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HYPERDOCTRINES, NATURAL DEDUCTION AND THE BECK CONDITION

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0. Introduction

In the late sixties F. W. LAWVERE showed that the logical connectives and quantifiers were examples of the categorical notion of adjointness. In [9] and [10] he amplified this notion by a more thorough discussion of the structure of a hyperdoctrine, which had much of the flavour of intuitionistic logic with equality. In this context it was natural to "stratify" formulae and proofs according to the free variables occurring in them, a procedure later to become standard in categorical logic. (See MAKKAI-REYES [12], FOURMAN [1], KOCK-REYES [7], for example.) In this paper, we make the relationship between hyperdoctrines and logic precise, showing that hyperdoctrines are naturally equivalent to first order intuitionistic theories with equality, where here "theory" is intended to include some proof theoretic structure, and not merely the notion of entailment. Moreover, we will show that this equivalence restricts to one giving a natural logical interpretation to the BECK (or CHEVALLEY) condition: in a given hyperdoctrine, the Beck condition for a pullback diagram is just the condition that the corresponding theory "recognizes" the pull back.

This work has an obvious relationship to LAMBEK [8], and to SZABO [21], but, apart from the evident difference in using natural deduction rather than the sequent calculus, one important variant must be noted. SZABO treats the quantifiers as infinite conjunctions and disjunctions, whereas here (following LAWVERE [9]) they are operations adjoint to substitution. This avoids any need to refer to infinitary logic, and more closely reflects their nature: the adjunctions are explicit in the rules for the quantifiers (in either Gentzen system).

There are also connections between hyperdoctrines and Dialectia interpretations (see P. SCOTT [17]) and realizability (see HYLAND, JOHNSTONE, PITTS [5]; note: a "tripos" is a po-hyperdoctrine with a generic predicate). We plan to explore these connections further in a sequel, particularly with respect to GIRARD's type theory (GIRARD [3]).

Basics of category theory may be found in MAC LANE [11] or GOLDBLATT [4].

1. First Order Logic

We base our logic, LPCE, on a natural deduction formulation of intuitionistic, multisorted, first order predicate calculus with equality. The main modification we must introduce (essentially to be able to allow interpretations with uninhabited sorts) is

[1]) These results are contained in the author's Ph. D. thesis, SEELY [18].

the "stratification" of formulae and derivations. So we suppose our language $\mathscr{L}$ to contain:

sort symbols: $X, Y, Z, \ldots$

free variables of each sort: $x, x', \ldots, y, y', \ldots, z, \ldots$

bound variables of each sort: $\xi, \xi', \ldots, \eta, \eta', \ldots, \zeta, \ldots$

sorted function symbols: $f, g, \ldots$

sorted predicate symbols: $P, Q, \ldots$

logical symbols: $\top, \bot, \wedge, \vee, \supset, \exists, \forall$.

(Among the predicate symbols we assume a binary predicate E_X for each sort X, which will be interpreted as equality for that sort.)

To be able to treat functions and predicates as if they were unary, and to simplify sorting terms and formulae, we introduce the meta-mathematical notion of "type": a type is a finite sequence (here written as a product) of sorts. So if X, Y are sorts, then $X \times Y$ is a type. The empty sequence of sorts is denoted 1. "A free variable of type $X \times Y$" is understood to mean a sequence $\langle x, y\rangle$ of free variables of sorts X, Y respectively, and similarly for other terms. The obvious convention gives equality predicates for all types: $E_{X\times Y}(\langle x, y\rangle, \langle x', y'\rangle)$ iff $E_X(x, x') \wedge E_Y(y, y')$, and so on. (Generally we will write $x = x'$ for $E_X(x, x')$.)

Note that every function and predicate symbol is typed: one type for the sorts of the arguments, and in addition a function symbol has a type for the sort of its "value". In the obvious way, this induces an assignment of types to all terms and formulae: we write $t\colon Y \to X$ and say t has *domain* Y and *codomain* X to mean that Y is the type giving the sorts of the free variables in t, and X is the sort of t. Similarly we write $\varphi{:}X$ (or $\varphi(x)$) and say φ *is over* X or *has type* X to mean that the free variables of φ have sorts given by the type X. (For example, "sentence" = "formula over 1"). *A technical point:* we want $\varphi{:}X \times Y$ (i.e. $\varphi = \varphi(x, y)$) to mean that the free variables of φ are *exactly* x, y of sorts X, Y, and not merely *among* x, y. Later we will want φ and ψ to have exactly the same free variables when we form, e.g., $\varphi \wedge \psi$. So that this is not too restrictive, we must be able to add "dummy" free variables to a formula. Perhaps the simplest way to do this is to add new function symbols to $\mathscr{L}$ corresponding to projections. (For example $\pi = \pi^{x\times y}\colon X \times Y \to X$, $\pi(x, y) = x$; $!_Y = \pi_1^Y\colon Y \to 1$, $!_Y(y) = *$, where $*$ is a (the) free variable of type 1.) Then e.g. the formula $x_1 = x_2 \wedge x_2 = x_3$ would actually read $E_X(\pi_1(x_1, x_2, x_3), \pi_2(x_1, x_2, x_3)) \wedge$ $\wedge\, E_X(\pi_2(x_1, x_2, x_3), \pi_3(x_1, x_2, x_3))$ where $\pi_1, \pi_2, \pi_3\colon X \times X \times X \to X$ are the evident projections.

The deduction rules and axioms are based on the standard natural deduction formulation of intuitionistic logic, as given in PRAWITZ [14], [15], with a few modifications. The basic rules are these:

$$(\wedge\mathrm{I})\ \frac{\varphi \quad \varphi'}{\varphi \wedge \varphi'} \qquad (\wedge\mathrm{E})_{\mathrm{L}}\ \frac{\varphi \wedge \varphi'}{\varphi} \qquad (\wedge\mathrm{E})_{\mathrm{R}}\ \frac{\varphi \wedge \varphi'}{\varphi'}$$

$$(\vee\mathrm{I})_{\mathrm{L}}\ \frac{\varphi}{\varphi \vee \varphi'} \qquad (\vee\mathrm{I})_{\mathrm{R}}\ \frac{\varphi'}{\varphi \vee \varphi'} \qquad (\vee\mathrm{E})\ \frac{\varphi \vee \varphi' \quad \begin{matrix}[\varphi]\\ \gamma\end{matrix} \quad \begin{matrix}[\varphi']\\ \gamma\end{matrix}}{\gamma}$$

$$(\supset\mathrm{I})\ \frac{\begin{matrix}[\varphi]\\ \varphi'\end{matrix}}{\varphi \supset \varphi'} \qquad (\supset\mathrm{E})\ \frac{\varphi \supset \varphi' \quad \varphi}{\varphi'}$$

$$(\forall\mathrm{I})\ \frac{\varphi(x)}{\forall\xi\varphi(\xi)} \qquad (\forall\mathrm{E})\ \frac{\forall\xi\varphi(\xi)}{\varphi(t)}$$

$$(\exists\mathrm{I})\ \frac{\varphi(t)}{\exists\xi\varphi(\xi)} \qquad (\exists\mathrm{E})\ \frac{\exists\xi\varphi(\xi) \quad \begin{matrix}[\varphi(x)]\\ \varphi'\end{matrix}}{\varphi'}$$

We add rules for $\top$ and $\bot$:

$$(!)\ \frac{\varphi}{\top_X} \qquad (\bot)\ \frac{\bot_X}{\varphi}$$

where φ is an atomic formula over X different from $\top_X$ (in (!)) or from $\bot_X$ (in ($\bot$)), as appropriate. $\top_X$, $\bot_X$ are $\top$, $\bot$ with a dummy free variable of type X.

