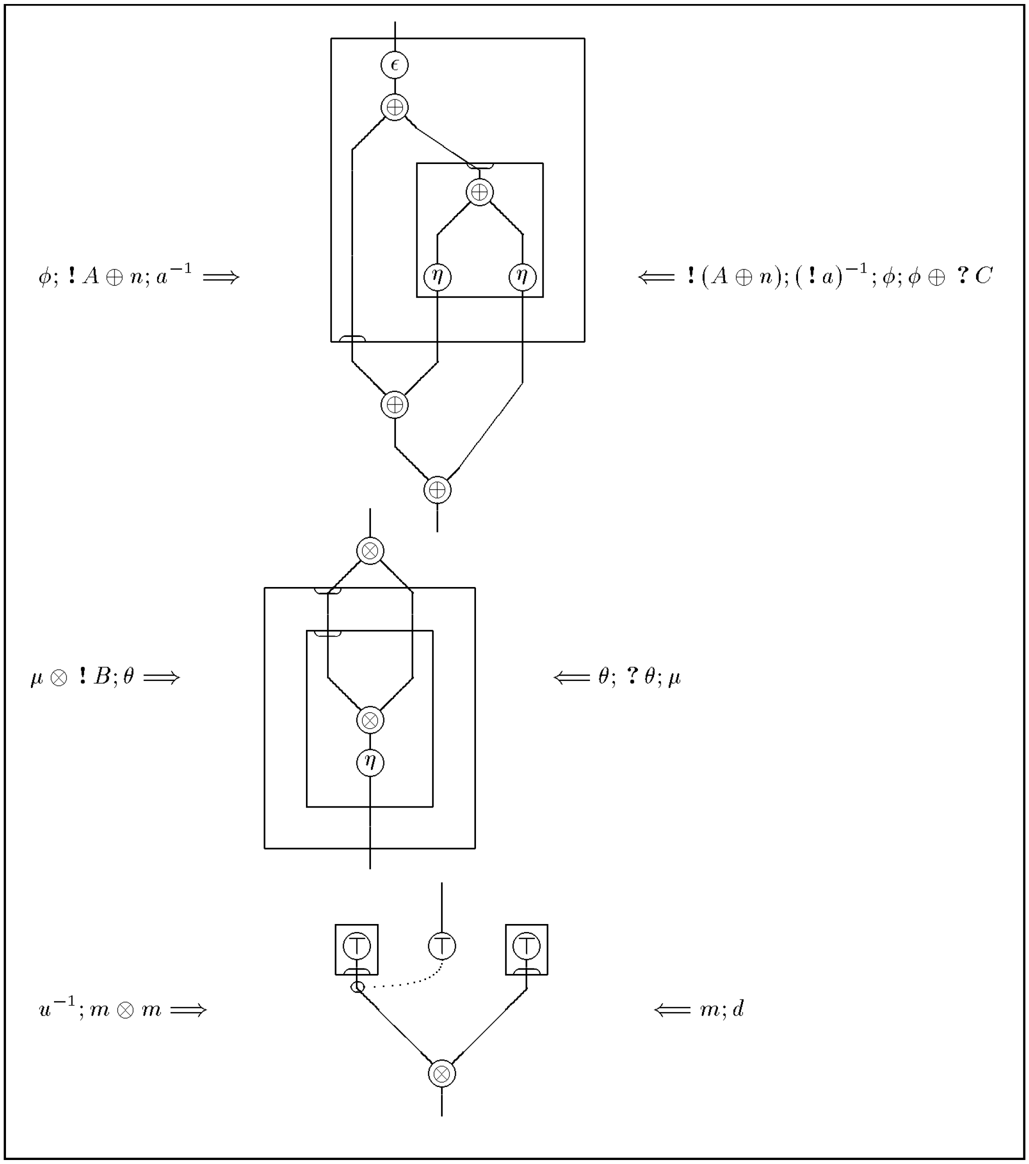
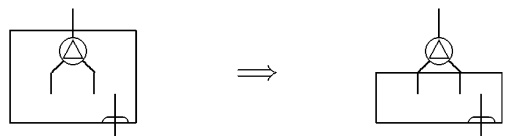


Fig. 7b. Some expanded normal forms

3.2. *The net rewrites are categorically sound*

We have also to check that the reductions and expansions of Tables 4, 5, 6, and 7 are valid: *i.e.* correspond to commutative diagrams. The only ones we need check are those involving the storage boxes. We shall generally use simplified instances of the rewrites to better illustrate the essence of each case—the more general instances then follow using soundness for weakly distributive categories.