We add equality rules:

$$(=\mathrm{I})\ \frac{\top_X}{t = t} \quad \text{for any term } t\colon X \to Y,$$

$$(=\mathrm{E})\ \frac{s = s' \ldots t = t' \quad \varphi(s, \ldots, t)}{\varphi(s', \ldots, t')}$$

for any atomic formula φ over $X \times \ldots \times Y$, and any terms

$$s, s'\colon X' \to X, \ldots, t, t'\colon Y' \to Y.$$

(We will also denote the evident derived rules by (!), ($\bot$), (=E).)

In all of the rules except ($\forall$I), ($\exists$E), the premises and conclusion must be formulae over the same type. (And so, we may as well require that φ, ψ be over the same type if $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \supset \psi$ are to be wffs.)

In ($\forall$I), x must not occur in any assumption on which $\varphi(x)$ depends (this is standard) except possibly as a dummy free variable, in which case the dummy occurrences of x may be discharged.

In ($\exists$E), x must not occur in $\exists\xi\varphi$, in φ', or in any assumption other than φ on which the upper occurrence of φ' depends, except possibly as a dummy variable in the *upper* occurrence of φ', and the assumptions on which that occurrence depends, in which case the dummy occurrences of x may be discharged.

Note that we have in effect stratified derivations: a derivation $P\colon \Gamma \vdash \varphi$ must have all $\psi \in \Gamma$ and φ over the same type X: we say P is over X too. The rules ($\forall$I), ($\exists$E) provide the only way to change levels, by the discharge of dummy variables. (We will not usually explicitly show dummy variables, however; they can be filled in from the syntactic rules and the context.)

Finally, we denote by (id) the "rule" $\dfrac{\varphi}{\varphi}$ (rewriting φ) which should be understood as being merely the top occurrence of φ.

2. Operations on derivations

As in SEELY [19], we will be interested in a number of operations on derivations: reductions, expansions, and permutations. Most of these are standard, and can be found in PRAWITZ [14], [15]. In addition, we have these new rules:

$$(!\ \mathrm{Red})\quad \begin{array}{c}\varphi\\ P\\ \top_X\end{array} \Rightarrow \frac{\varphi}{\top_X}\,(!)\quad (\varphi \text{ the sole assumption of } P)$$

$$(\bot\ \mathrm{Red})\quad \begin{array}{c}\bot_X\\ P\\ \varphi\end{array} \Rightarrow \frac{\bot_X}{\varphi}\,(\bot)\quad (\bot_X \text{ the sole assumption of } P)$$

(Strictly, (! Red) and ($\bot$ Red) apply only when φ is atomic and different from $\top_X$ (respectively $\bot_X$). If φ is $\top_X$ (respectively $\bot_X$) these reductions will be understood to be to the identity derivation. Also, if φ is not atomic, the reductions will be to the appropriate derived rule.)

$$(=\mathrm{Red})\quad \frac{\dfrac{\top}{t=t}\quad \begin{array}{c}P\\ \varphi(t)\end{array}}{\varphi(t)} \Rightarrow \begin{array}{c}P\\ \varphi(t)\end{array}\quad (\text{atomic } \varphi)$$

$$(=\mathrm{Exp})\quad \begin{array}{c}P\\ t=t'\\ Q(t,t')\\ \\ \varphi(t,t')\end{array} \Rightarrow \frac{\begin{array}{c}P\\ t=t'\end{array}\quad \begin{array}{c}\dfrac{\top}{t=t}\\ Q(t,t)\\ \varphi(t,t)\end{array}}{\varphi(t,t')}$$

(provided the RHS is a derivation)

$$(\vee\ \mathrm{Perm})\quad \begin{array}{c}\dfrac{\begin{array}{c}P\\ \varphi\vee\psi\end{array}\quad \begin{array}{c}[\varphi]\\ Q\\ \theta\end{array}\quad \begin{array}{c}[\psi]\\ R\\ \theta\end{array}}{[\theta]}\\ S\\ \gamma\end{array} \Rightarrow \frac{\begin{array}{c}P\\ \varphi\vee\psi\end{array}\quad \begin{array}{c}[\varphi]\\ Q\\ [\theta]\\ S\\ \gamma\end{array}\quad \begin{array}{c}[\psi]\\ R\\ [\theta]\\ S\\ \gamma\end{array}}{\gamma}$$

$$(\exists\ \mathrm{Perm})\quad \begin{array}{c}\dfrac{\begin{array}{c}P\\ \exists\xi\varphi(\xi)\end{array}\quad \begin{array}{c}[\varphi(x)]\\ Q\\ \theta\end{array}}{[\theta]}\\ S\\ \gamma\end{array} \Rightarrow \frac{\begin{array}{c}P\\ \exists\xi\varphi(\xi)\end{array}\quad \begin{array}{c}[\varphi(x)]\\ Q\\ [\theta]\\ S\\ \gamma\end{array}}{\gamma}$$

And a "coherence" rule for equality:

$$(\mathrm{R\ Coh})\quad \dfrac{\dfrac{\begin{array}{c}P\\ t=t\end{array}}{\top}}{t=t} \Rightarrow \begin{array}{c}P\\ t=t\end{array}$$

Remarks. (1) As shown in SEELY [19] these permutation rules amount to strengthening PRAWITZ' $\vee$ expansion and $\exists$ expansion to:

$$(\vee\ \mathrm{Exp})\qquad \begin{array}{c} P_0 \\ [\varphi \vee \psi] \\ P_1 \\ \theta \end{array} \Rightarrow \dfrac{\begin{array}{c} P_0 \\ \varphi \vee \psi \end{array} \quad \begin{array}{c} \dfrac{[\varphi]}{\varphi \vee \psi} \\ P_1 \\ \theta \end{array} \quad \begin{array}{c} \dfrac{[\psi]}{\varphi \vee \psi} \\ P_1 \\ \theta \end{array}}{\theta}$$

(provided the RHS is a derivation)

$$(\exists\ \mathrm{Exp})\qquad \begin{array}{c} P_0 \\ [\exists\xi\varphi(\xi)] \\ P_1 \\ \theta \end{array} \Rightarrow \dfrac{\begin{array}{c} P_0 \\ \exists\xi\varphi(\xi) \end{array} \quad \begin{array}{c} \dfrac{[\varphi(x)]}{[\exists\xi\varphi(\xi)]} \\ P_1 \\ \theta \end{array}}{\theta}$$

(provided the RHS is a derivation).
(There is a (= Perm) in the same spirit, but we won't need to refer to it.)

(2) There are some derived rules of interest:

In (= E), some of the terms s' may be identical to the corresponding s, and, adopting the convention that $\top$ may be dropped as an assumption whenever it occurs, so too can assumptions of the form $s = s$.

Some special cases of (= E) should be noted: for any terms t, t', t'', typed $X \to Y$, say, the following are (derived) rules:

$$(\mathrm{S})\qquad \frac{t = t'}{t' = t} \qquad \left(\mathrm{viz.}\ \frac{t = t' \quad \dfrac{\top}{t = t} \quad \dfrac{\top}{t = t}\,(=\mathrm{I})}{t' = t}\,(=\mathrm{E})\right)$$

$$(\mathrm{T})\qquad \frac{t = t' \quad t' = t''}{t = t''} \qquad \left(\mathrm{viz.}\ \frac{\dfrac{\top}{t = t}\,(=\mathrm{I}) \quad t' = t'' \quad t = t'}{t = t''}\,(=\mathrm{E})\right)$$

(In keeping with these notations, we shall also use (R) for (= I) and (sub) for (= E)).

(3) We will be casual in using the meta-notation for types and "terms of a given type". For example, (= E) will frequently appear as

$$\frac{t = t' \quad \varphi(t)}{\varphi(t')}.$$

(4) A given theory in LPCE may impose further operations on derivations, in addition to non logical rules. We regard a theory as given by its language, non-logical deduction rules, and operations on derivations.

We define an equivalence relation $\equiv$ on derivations in the natural way: $\equiv$ is the smallest equivalence relation making all of the given operations equivalences. Explicitly, derivations P, P' are equivalent iff there are derivations $P = P_1, P_2, \ldots, P_k = P'$ ($k \geq 1$) so that for each $i < k$, either P_{i+1} is obtained from P_i by replacing a subtree of P_i by the result of applying an operation to the subtree, or P_i is obtained from P_{i+1} this way.

Three equivalence schema are useful:

Lemma. *If $Q_i(t, t'): t = t' \vdash \varphi(t, t')$ $(i = 0, 1)$ are derivations such that $Q_0(t, t) \equiv \equiv Q_1(t, t)$, for any t, then $Q_0(t, t') \equiv Q_1(t, t')$, for any t, t'.*

Corollary 1. *For any derivations $Q_i(t, t'): \Gamma \vdash t = t'$ $(i = 0, 1)$, $Q_0(t, t') \equiv Q_1(t, t')$.*

Corollary 2. *For any derivations P, P_1, P_2, P_3 there are equivalences:*

(i) (S Coh): $$\text{(S)}\,\dfrac{\text{(S)}\,\dfrac{\overset{P_1}{t = t'}}{t' = t}}{t = t'} \equiv \overset{P_1}{t = t'}$$

(ii) (T Coh): $$\dfrac{\text{(T)}\,\dfrac{\overset{P_1}{t = t'} \quad \overset{P_2}{t' = t''}}{t = t''} \qquad t'' = t'''}{t = t'''} \equiv \dfrac{\overset{P_1}{t = t'} \quad \dfrac{\overset{P_2}{t' = t''} \quad \overset{P_3}{t'' = t'''}}{t' = t'''}\,\text{(T)}}{t = t'''}\,\text{(T)}$$

(iii) (= Simp): *If (φ is atomic and) t, t' appear in a dummy position, (so P is a derivation of $\varphi(t')$)*

$$\dfrac{\overset{P_1}{t = t'} \quad \overset{P}{\varphi(t)}}{\varphi(t')} \equiv \overset{P}{\varphi(t')}.$$

Proofs. The Lemma is an immediate consequence of (= Exp). Then Corollary 1 follows, since $t = t$ is a "terminal object": there is, up to equivalence, exactly one derivation $\Gamma \vdash t = t$, for any Γ. This can be shown directly, via (! Red) and (R Coh), or by using the methods and results of § 4.

Similarly, Corollary 2(i) and (ii) are immediate. To prove (= Simp), replace t' by t and apply (= Red) to see

$$\dfrac{\overset{P_1(t,t)}{t = t} \quad \overset{P}{\varphi(t)}}{\varphi(t)} \equiv \overset{P}{\varphi(t)}. \quad \square$$

§ 3. Hyperdoctrines

Definition. A $\boldsymbol{T}$-category $\boldsymbol{P}$ is a *hyperdoctrine* iff

(0) $\boldsymbol{T}$ has finite products and terminal object 1,
(1) $\boldsymbol{P}$ is an indexed category over $\boldsymbol{T}$ ("a $\boldsymbol{T}$-category"),
(2) for each object X of $\boldsymbol{T}$, the fibre $\boldsymbol{P}(X)$ is cartesian closed, and furthermore,
(3) has finite coproducts and an initial object 0_X,
(4) for each morphism t of $\boldsymbol{T}$, the "inverse image" functor t^* preserves the structure of (2), (3),
(5) $\boldsymbol{P}$ has $\boldsymbol{T}$-sums and $\boldsymbol{T}$-products.

As this definition is non-standard, (it differs from that given in LAWVERE [10]) perhaps we should elaborate. This will also allow us to fix notation. The definition of an indexed category is standard enough, (although we only suppose finite products in the base category, and not arbitrary finite limits). (See PARÉ-SCHUMACHER [13].) The point is this: an indexed category is equivalent to a (pseudo)-functor $\boldsymbol{T}^{\mathrm{op}} \to \mathscr{C}at$. Hence the notation $\boldsymbol{P}(X)$ for the "fibre" over X. The "inverse image" functor t^*, for $t\colon X \to Y$ a morphism of $\boldsymbol{T}$, is then given as $t^* = \boldsymbol{P}(t)\colon \boldsymbol{P}(Y) \to \boldsymbol{P}(X)$. Clauses (2)–(4) are straightforward. Clause (5) can be replaced by:

(5′) (i) for each $t\colon X \to Y$ in $\boldsymbol{T}$, t^* has adjoints

$$\Sigma_t \dashv t^* \dashv \Pi_t,$$

and

(5′) (ii) if
$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "r"'] & Y \arrow[d, "s"] \\ X' \arrow[r, "t'"] & Y' \end{tikzcd}$$
is a pullback in $\boldsymbol{T}$, and $\varphi \in |\boldsymbol{P}(Y)|$,

then the morphism $\Sigma_r t^* \varphi \to t'^* \Sigma_s \varphi$ is an isomorphism.

Note that then we can replace

(4) by:

(4′) the "inverse image" functors t^* preserve exponentiation.

(All the rest of the structure of (2) and (3) must be preserved, because of the existence of both adjoints.) This is equivalent to

(4″) for each $t\colon X \to Y$ in $\boldsymbol{T}$, $\varphi \in |\boldsymbol{P}(X)|$, $\psi \in |\boldsymbol{P}(Y)|$, the morphism

$$\Sigma_t(t^*\psi \wedge \varphi) \to \psi \wedge \Sigma_t \varphi$$

is an isomorphism.

Condition (4′) (or (4″)) is called *Frobenius Reciprocity*; condition (5′) (ii) is called *Beck condition*. Note that the analogous condition for Π must hold; just consider adjoints.

Some remarks on pullbacks and the Beck condition: Any category with finite products must have the following types of pullbacks:

(a)
$$\begin{tikzcd} X \arrow[r, "\langle X, t\rangle"] \arrow[d, "t"] & X \times Y \arrow[d, "t \times X"] \\ Y \arrow[r, "\Delta_Y"] & Y \times Y \end{tikzcd}$$

(b)
$$\begin{tikzcd} A \times X \arrow[r, "A \times t"] \arrow[d, "r \times X"'] & A \times Y \arrow[d, "r \times Y"] \\ B \times X \arrow[r, "B \times t"] & B \times Y \end{tikzcd}$$

(c)
$$\begin{tikzcd} X \arrow[r, "X"] \arrow[d, "X"'] & X \arrow[d, "\Delta_X"] \\ X \arrow[r, "\Delta_X"] & X \times X \end{tikzcd}$$

(d)
$$\begin{tikzcd} A \times X \arrow[r, "r \times t"] \arrow[d, "s \wr"'] & B \times Y \arrow[d, "\wr s"] \\ X \times A \arrow[r, "t \times r"] & Y \times B \end{tikzcd}$$

where s is the "switch coordinates" isomorphism.

Also, given a pullback

(D)
$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "r"'] & Y \arrow[d, "s"] \\ X' \arrow[r, "t'"] & Y' \end{tikzcd}$$

then for any object Z,

(i)$_D$
$$\begin{tikzcd} X \times Z \arrow[r, "t \times Z"] \arrow[d, "r \times Z"'] & Y \times Z \arrow[d, "s \times Z"] \\ X' \times Z \arrow[r, "t' \times Z"] & Y' \times Z \end{tikzcd}$$

("Preservation under products") is a pullback. Finally, if

(D′)
$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[d, "s"'] & Z \arrow[d, "q"] \\ Y' \arrow[r, "p'"] & Z' \end{tikzcd}$$

is a pullback, then so is

(ii)$_{DD'}$
$$\begin{tikzcd} X \arrow[r, "pt"] \arrow[d, "r"'] & Z \arrow[d, "q"] \\ X' \arrow[r, "p't'"] & Z' \end{tikzcd}$$

("Preservation under composition").