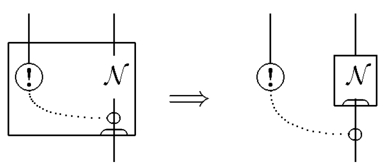
corresponds to the sequent rewrite (we simplify the sequents somewhat)

$$\dfrac{\dfrac{!A \otimes\, !A \xrightarrow{f} B}{!A \longrightarrow B}}{!A \longrightarrow\, !B} \quad\Longrightarrow\quad \dfrac{\dfrac{!A \otimes\, !A \xrightarrow{f} B}{!A \otimes\, !A \longrightarrow\, !B}}{!A \longrightarrow\, !B}$$

It is easy to verify this amounts to

$$\delta;\, !d;\, !f = d; \delta \otimes \delta; m;\, !f$$

which follows from d being an algebra map. (The other is dual.)



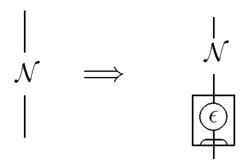
amounts to

$$\dfrac{\dfrac{!A \xrightarrow{f} B}{!C \otimes\, !A \longrightarrow B}}{!C \otimes\, !A \longrightarrow\, !B} \quad\Longrightarrow\quad \dfrac{\dfrac{!A \xrightarrow{f} B}{!A \longrightarrow\, !B}}{!C \otimes\, !A \longrightarrow\, !B}$$

which can be seen to commute from this analysis of the outer square:

$$\begin{tikzcd}
!C \otimes\, !A \arrow[r, "e \otimes 1"] \arrow[d, "\delta \otimes \delta"'] \arrow[dr, phantom, "\text{coalg.}"] & \top \otimes\, !A \arrow[r, "\cong"] \arrow[d, "m \otimes \delta"] \arrow[ddr, phantom, "\text{comon.}"] & !A \arrow[dd, "\delta"] \\
!!C \otimes\, !!A \arrow[r, "!e \otimes 1"] \arrow[d, "m"'] \arrow[dr, phantom, "\text{nat.}"] & !\top \otimes\, !!A \arrow[d, "m"] & \\
!(!C \otimes\, !A) \arrow[r, "!(e \otimes 1)"] & !(\top \otimes\, !A) \arrow[r, "\cong"] & !!A
\end{tikzcd}$$

The square marked "coalg." commutes because δ is a coalgebra map, and the rectangle "comon." commutes because $!$ is comonoidal, and δ is natural. The square "nat." is another instance of naturality.



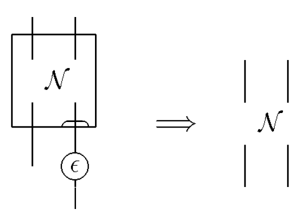
amounts to

$$!B \xrightarrow{f} !A \quad \Longrightarrow \quad \frac{\dfrac{!B \xrightarrow{f} !A \qquad \dfrac{A \longrightarrow A}{!A \longrightarrow A}}{!B \longrightarrow A}}{!B \longrightarrow !A}$$

which means $\delta; !f; \epsilon_{!} = f$:

$$\begin{tikzcd}
!B \arrow[d, "1"'] \arrow[dr, "\delta"] & \\
!B & !!B \arrow[l, "\epsilon_{!}"'] \\
!A \arrow[from=u, "f"'] & !!A \arrow[l, "\epsilon_{!}"'] \arrow[from=u, "!f"]
\end{tikzcd}$$

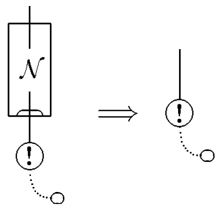
this commutes since $!$ is a cotriple and δ, ϵ are its structure (natural) transformations.



amounts to

$$\frac{\dfrac{!C \xrightarrow{f} B \oplus ?D}{!C \longrightarrow !B \oplus ?D}}{!C \longrightarrow B \oplus ?D} \quad \Longrightarrow \quad !C \xrightarrow{f} B \oplus ?D$$