If we are only interested in the pure logic with equality (and not in any other rules a theory in LPCE might introduce), it will suffice if we suppose the Beck condition only for the cases (a), (b), (c), and (d), and that it is preserved under the pullback formation rules (i), (ii), given above. In fact, some of this is automatic: Beck must hold in case (d), since for any isomorphism s, $(s^{-1})^* = \Sigma_s$; also Beck must be preserved under (ii). (In fact, on most of those occasions we actually use the Beck condition, it will be the case that it is preserved under (i) also, (e.g., when the $\varphi \in |\boldsymbol{P}(Y \times Z)|$ is of the form $\pi_Y^* \psi$, for $\psi \in |\boldsymbol{P}(Y)|$).) So we need only assert Beck for cases (a), (b), and (c), and its preservation under (i). We shall return to this point in § 8.

§ 4. Construction 1: LPCE → Hyperdoctrine

For any theory T expressed in LPCE, there is a corresponding hyperdoctrine $\boldsymbol{P}_T$ (over $\boldsymbol{T}_T$). We'll construct $\boldsymbol{P}_T := \boldsymbol{P}_0$ for the case of the empty theory T_0 (so, insofar as $\boldsymbol{P}_T$ "is" T, the resulting $\boldsymbol{P}_0$ "is" LPCE; the more general construction will then

(I hope) be clear). Objects of $\boldsymbol{T}_0$ are the types of the language $\mathscr{L}$. Morphisms of $\boldsymbol{T}_0$ are essentially the terms. (Some technical fiddling is necessary here so that $\boldsymbol{T}_0$ is a category with finite products, but this is straightforward. For details see SEELY [18].) For any type X, the objects of the fibre over X are formulae whose free variables are of type X. Morphisms of the fibre are proofs (i.e. equivalence classes of derivations) – again the domains and codomains are obvious. (For arbitrary T, we can expect to get more morphisms in the fibres; this will be the only change. Of course, changing $\mathscr{L}$ will affect $\boldsymbol{T}$ and $|\boldsymbol{P}|$ also.)

Theorem. *$\boldsymbol{P}_0$ as constructed above is a hyperdoctrine.*

Proof. (0) $\boldsymbol{T}_0$ has by construction all finite products, including the empty product 1.

(1) For a term $t\colon X \to Y$, we must construct $t^*\colon \boldsymbol{P}_0(Y) \to \boldsymbol{P}_0(X)$: this is just "substitute t for y", i.e. $t^*\varphi$ is $\varphi(t)$.

(2) The cartesian closed structure is given using $\wedge$ for products and $\supset$ for exponentiation. The deduction rules and definition of equivalence account for the proper structure; this is more or less proven in SEELY [19].

(3) Similarly, $\vee$ for coproduct and $\perp_X$ for 0_X give the required structure.

(4′) This is trivial by definition of substitution:

$$\varphi(t) \supset \varphi'(t) = (\varphi \supset \varphi')\,(t).$$

(5′) (i) For $t\colon X \to y$, φ over X, we define

$$\Sigma_t\varphi =_{\mathrm{df}} \exists\xi(t\xi = y \wedge \varphi(\xi)), \qquad \Pi_t\varphi =_{\mathrm{df}} \forall\xi(t\xi = y \supset \varphi(\xi)).$$

Σ_t, Π_t are defined on proofs as follows: suppose $\gamma \xrightarrow{\bar{P}} \varphi$ is a proof over X, represented by a derivation P. Then $\Sigma_t\bar{P}$ is the proof represented by the derivation:

(i)

$$
\dfrac{\exists\xi(t\xi = y \wedge \gamma(\xi)) \qquad \dfrac{\dfrac{\begin{array}{c}(\wedge\,\mathrm{E})\ \dfrac{tx = y \wedge \gamma(x)}{[\gamma(x)]}\\ P\\ \varphi(x)\end{array} \qquad \dfrac{tx = y \wedge \gamma(x)}{tx = y}\,(\wedge\,\mathrm{E})}{tx = y \wedge \varphi(x)}\,(\wedge\,\mathrm{I})}{\exists\xi(t\xi = y \wedge \varphi(\xi))}\,(\exists\,\mathrm{I})}{\exists\xi(t\xi = y \wedge \varphi(\xi))}\,(\exists\,\mathrm{E})
$$

$\Pi_t\bar{P}$ is defined similarly.

These are the required adjoints: we must show, for γ over X, φ over Y, bijections:

$$\frac{\Sigma_t\gamma \to \varphi \quad (\text{over } Y)}{\gamma \to t^*\varphi \quad (\text{over } X)} \qquad \frac{\varphi \to \Pi_t\gamma \quad (\text{over } Y)}{t^*\varphi \to \gamma \quad (\text{over } X)}.$$

We do this for Π, the proof for Σ is similar.

Given $\varphi(y) \xrightarrow{\bar{P}} \forall\xi(t\xi = y \supset \gamma(\xi))$ over Y, with P a derivation in $\bar{P}$, we obtain $\varphi(t(x)) \to \gamma(x)$, represented by the derivation:

(ii)

$$\cfrac{(\supset \mathrm{E})\ \cfrac{\begin{array}{c}\varphi(t(x))\\ P(t)\\ \forall\xi(t\xi = t(x) \supset \gamma(\xi))\end{array}}{t(x) = t(x) \supset \gamma(x)}\,(\forall \mathrm{E}) \qquad \cfrac{\top}{t(x) = t(x)}\,(\mathrm{R})}{\gamma(x)}$$

Conversely, given $\varphi(t(x)) \xrightarrow{\bar{P}} \gamma(x)$ over X, and P in $\bar{P}$, we define a proof

$$\varphi(y) \to \forall\xi(t\xi = y \supset \gamma(\xi))$$

represented by:

(iii)

$$\cfrac{\cfrac{\begin{array}{c}\cfrac{\cfrac{\overset{(1)}{t(x) = y}}{y = t(x)}\,(\mathrm{S}) \quad \varphi(y)}{\varphi(t(x))}\,(\mathrm{sub})' \\ P \\ \gamma(x)\end{array}}{t(x) = y \supset \gamma(x)}\,(\supset \mathrm{I})\ (1)}{\forall\xi(t\xi = y \supset \gamma(\xi))}\,(\forall \mathrm{I})$$

((sub)$'$ is the suitable derived rule.) It must be checked that these processes are in fact well-defined with respect to the equivalence relation, but that this is so should be clear from the forms of the above derivations. (This remark will also hold when we come to consider the Beck conditions below.)

Also, we must check that these correspondences do in fact determine an adjunction. Perhaps the simplest way is to verify the triangle equalities for the induced natural transformations $\eta_\Sigma\colon I \to t^*\Sigma_t$, $\varepsilon_\Sigma\colon \Sigma_t t^* \to I$, $\eta_\Pi\colon I \to \Pi_t t^*$, $\varepsilon_\Pi\colon t^*\Pi_t \to I$, checking also that these are in fact natural transformations. This involves writing out various derivations in full, and collapsing and expanding them according to the operations given above: we sketch one part of the proof as an example: namely, that the triangle equality $\Pi_t\varepsilon_\varphi\eta_{\Pi_t\varphi} = \mathrm{id}_{\Pi_t\varphi}$ holds for the adjunction $t^* \dashv \Pi_t$, for φ over X. We shall not define ε, η here—the definitions are immediate from the correspondences (ii), (iii) above—but when they are written out in full, and when the definition of $\Pi_t\bar{P}$, for a proof $\bar{P}$, is also written in full, it would be seen that the composition $\Pi_t\varepsilon_\varphi \cdot \eta_{\Pi_t\varphi}$ is represented by the following derivation (perhaps it should be mentioned that the subderivation above the topmost occurrence of $\forall\xi(t\xi = tx \supset \varphi(\xi))$ is the expanded form of

$$\cfrac{\cfrac{tx = y}{y = tx}\,(\mathrm{S}) \qquad \forall\xi(t\xi = y \supset \varphi(\xi))}{\forall\xi(t\xi = tx \supset \varphi(\xi))}\,(\mathrm{sub})'$$