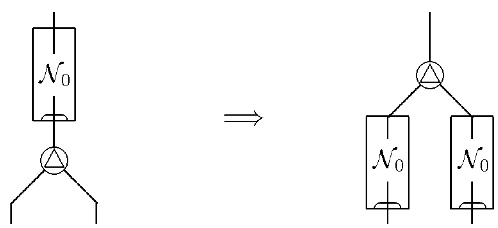
i.e. that $\delta; !f; \phi; \epsilon \oplus ?D = f$. To see this, we use the commuting diagram $\delta; !f; !\epsilon = f$ plus $\phi; \epsilon \oplus ?D = \epsilon$, which commutes because ϕ is a costrength.



amounts to

$$\dfrac{\dfrac{\,!\,C \xrightarrow{f} A\,}{\,!\,C \longrightarrow\ !\,A\,} \qquad \dfrac{X \xrightarrow{g} Y}{\,!\,A \otimes X \longrightarrow Y\,}}{\,!\,C \otimes X \longrightarrow Y\,} \qquad \Longrightarrow \qquad \dfrac{X \xrightarrow{g} Y}{\,!\,C \otimes X \longrightarrow Y\,}$$

which becomes $\delta \otimes X;\ !\,f \otimes X; e \otimes X; u; g = e \otimes X; u; g$. This commutes because $\delta; e = e$ (since δ is a comonoid map) and $!\,f; e = e$ (since e is natural).



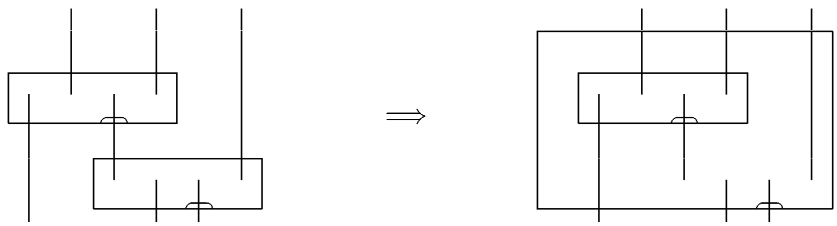
amounts to

$$\dfrac{\dfrac{\,!\,C \xrightarrow{f} B\,}{\,!\,C \longrightarrow\ !\,B\,} \qquad !\,B \longrightarrow\ !\,B \otimes\, !\,B}{\,!\,C \longrightarrow\ !\,B \otimes\, !\,B\,}$$

$$\Longrightarrow \qquad \dfrac{!\,C \longrightarrow\ !\,C \otimes\, !\,C \qquad \dfrac{\dfrac{\,!\,C \xrightarrow{f} B\,}{\,!\,C \longrightarrow\ !\,B\,} \quad \dfrac{\,!\,C \xrightarrow{f} B\,}{\,!\,C \longrightarrow\ !\,B\,}}{\,!\,C \otimes\, !\,C \longrightarrow\ !\,B \otimes\, !\,B\,}}{\,!\,C \longrightarrow\ !\,B \otimes\, !\,B\,}$$

We abbreviate the application (to identity) of the left contraction rule that generates the derivable sequent $!\,X \longrightarrow\ !\,X \otimes\, !\,X$. This commutes by naturality:

$$\begin{tikzcd}
!\,C \arrow[r, "\delta"] \arrow[d, "d"'] & !!\,C \arrow[r, "!\,f"] \arrow[d, "d"'] & !\,B \arrow[d, "d"] \\
!\,C \otimes\, !\,C \arrow[r, "\delta \otimes \delta"] & !!\,C \otimes\, !!\,C \arrow[r, "!\,f \otimes\, !\,f"] & !\,B \otimes\, !\,B
\end{tikzcd}$$



amounts to

$$\dfrac{\dfrac{!X \otimes !Y \xrightarrow{f} B}{!X \otimes !Y \longrightarrow !B} \qquad \dfrac{!B \otimes !Z \xrightarrow{g} A}{!B \otimes !Z \longrightarrow !A}}{!X \otimes !Y \otimes !Z \longrightarrow !A}$$

$$\Longrightarrow \quad \dfrac{\dfrac{\dfrac{!X \otimes !Y \xrightarrow{f} B}{!X \otimes !Y \longrightarrow !B} \qquad !B \otimes !Z \xrightarrow{g} A}{!X \otimes !Y \otimes !Z \longrightarrow A}}{!X \otimes !Y \otimes !Z \longrightarrow !A}$$

Actually, we ought to allow f, g to be a bit more general: $!X \otimes !Y \xrightarrow{f} ?U \oplus B$ and $!B \otimes !Z \xrightarrow{g} ?V \oplus A$, but we shall leave this extra step to the reader.

To verify the validity of this, we must check the commutativity of the following outer rectangle; essentially this amounts to δ being monoidal: the diagram in the middle of the displayed decomposition below, flanked by naturality squares around it. In the following diagram, we abbreviate $\delta \otimes \delta$ by δ_2, (and similarly $\delta_3 = \delta \otimes \delta \otimes \delta$), and $m_3 = m \otimes 1; m$, as in the center-right triangle in the diagram.

$$
\begin{tikzcd}
{!X \otimes !Y \otimes !Z} \arrow[rr, "\delta_3"] \arrow[d, "\delta_2 \otimes 1"'] & & {!!X \otimes !!Y \otimes !!Z} \arrow[r, "m_3"] \arrow[d, "\delta_2 \otimes 1"'] & {!(!X \otimes !Y \otimes !Z)} \arrow[d, "!(\delta_2 \otimes 1)"] \\
{!!X \otimes !!Y \otimes !Z} \arrow[r, "1 \otimes 1 \otimes \delta"] \arrow[dd, "m \otimes 1"'] & {!!X \otimes !!Y \otimes !!Z} \arrow[r, "\delta \otimes \delta \otimes 1"] \arrow[dd, "m \otimes 1"'] & {!!!X \otimes !!!Y \otimes !!Z} \arrow[r, "m_3"] \arrow[d, "m \otimes 1"'] & {!(!!X \otimes !!Y \otimes !Z)} \arrow[dd, "!(m \otimes 1)"] \\
& & {!(!!X \otimes !!Y) \otimes !!Z} \arrow[ur, "m"] \arrow[d, "!m \otimes 1"'] & \\
{!(!X \otimes !Y) \otimes !Z} \arrow[r, "1 \otimes \delta"] \arrow[d, "!f \otimes 1"'] & {!(!X \otimes !Y) \otimes !!Z} \arrow[r, "\delta \otimes 1"] & {!!(!X \otimes !Y) \otimes !!Z} \arrow[r, "m"] \arrow[d, "!!f \otimes 1"'] & {!(!(!X \otimes !Y) \otimes !Z)} \arrow[d, "!(!f \otimes 1)"] \\
{!B \otimes !Z} \arrow[rr, "\delta_2"] & & {!!B \otimes !!Z} \arrow[r, "m"] & {!(!B \otimes !Z)} \arrow[d, "!g"] \\
& & & {!A}
\end{tikzcd}
$$

The more general case (with $?U$, $?V$) just adds some extra steps involving weak distributivities—the diagrams are a bit larger, but the essence remains the same.

4. Coherence

The "completeness" we have so far is in a sense mere formalism—it only begins to gain some significance when we extend it to include the connection between coherence for the categories and normalization for the proof circuits (nets). We find that most of the ideas of (Blute *et al* 1992) carry over to the present context, and that we can apply the Empire Rewiring Theorem here.

In this paper we shall attempt a more modest goal than we did in (Blute *et al* 1992). There we gave a very detailed analysis of the rewiring steps, and showed that the reduction, expansion, and rewiring rules form a reduction/expansion system modulo equations, which in particular implies uniqueness of expanded normal forms modulo the equivalences given by the rewirings. The main problems in deriving these results in (Blute *et al* 1992) arose in considering the noncommutative case, which is not relevant in the context of the present paper. As a result, it is not difficult to apply the techniques developed in (Blute *et al* 1992) to extend the analysis to the present context. However, we shall leave that as an exercise for the interested reader, and instead content ourselves with a summary of the Empire Rewiring Theorem as it extends to weakly distributive categories with storage. We shall just call it the Rewiring Theorem from now on.

The first step in deriving a coherence result for weakly distributive categories with storage is an analysis of the normalization procedure for the proof net system. The cut-elimination steps, *viz.* the reductions and expansions of Tables 4, 6, 7, are just those of (Danos 1990), and so form a confluent system for which we have strong normalization. The "box-expansion" rules of Table 5 create no new cuts, so we arrive at our normalization process by appending these rewrites to the Danos process and modifying the definition of "normal form" accordingly.

We state the following result, essentially due to V. Danos:

Theorem 4.1. The system of reductions and expansions contained in Tables 4, 5, 6, and 7 is confluent and strongly normalizing, modulo the positioning of the thinning links.