(a derived rule), that one might expect from the correspondence (iii) above; recall that (T) is a special case of (= E) or (sub)):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall\xi(t\xi = y \supset \varphi(\xi))}{tx' = y \supset \varphi(x')}\,(\forall\,\mathrm{E}) \qquad \dfrac{\overset{(1)}{tx' = tx} \quad \overset{(2)}{tx = y}}{tx' = y}\,(\mathrm{T})}{\varphi(x')}\,(\supset\mathrm{E})}{tx' = tx \supset \varphi(x')}\,(\supset\mathrm{I})\ (1)}{\forall\xi(t\xi = tx \supset \varphi(\xi))}\,(\forall\,\mathrm{I})}{tx = y \supset \forall\xi(t\xi = tx \supset \varphi(\xi))}\,(\supset\mathrm{I})\ (2)}{\forall\xi'(t\xi' = y \supset \forall\xi(t\xi = t\xi' \supset \varphi(\xi)))}\,(\forall\,\mathrm{I})}{tx = y \supset \forall\xi(t\xi = tx \supset \varphi(\xi))}\,(\forall\,\mathrm{E})\ (3) \quad tx = y}{\forall\xi(t\xi = tx \supset \varphi(\xi))}\,(\supset\mathrm{E})}{tx = tx \supset \varphi(x)}\,(\forall\,\mathrm{E}) \qquad \dfrac{\top}{tx = tx}\,(=\mathrm{I})}{\varphi(x)}\,(\supset\mathrm{E})}{tx = y \supset \varphi(x)}\,(\supset\mathrm{I})\ (3)}{\forall\xi(t\xi = y \supset \varphi(\xi))}\,(\forall\,\mathrm{I})
$$

We must show this is equivalent to the identity derivation. Straightforward uses of ($\forall$ Red) and ($\supset$ Red) give us the normal form of the above:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\forall\xi(t\xi = y \supset \varphi(\xi))}{tx = y \supset \varphi(x)} \qquad \dfrac{\dfrac{\top}{tx = tx} \quad tx = y}{tx = y}}{\varphi(x)}}{tx = y \supset \varphi(x)}}{\forall\xi(t\xi = y \supset \varphi(\xi))}
$$

Finally, ($=$ Exp), ($\supset$ Exp), and ($\forall$ Exp) show this is equivalent to (id) as required.

The other parts of the proof that we have defined an adjunction proceed much like this, so we shall omit the rest of the details.

(5′) (ii) As we remarked earlier, there are only three cases we need consider. (For a more general theory T, there may be others, but then the axioms and rules of T that give rise to further pullbacks will permit a similar proof to go through.) In fact (as the categorist will have suspected from adjointness considerations) one direction is automatic: for any commutative diagram

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "r"'] & Y \arrow[d, "s"] \\ X' \arrow[r, "t'"'] & Y' \end{tikzcd}$$

and any formula φ over Y, there's a proof $\Sigma_r t^*\varphi \to t'^*\Sigma_s\varphi$ represented by the derivation:

$$\dfrac{\exists\xi(r\xi = x' \wedge \varphi(t\xi)) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{rx = x' \wedge \varphi(tx)}{rx = x'}\,(\wedge\,\mathrm{E})}{t'rx = t'x'}\,(\mathrm{ap})}{stx = t'x'}\,(\mathrm{id})' \qquad \dfrac{rx = x' \wedge \varphi(tx)}{\varphi(tx)}\,(\wedge\,\mathrm{E})}{stx = t'x' \wedge \varphi(tx)}\,(\wedge\,\mathrm{I})}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\,(\exists\,\mathrm{I})}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\,(\exists\,\mathrm{E})$$

(ap) is a derived rule: if $t\colon X \to Y$, $s_0, s_1\colon Z \to X$ are terms typed as indicated, (ap) is the derivation

$$\dfrac{s_0 = s_1 \qquad \dfrac{}{\top}\,(\mathrm{R}) \quad ts_0 = ts_0}{ts_0 = ts_1}\,(\mathrm{sub})$$

(id)′ is either the identity derivation (if st and $t'r$ are identical) or is a derived rule (if $(P)\colon \vdash st = t'r$ in T):

$$\dfrac{\begin{array}{l} P(x) \\ stx = t'rx \end{array} \qquad t'rx = t'x'}{stx = t'x'}\,(\mathrm{T})$$

We give the inverse proofs for pullbacks of types (a), (b); note that type (a) actually gives two cases because of its lack of symmetry. Type (c) essentially follows from the isomorphism $\top_X \cong (x = x)$, in view of a later result, so we will not discuss it here. Also, the preservation of the Beck condition under products holds in the logic, so for the empty theory our job will be done with type (a) and (b) pullbacks. As for the situation for more general theories, this will be more fully discussed in § 8.

Type (a) (i). $\Delta_Y^* \underset{t\times Y}{\Sigma} \varphi \to \underset{t}{\Sigma} \langle X, t\rangle^* \varphi$: This is represented by the derivation (I) on p. 517. (Recall that the pair notation is a meta-notation, so that (pE), for example, is merely a special case of ($\wedge$ E); and $\exists\langle\xi, \eta\rangle\, \psi$ is really $\exists\xi\, \exists\eta\psi$. Hence the above use of ($\exists$ E), for example, actually indicates two such uses. This remark will also cover what follows.)

(I)

$$\dfrac{\exists\langle\xi,\eta\rangle\,(\langle t\xi,\eta\rangle = \langle y,y\rangle \wedge \varphi(\xi,\eta)) \qquad \dfrac{\dfrac{(\wedge\mathrm{E})\,\dfrac{(\mathrm{pE})\,\dfrac{\overset{(1)}{\langle tx,y'\rangle = \langle y,y\rangle \wedge \varphi(x,y')}}{\langle tx,y'\rangle = \langle y,y\rangle}}{tx = y} \qquad \dfrac{(\mathrm{T})\,\dfrac{(\wedge\mathrm{E})\,\dfrac{(\mathrm{pE})\,\dfrac{(1)}{\langle tx,y'\rangle = \langle y,y\rangle}}{y' = y} \qquad \dfrac{\dfrac{\dfrac{\overset{(1)}{\langle tx,y'\rangle = \langle y,y\rangle \wedge \varphi(x,y')}}{\langle tx,y'\rangle = \langle y,y\rangle}\,(\wedge\mathrm{E})}{tx = y}\,(\mathrm{pE})}{y = tx}\,(\mathrm{s})}{y' = tx} \qquad \dfrac{(1)}{\varphi(x,y')}\,(\wedge\mathrm{E})}{\varphi(x,tx)}\,(\mathrm{sub})}{tx = y \wedge \varphi(x,tx)}\,(\wedge\mathrm{I})}{\exists\xi(t\xi = y \wedge \varphi(\xi,t\xi))}\,(\exists\mathrm{I})}{\exists\xi(t\xi = y \wedge \varphi(\xi,t\xi))}\,(\exists\mathrm{E})\ (1)$$