Remark: In the original Danos system, there were no thinning links, and so there a genuine confluent, strongly normalizing system is obtained. However, the thinning links are necessary to obtain coherence.

Proof. This system differs from that of the net system for the pure multiplicative fragment due to the non-local nature of the box rules. Thus there are a number of critical pairs to be checked. This is of course fairly tedious, and we choose to illustrate the process with two examples, rather than check all of the pairs, in Figures 8, 9.
Similarly, one can show confluence for all of the critical pairs, thus obtaining confluence of the system. The strong normalization is obvious. □

Next we recall briefly the notion of expanded normal form. Intuitively, we treat storage boxes as if they were "black boxes" at one level, and then consider a net to be in (expanded) normal form if it is in such form (understood naively) without considering the contents of the storage boxes, and in addition, if each subnet inside a storage box is itself in such form. So a net is in normal form if no further reductions may be done on the net, and in expanded normal form if it is in normal form and if any expansion of a non-atomic "wire" could be removed by a series of reductions, so that any expansion would immediately cause the net to no longer be in normal form.

Now we recall the definition of empire, extended to include the modalities. For our purposes it is sufficient to suppose the net is in expanded normal form.

Definition 4.2. (empires) Let P be a proof circuit in expanded normal form, and let A be an occurrence of a formula in the proof circuit. The empire $e(A)$ is a set of (occurrences of) formulas of the circuit, defined as follows:

— $A \in e(A)$.
— If $B \in e(A)$ and B is linked to a formula C *via* an axiom (*i.e.* $(\top\ I)$ or $(\perp\ E)$), thinning, or dereliction link, then $C \in e(A)$.
— If $B = C \odot D$ and $B \in e(A)$, then $C, D \in e(A)$.
— If $C \in e(A), C \neq A$ *or* $D \in e(A), D \neq A$, and $B = C \odot D$ is an occurrence of a (non-switchable) link $(\oplus\ E)$ or $(\otimes\ I)$, then $B \in e(A)$.
— If $C \in e(A), C \neq A$ *and* $D \in e(A), D \neq A$, and $B = C \odot D$ is an occurrence of a (switchable) link $(\otimes\ E)$ or $(\oplus\ I)$, then $B \in e(A)$.
— At a $(contr\ I)$ link, the conclusion occurrence $?B \in e(A)$ if and only if every input occurrence $?B \in e(A)$.
— Dually, at a $(contr\ E)$ link, the input instance $!B \in e(A)$ if and only if every conclusion occurrence $!B \in e(A)$.
— If $B \in e(A)$ and if B and C are both ports (principal or auxiliary) of the same storage box, then $C \in e(A)$.

Note that at switchable links, the principal port is in the empire if and only if every

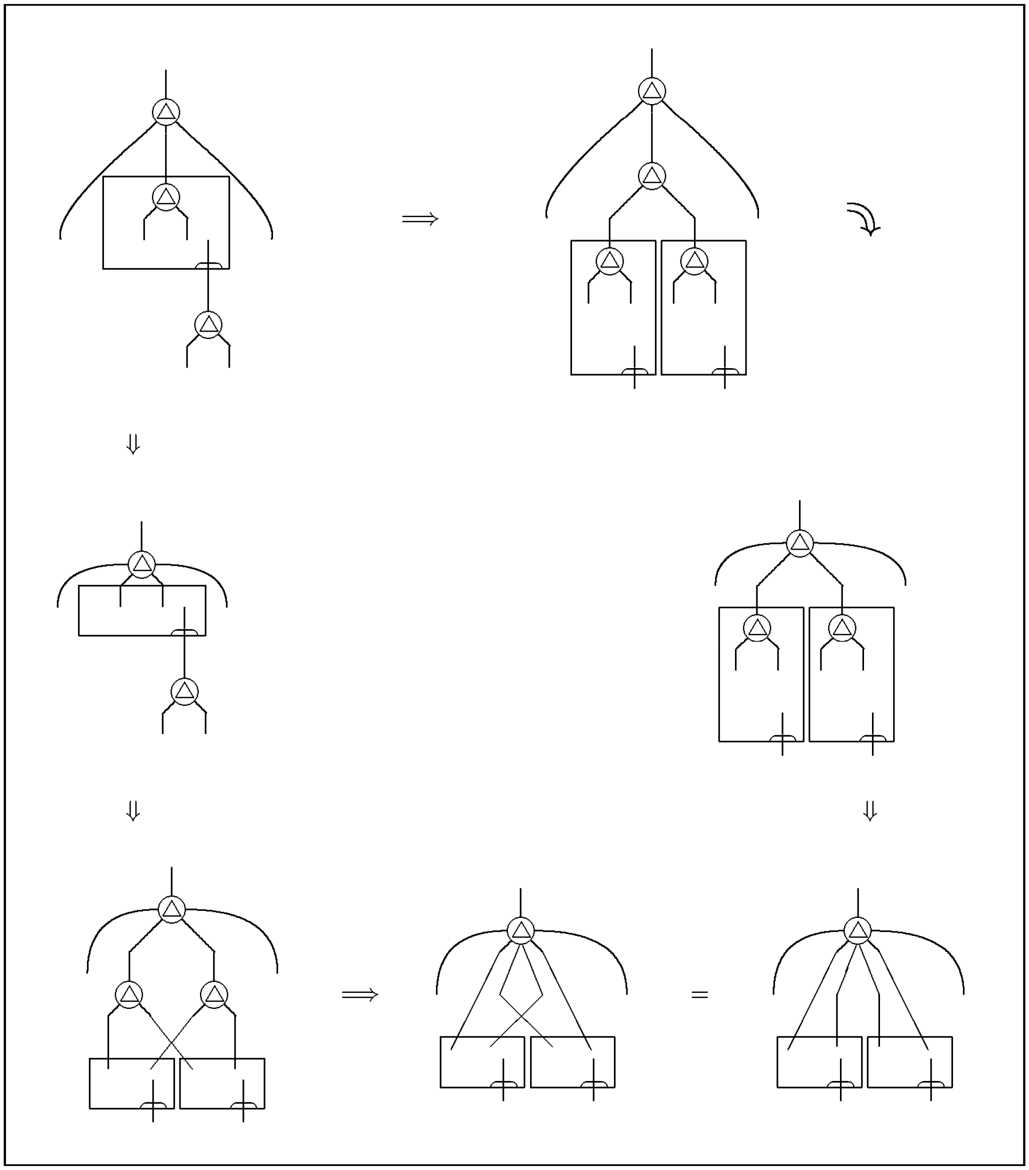


Fig. 8. Confluence: an example of a critical pair

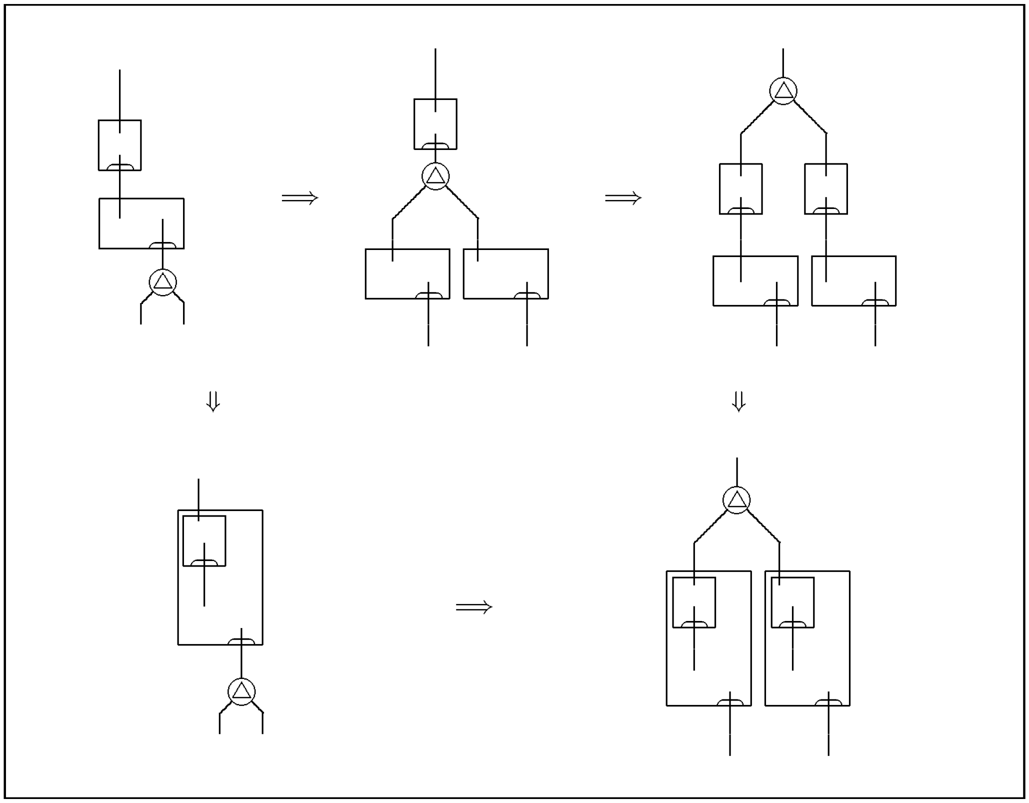


Fig. 9. Confluence: another example of a critical pair

auxiliary port is in the empire, and at non-switchable links some port is in the empire if and only if every port is in the empire.

As shown in (Girard 1987), $e(A)$ may be formed by intersecting connected components of A, each within a subgraph obtained by removing all instances of formulas containing that instance of A as a proper subformula. The intersection is taken over all nets, *i.e.* over all choices of switch at the switchable links. Since $e(A)$ is connected, it is itself a proof net. The empire $e(A)$ may also be described as the largest subnet for which A is an extremal formula (*i.e.* premise or conclusion as appropriate).

We now define the permissible moves for thinning links.

Definition 4.3. (rewiring formulas) A rewiring of a formula introduced by thinning is a reconnection of the thinning link attached to that formula to any other formula in the empire of the original formula. In other words, this amounts to changing a thinning link to link with another formula in that formula's empire.

Note that rewiring a thinning link preserves the proof net property. Conversely, it can be seen that the reconnection of a formula to a formula outside of its empire will produce a proof structure which is not a proof net. We shall often refer to such a rewiring as the rewiring of a thinning link, rather than of the formula corresponding to the link.

It must be understood in Definition 4.3, that a formula cannot be rewired to itself, even though it lies in its own empire.

Definition 4.4. (rewiring nets) A rewiring of a net is a series of rewirings of the thinning links in the net. Note that this notion is not parallel but sequential—rewiring one link may well alter the empire of another.

Theorem 4.5. (Rewiring Theorem) Two nets with the same expanded normal form, and with possibly different thinning links are equivalent, in the sense that they represent the same morphism in the free category, if and only if one is a rewiring of the other.

Proof. The "only if" part of the theorem involves a straightforward check that for each appropriate defining commutative diagram, each composite is represented by a net whose expanded normal form is a rewiring of the expanded normal form of the other composite. The number of cases that need to be checked is quite small, as one only needs to check those diagrams from Section 2 that involve thinning.