(II)

$$\dfrac{\exists\eta(\langle\eta,\eta\rangle = \langle tx,y\rangle \wedge \varphi(\eta)) \qquad \dfrac{\dfrac{\dfrac{(\mathrm{R})\,\dfrac{\top}{x = x} \qquad (\mathrm{T})\,\dfrac{(\mathrm{S})\,\dfrac{(\mathrm{pE})\,\dfrac{(\wedge\mathrm{E})\,\dfrac{\overset{(1)}{\langle y',y'\rangle = \langle tx,y\rangle \wedge \varphi(y')}}{\langle y',y'\rangle = \langle tx,y\rangle}}{y' = tx}}{tx = y'} \qquad \dfrac{\dfrac{\overset{(1)}{\langle y',y'\rangle = \langle tx,y\rangle \wedge \varphi(y')}}{\langle y',y'\rangle = \langle tx,y\rangle}\,(\wedge\mathrm{E})}{y' = y}\,(\mathrm{pE})}{tx = y}}{\langle x,tx\rangle = \langle x,y\rangle}\,(\mathrm{pI}) \qquad \dfrac{(\mathrm{pE})\,\dfrac{\dfrac{(1)}{\langle y',y'\rangle = \langle tx,y\rangle}\,(\wedge\mathrm{E})}{y' = tx} \qquad \dfrac{(1)}{\varphi(y')}\,(\wedge\mathrm{E})}{\varphi(tx)}\,(\mathrm{sub})}{\langle x,tx\rangle = \langle x,y\rangle \wedge \varphi(tx)}\,(\wedge\mathrm{I})}{\exists\xi(\langle\xi,t\xi\rangle = \langle x,y\rangle \wedge \varphi(t\xi))}\,(\exists\mathrm{I})}{\exists\xi(\langle\xi,t\xi\rangle = \langle x,y\rangle \wedge \varphi(t\xi))}\,(\exists\mathrm{E})\ (1)$$

Type (a) (ii). $(t \times Y)^* \underset{\Delta_Y}{\Sigma} \varphi \to \underset{\langle X, t\rangle}{\Sigma} t^*\varphi$: This is represented by the derivation (II) on p. 517.

Type (b). $(B \times t)^* \underset{r \times Y}{\Sigma} \varphi \to \underset{r \times X}{\Sigma} (A \times t)^* \varphi$: This is represented by the following derivation (III):

(III)

$$
\dfrac{
\exists\langle\alpha,\eta\rangle\,(\langle r\alpha,\eta\rangle = \langle b,tx\rangle \wedge \varphi(\alpha,\eta))
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{\overset{(1)}{\langle ra,y\rangle = \langle b,tx\rangle \wedge \varphi(a,y)}}{\langle ra,y\rangle = \langle b,tx\rangle}\,{\scriptstyle(\wedge\mathrm{E})}}{ra = b}\,{\scriptstyle(\mathrm{pE})}
\quad
\dfrac{\top}{x = x}\,{\scriptstyle(\mathrm{R})}
}{\langle ra,x\rangle = \langle b,x\rangle}\,{\scriptstyle(\mathrm{pI})}
\quad
\dfrac{
\dfrac{\dfrac{(1)}{\langle ra,y\rangle = \langle b,tx\rangle}\,{\scriptstyle(\wedge\mathrm{E})}}{y = tx}\,{\scriptstyle(\mathrm{pE})}
\quad
\dfrac{(1)}{\varphi(a,y)}\,{\scriptstyle(\wedge\mathrm{E})}
}{\varphi(a,tx)}\,{\scriptstyle(\mathrm{sub})}
}{\langle ra,x\rangle = \langle b,x\rangle \wedge \varphi(a,tx)}\,{\scriptstyle(\wedge\mathrm{I})}
}{\exists\langle\alpha,\xi\rangle\,(\langle r\alpha,\xi\rangle = \langle b,x\rangle \wedge \varphi(\alpha,t\xi))}\,{\scriptstyle(\exists\mathrm{I})}
}{\exists\langle\alpha,\xi\rangle\,(\langle r\alpha,\xi\rangle = \langle b,x\rangle \wedge \varphi(\alpha,t\xi))}\,{\scriptstyle(\exists\mathrm{E})\ (1)}
$$

(The "dual" case $(r \times Y)^* \underset{B \times t}{\Sigma} \varphi \to \underset{A \times t}{\Sigma} (r \times X)^* \varphi$ follows immediately from this, under evident substitutions.)

It must be checked that these are inverse to the morphisms $\Sigma_r t^*\varphi \to t'^*\Sigma_s\varphi$ (for suitable t, t', r, s)—this amounts to writing out the various compositions and reducing them to the identity derivations. The details are tedious: we give one example:

$$\Delta_Y^* \underset{t \times Y}{\Sigma} \varphi \to \Sigma_t \langle X, t\rangle^* \varphi \to \Delta_Y^* \underset{t \times Y}{\Sigma} \varphi = \mathrm{id}.$$

We must show that the derivation (IV) on p. 519 is equivalent to (id). (We will suppose φ atomic, so that the use of (sub) is correct.) It is not difficult to reduce this to its normal form (V) on p. 520 (($\exists$ Perm), ($\exists$ Red), and ($\wedge$ Red)). Using (T Coh), ($=$ Exp), and (S Coh), this is equivalent to (VI) on p. 520. This in turn (using ($=$ Exp)) is equivalent to (VII) on p. 521. Using ($=$ Simp), ($=$ Red), and (S Coh), we arrive at the equivalent derivation (VIII) on p. 521, and this, using ($\wedge$ Exp) and ($\exists$ Exp), is easily shown to be equivalent to (id).

The other parts of the proof are similar: moreover, we shall have more to say about this in § 8 so we will not give further details here. So this completes the proof of the theorem.

§ 5. Construction 2: Hyperdoctrine → LPCE

For any hyperdoctrine $\boldsymbol{P}$ (over $\boldsymbol{T}$) there is a corresponding theory $T_{\boldsymbol{P}}$ in LPCE. Briefly, the theory $T := T_{\boldsymbol{P}}$ is constructed as follows: the sorts of the language $\mathscr{L}$ of T are the objects of $\boldsymbol{T}$, the function symbols are morphisms of $\boldsymbol{T}$, predicate symbols are objects of $\boldsymbol{P}$ (sorted by fibres) and rules of inference the morphisms of $\boldsymbol{P}$. If $gf = h$, then we add a reduction $\dfrac{P_f}{P_g} \Rightarrow P_h$, where P_i is the rule given by i.

(IV)

$$(1)$$

$$\underline{\langle tx, y' \rangle = \langle y, y \rangle \wedge \varphi(x, y')}\ (\wedge\,\mathrm{E})$$

$$(\wedge\,\mathrm{E})\ \underline{\quad(1)\quad}\qquad \underline{\langle tx, y' \rangle = \langle y, y \rangle}\ (\mathrm{pE})$$

$$(\mathrm{pE})\ \underline{\langle tx, y' \rangle = \langle y, y \rangle}\qquad \underline{tx = y}\ (\mathrm{S})$$

$$(\wedge\,\mathrm{E})\ \underline{\quad(1)\quad}\quad \underline{y' = y\quad y = tx}\ (\mathrm{T})\quad \underline{\quad(1)\quad}\ (\wedge\,\mathrm{E})\qquad\qquad (2)$$

$$(\mathrm{pE})\ \underline{\langle tx, y' \rangle = \langle y, y \rangle}\quad \underline{y' = tx\quad \varphi(x, y')}\ (\mathrm{sub})\qquad \underline{tx = y \wedge \varphi(x, tx)}\quad \underline{\quad(2)\quad}\ (\wedge\,\mathrm{E})$$

$$\underline{tx = y \qquad \varphi(x, tx)}\ (\wedge\,\mathrm{I})\qquad \underline{tx = y\quad tx = y}\quad \underline{\quad(2)\quad}\ (\wedge\,\mathrm{E})$$

$$\underline{tx = y \wedge \varphi(x, tx)}\ (\exists\,\mathrm{I})\qquad \underline{\langle tx, tx \rangle = \langle y, y \rangle\quad \varphi(x, tx)}\ (\wedge\,\mathrm{I})$$

$$\underline{\exists\langle \xi, \eta \rangle\,(\langle t\xi, \eta \rangle = \langle y, y \rangle \wedge \varphi(\xi, \eta))\ \ \exists\xi(t\xi = y \wedge \varphi(\xi, t\xi))}\ (\exists\,\mathrm{E})\ (1)\qquad \underline{\langle tx, tx \rangle = \langle y, y \rangle \wedge \varphi(x, tx)}\ (\exists\mathrm{I})$$

$$\underline{\exists\xi(t\xi = y \wedge \varphi(\xi, t\xi))\qquad\qquad \exists\langle \xi, \eta \rangle\,(\langle t\xi, \eta \rangle = \langle y, y \rangle \wedge \varphi(\xi, \eta))}\ (\exists\,\mathrm{E})\ (2)$$

$$\exists\langle \xi, \eta \rangle\,(\langle t\xi, \eta \rangle = \langle y, y \rangle \wedge \varphi(\xi, \eta))$$