"If" is an induction on the size of the net. We check that if a thinning link is rewired to another formula in its empire, the resulting net is equivalent to the original net. In fact, this may be reduced to the case where the terminal links are all either $(\otimes\ I)$ or $(\oplus\ E)$—all other cases are easily handled by the induction hypothesis. For example, we may suppose there are no terminal switchable links, because for a terminal switchable link, if the compound formula is in the empire of the thinning link, so must the components be. It is also easy to check that if there is a terminal "exponential" link, then again we can quickly reduce to a smaller net without that link, and so by induction we are done. For example, if there is a terminal dereliction link, then any thinning attached there may be moved past the link (essentially by functoriality). Likewise, a terminal thinning link is trivially handled since one can "slide" thinning links along other thinning links. A terminal storage link may be handled by a method that depends on the type of thinning link involved. For an exponential thinning, the box rewrite rule in Table 5 allows us to move the thinning link outside the box. For a unit thinning, the thinning link and the empire of the unit lie either completely inside or completely outside the storage box—in either case the induction assumption is easily applied.
So without loss in generality, we suppose all terminal links are either of the form $(\otimes\ I)$ or $(\oplus\ E)$. We shall sketch the proof, assuming the thinning link to be attached to a unit I. The proof is similar for the case where it is attached to an exponential formula. Denote the original (respectively, new) thinning link to I by I — C (respectively I — D). Note that both these thinning links lie inside the empire $e(I)$, calculated in the original net, since we suppose the rewiring is valid. We can find a splitting terminal formula E; we suppose for definiteness $E = A \otimes B$. (The other cases can be left as an exercise.) If E is not in the empire $e(I)$ of I (in the original net) then $e(I) \subseteq e(X)$ for $X = A$ or B, and so we have the result by induction, since the rewiring takes place inside the smaller net $e(X)$.
If $E \in e(I)$ then it is possible that the formulas C, D are "split": $I, C \in e(A),\ D \in e(B)$, say. (If not, argue as above.) We shall effect the rewiring in stages that preserve the Lambek equivalence. The key idea is that if $C = A,\ D = B$, the rewiring through the $\otimes$