(V)

$$
\begin{prooftree}
\AxiomC{$\exists\langle\xi,\eta\rangle\,(\langle t\xi,\eta\rangle = \langle y,y\rangle \wedge \varphi(\xi,\eta))$}
\AxiomC{(1)}
\noLine
\UnaryInfC{$\langle tx,y'\rangle = \langle y,y\rangle \wedge \varphi(x,y')$}
\LeftLabel{($\wedge$ E)}
\UnaryInfC{$\langle tx,y'\rangle = \langle y,y\rangle$}
\RightLabel{(pE)}
\UnaryInfC{$tx = y$}
\AxiomC{(1)}
\LeftLabel{($\wedge$ E)}
\UnaryInfC{$\langle tx,y'\rangle = \langle y,y\rangle$}
\LeftLabel{(pE)}
\UnaryInfC{$tx = y$}
\LeftLabel{(pI)}
\BinaryInfC{$\langle tx,tx\rangle = \langle y,y\rangle$}
\AxiomC{(1)}
\UnaryInfC{$\langle tx,y'\rangle = \langle y,y\rangle$}
\LeftLabel{(pE)}
\UnaryInfC{$y' = y$}
\AxiomC{(1)}
\RightLabel{($\wedge$ E)}
\UnaryInfC{$\langle tx,y'\rangle = \langle y,y\rangle$}
\RightLabel{(pE)}
\UnaryInfC{$tx = y$}
\RightLabel{(S)}
\UnaryInfC{$y = tx$}
\LeftLabel{(T)}
\BinaryInfC{$y' = tx$}
\AxiomC{(1)}
\RightLabel{($\wedge$ E)}
\UnaryInfC{$\varphi(x,y')$}
\RightLabel{(sub)}
\BinaryInfC{$\varphi(x,tx)$}
\RightLabel{($\wedge$ I)}
\BinaryInfC{$\langle tx,tx\rangle = \langle y,y\rangle \wedge \varphi(x,tx)$}
\RightLabel{($\exists$ I)}
\UnaryInfC{$\exists\langle\xi,\eta\rangle\,(\langle t\xi,\eta\rangle = \langle y,y\rangle \wedge \varphi(\xi,\eta))$}
\RightLabel{($\exists$ E) (1)}
\BinaryInfC{$\exists\langle\xi,\eta\rangle\,(\langle t\xi,\eta\rangle = \langle y,y\rangle \wedge \varphi(\xi,\eta))$}
\end{prooftree}
$$

(VI)

$$
\begin{prooftree}
\AxiomC{$\exists\langle\xi,\eta\rangle\,(\langle t\xi,\eta\rangle = \langle y,y\rangle \wedge \varphi(\xi,\eta))$}
\AxiomC{(1)}
\noLine
\UnaryInfC{$\langle tx,y'\rangle = \langle y,y\rangle \wedge \varphi(x,y')$}
\LeftLabel{($\wedge$ E)}
\UnaryInfC{$\langle tx,y'\rangle = \langle y,y\rangle$}
\LeftLabel{(pE)}
\UnaryInfC{$tx = y$}
\AxiomC{(1)}
\LeftLabel{($\wedge$ E)}
\UnaryInfC{$\langle tx,y'\rangle = \langle y,y\rangle$}
\LeftLabel{(pE)}
\UnaryInfC{$y' = y$}
\LeftLabel{(S)}
\UnaryInfC{$y = y'$}
\AxiomC{(1)}
\UnaryInfC{$\langle tx,y'\rangle = \langle y,y\rangle$}
\UnaryInfC{$y' = y$}
\AxiomC{(1)}
\RightLabel{($\wedge$ E)}
\UnaryInfC{$\langle tx,y'\rangle = \langle y,y\rangle$}
\RightLabel{(pE)}
\UnaryInfC{$tx = y$}
\RightLabel{(S)}
\UnaryInfC{$y = tx$}
\RightLabel{(T)}
\BinaryInfC{$y' = tx$}
\RightLabel{(T)}
\BinaryInfC{$y = tx$}
\RightLabel{(S)}
\UnaryInfC{$tx = y$}
\LeftLabel{(pI)}
\BinaryInfC{$\langle tx,tx\rangle = \langle y,y\rangle$}
\AxiomC{(1)}
\LeftLabel{($\wedge$ E)}
\UnaryInfC{$\langle tx,y'\rangle = \langle y,y\rangle$}
\LeftLabel{(pE)}
\UnaryInfC{$y' = y$}
\AxiomC{(1)}
\RightLabel{($\wedge$ E)}
\UnaryInfC{$\langle tx,y'\rangle = \langle y,y\rangle$}
\RightLabel{(pE)}
\UnaryInfC{$tx = y$}
\RightLabel{(S)}
\UnaryInfC{$y = tx$}
\LeftLabel{(T)}
\BinaryInfC{$y' = tx$}
\AxiomC{(1)}
\RightLabel{($\wedge$ E)}
\UnaryInfC{$\varphi(x,y')$}
\RightLabel{(sub)}
\BinaryInfC{$\varphi(x,tx)$}
\RightLabel{($\wedge$ I)}
\BinaryInfC{$\langle tx,tx\rangle = \langle y,y\rangle \wedge \varphi(x,tx)$}
\RightLabel{($\exists$ I)}
\UnaryInfC{$\exists\langle\xi,\eta\rangle\,(\langle t\xi,\eta\rangle = \langle y,y\rangle \wedge \varphi(\xi,\eta))$}
\RightLabel{($\exists$ E) (1)}
\BinaryInfC{}
\end{prooftree}
$$

(VII)

$$
\dfrac{\exists\langle\xi,\eta\rangle\,(\langle t\xi,\eta\rangle=\langle y,y\rangle\wedge\varphi(\xi,\eta))
\qquad
\dfrac{\dfrac{
\dfrac{\dfrac{\dfrac{(1)}{\langle tx,y'\rangle=\langle y,y\rangle\wedge\varphi(x,y')}\,(\wedge\mathrm{E})}{\dfrac{\langle tx,y'\rangle=\langle y,y\rangle}{y'=y}\,(\mathrm{pE})\quad \dfrac{\dfrac{\langle tx,y'\rangle=\langle y,y\rangle}{tx=y}\,(\mathrm{pE})}{y=tx}\,(\mathrm{S})}\,(\wedge\mathrm{E})}{y'=tx}\,(\mathrm{T})
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(1)}{\langle tx,y'\rangle=\langle y,y\rangle}\,(\wedge\mathrm{E})}{tx=y}\,(\mathrm{pE})\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(1)}{\langle tx,y'\rangle=\langle y,y\rangle}\,(\wedge\mathrm{E})}{y'=y}\,(\mathrm{pE})}{y=y'}\,(\mathrm{S})\quad \dfrac{\top}{y'=y'}\,(\mathrm{R})}{y=y'}\,(\mathrm{T})}{y'=y}\,(\mathrm{S})}{\langle tx,y'\rangle=\langle y,y\rangle}\,(\mathrm{pI})\quad \dfrac{\dfrac{\top}{y'=y'}\,(\mathrm{R})\quad \dfrac{(1)}{\varphi(x,y')}\,(\wedge\mathrm{E})}{\varphi(x,y')}\,(\mathrm{sub})}{\langle tx,y'\rangle=\langle y,y\rangle\wedge\varphi(x,y')}\,(\wedge\mathrm{I})}{\exists\eta(\langle tx,\eta\rangle=\langle y,y\rangle\wedge\varphi(x,\eta))}\,(\exists\mathrm{I})
}{\exists\eta(\langle tx,\eta\rangle=\langle y,y\rangle\wedge\varphi(x,\eta))}\,(\mathrm{sub})}{\exists\xi\,\exists\eta(\langle t\xi,\eta\rangle=\langle y,y\rangle\wedge\varphi(\xi,\eta))}\,(\exists\mathrm{I})
}{\exists\langle\xi,\eta\rangle\,(\langle t\xi,\eta\rangle=\langle y,y\rangle\wedge\varphi(\xi,\eta))}\,(\exists\mathrm{E})\ (1)
$$