link preserves equivalence since it corresponds to the following sequent rewrite:

$$\frac{\dfrac{\Gamma_1 \longrightarrow \Gamma_2, A}{\Gamma_1, \top \longrightarrow \Gamma_2, A} \quad \Delta_1 \longrightarrow \Delta_2, B}{\Gamma_1, \top, \Delta_1 \longrightarrow \Gamma_2, \Delta_2, A \otimes B} \quad \Rightarrow \quad \frac{\Gamma_1 \longrightarrow \Gamma_2, A \quad \dfrac{\Delta_1 \longrightarrow \Delta_2, B}{\top, \Delta_1 \longrightarrow \Delta_2, B}}{\Gamma_1, \top, \Delta_1 \longrightarrow \Gamma_2, \Delta_2, A \otimes B}$$

We then reduce the general case to this one by induction: certainly we can rewire I from C to A preserving the equivalence, since this takes place in the smaller net $e(A)$. There is a small problem in doing the same thing to D, since rewiring I may change the empires. However, for any setting of the switches there must be a path (in the original net) from C to D, and moreover, since $A \otimes B$ is splitting, such paths must go through this link. Hence even after rewiring I to D we will have $B \in e(I)$, and so we can rewire I from D to B preserving equivalence. □

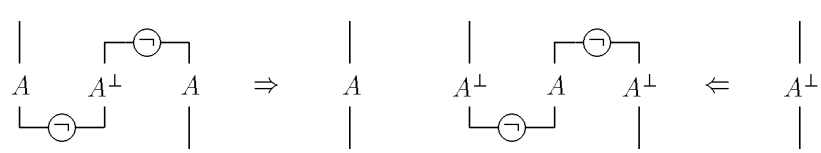
We should conclude by pointing out once again that an alternate proof of this result may be obtained by following the methods used in (Blute *et al* 1992). (In fact, once the machinery from that paper is in place, the Rewiring Theorem becomes a trivial corollary.)

Remark 4.6. (The coherence theorem)

We might remark that if two nets have no thinning links, they are equivalent if and only if they have the same expanded normal form. We can then see that to determine if two morphisms are equal (*i.e.* if two nets are equivalent), all we need do is to first get the expanded normal form of each net, not doing any rewiring of thinning links other than that which occurs naturally as part of the reduction process; these must be the same apart from the wiring of thinning links if the nets are to be equivalent. The Rewiring Theorem then decides the equivalence of the nets at this point. □

5. Adding negation

As in the earlier papers (Cockett and Seely 1991; Blute *et al* 1992) we can extend the current context to include negation, by adding two new links and two new rewrites, namely, one reduction and one expansion:

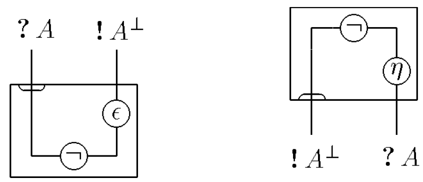


What is of interest in this case is that by adding negation to weakly distributive categories with storage, we find that the tensorial strength is just sufficient to force the two storage operators ! and ? to be de Morgan duals.

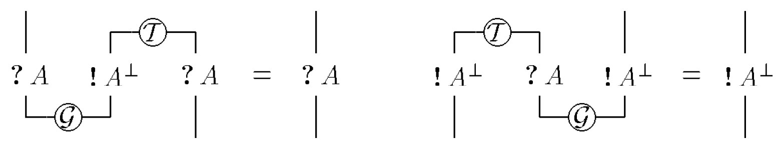
Proposition 5.1. In a weakly distributive category with storage and with negation, $(\,?\,A)^\perp \simeq\ !\,A^\perp$.

Proof. There are circuits

namely,



having the property that



We can verify the necessary reductions as a simple consequence of the negation rewrites, and the "box-swallows-box" reduction. It is a simple exercise to verify that this characterizes negation, essentially because these rewrites define negation in terms of adjunctions. $\square$

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