(VIII)

$$
\dfrac{\exists\langle\xi,\eta\rangle\,(\langle t\xi,\eta\rangle=\langle y,y\rangle\wedge\varphi(\xi,\eta))
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(1)}{\langle tx,y'\rangle=\langle y,y\rangle}\,(\wedge\mathrm{E})}{tx=y}\,(\mathrm{pE})\quad \dfrac{\dfrac{(1)}{\langle tx,y'\rangle=\langle y,y\rangle}\,(\wedge\mathrm{E})}{y'=y}\,(\mathrm{pE})}{\langle tx,y'\rangle=\langle y,y\rangle}\,(\mathrm{pI})\quad \dfrac{\dfrac{(1)}{\langle tx,y'\rangle=\langle y,y\rangle\wedge\varphi(x,y')}}{\varphi(x,y')}\,(\wedge\mathrm{E})}{\langle tx,y'\rangle=\langle y,y\rangle\wedge\varphi(x,y')}\,(\wedge\mathrm{I})}{\exists\langle\xi,\eta\rangle\,(\langle t\xi,\eta\rangle=\langle y,y\rangle\wedge\varphi(\xi,\eta))}\,(\exists\mathrm{I})
}{\exists\langle\xi,\eta\rangle\,(\langle t\xi,\eta\rangle=\langle y,y\rangle\wedge\varphi(\xi,\eta))}\,(\exists\mathrm{E})\ (1)
$$

This is not quite enough, however—intuitively, we expect $\varphi \times \psi$ to be the same as $\varphi \wedge \psi$, but $\varphi \times \psi$ is an atomic formula, whereas $\varphi \wedge \psi$ certainly is not. It is easy to see that there is a derivation $\varphi \times \psi \vdash \varphi \wedge \psi$ (viz.

$$\frac{\begin{array}{cc} \varphi \times \psi & \varphi \times \psi \\ P_{\pi_1} & P_{\pi_2} \\ \varphi & \psi \end{array}}{\varphi \wedge \psi}$$

where P_{π_1}, P_{π_2} are the derivations (rules of inference, in fact) corresponding to the projections $\pi_1 \colon \varphi \times \psi \to \varphi$, $\pi_2 \colon \varphi \times \psi \to \psi$). But there is no reason to suppose that there is an inverse derivation $\varphi \wedge \psi \vdash \varphi \times \psi$, so if we want these to be equivalent, we must bodily add such a rule to T. Moreover, if we want this equivalence to be of the nature of an isomorphism, we must add reductions to the effect that the two "compositions" $\varphi \wedge \psi \to \varphi \times \psi \to \varphi \wedge \psi$ and $\varphi \times \psi \to \varphi \wedge \psi \to \varphi \times \psi$ both reduce to the identity. So we add to T the rule $\dfrac{\varphi \wedge \psi}{\varphi \times \psi}$ $(\times \mathrm{I})$, the reduction

$$(\times\ \mathrm{Red}) \qquad \frac{\begin{array}{cc} \begin{array}{c} P \\ \dfrac{\varphi \wedge \psi}{\varphi \times \psi} \\ P_{\pi_1} \\ \varphi \end{array} & \begin{array}{c} P \\ \dfrac{\varphi \wedge \psi}{\varphi \times \psi} \\ P_{\pi_2} \\ \psi \end{array} \end{array}}{\varphi \wedge \psi} \qquad \Rightarrow \begin{array}{c} P \\ \varphi \wedge \psi \end{array}$$

and the expansion

$$(\times\ \mathrm{Exp}) \qquad \begin{array}{c} P \\ \varphi \times \psi \end{array} \Rightarrow \quad \cfrac{\cfrac{\begin{array}{cc} \begin{array}{c} P \\ \varphi \times \psi \\ P_{\pi_1} \\ \varphi \end{array} & \begin{array}{c} P \\ \varphi \times \psi \\ P_{\pi_2} \\ \psi \end{array} \end{array}}{\varphi \wedge \psi}}{\varphi \times \psi}$$

We treat the remaining logical connectives and quantifiers in a similar fashion, so that $\times = \wedge$, $\bigsqcup = \vee$, $\exp = \supset$, $\Pi_\pi = \forall$, $\Sigma_\pi = \exists$, $0 = \bot$, $1 = \top$. (By $\Pi_\pi = \forall$ we mean $\forall$ is given by the "Π-image" along a projection; similarly for $\Sigma_\pi = \exists$.) Finally, we want t^* to mean substitution of t, so we must add the rules

$$\frac{\varphi(t)}{t^*\varphi}\ (*\ \mathrm{I}), \qquad \frac{t^*\varphi}{\varphi(t)}\ (*\ \mathrm{E}),$$

the reduction

$$(*\ \mathrm{Red}) \qquad \cfrac{\cfrac{\begin{array}{c} P \\ \varphi(t) \end{array}}{t^*\varphi}}{\varphi(t)} \Rightarrow \begin{array}{c} P \\ \varphi(t), \end{array}$$

and the expansion

$$(^*\,\mathrm{Exp})\qquad \begin{array}{c} P \\ t^*\varphi \end{array} \Rightarrow \dfrac{\dfrac{\begin{array}{c} P \\ t^*\varphi \end{array}}{\varphi(t)}}{t^*\varphi}$$

The equality predicate of sort X, for any $X \in |\boldsymbol{T}|$, is $\Sigma_{\Delta_X} \top_X$. (By an abuse of notation that is justified by the above, the terminal object of $\boldsymbol{P}(X)$ will be denoted $\top_X$. Similarly, products, coproducts, exponentiation, and the initial object in $\boldsymbol{P}(X)$ will henceforth be denoted $\wedge$, $\vee$, $\supset$, and $\perp_X$ respectively. The categorical structure of $\boldsymbol{T}$, on the other hand, will retain the more customary notation.)

We must now make sure that all this is justified: that in fact $\varphi \wedge \psi$ has the the logical properties we expect of $\varphi \wedge \psi$, and so on for the rest of the structure. This is well known for the propositional connectives: see SEELY [19]. That the (ordinary) quantifiers $\exists, \forall$ are given by the adjoints to projections – if φ is over $X \times Y$, then $\forall \xi \varphi$ (respectively $\exists \xi \varphi$), over Y, is $\Pi_{\pi_Y} \varphi$ (respectively $\Sigma_{\pi_Y} \varphi$), where $\pi_Y \colon X \times Y \to Y$ – is straightforward. We must check that the introduction and elimination rules for $\exists$ and $\forall$ are satisfied: for example, ($\exists$ E) becomes the assertation that given φ over $X \times Y$, φ' over Y, for types X, Y, and a proof (derivation) $\varphi \to \pi_Y^* \varphi'$, there is a proof (derivation) $\Sigma_{\pi_Y} \varphi \to \varphi'$. This is obviously so by adjointness. Also, ($\exists$ I) asserts the existence of a proof (derivation) $\varphi \to \pi_Y^* \Sigma_{\pi_Y} \varphi$, which again exists by adjointness. (Actually we should allow for a term $t\colon = \langle t', Y \rangle \colon Z \times Y \to X \times Y$. Applying t^* to the above gives the required $t^*\varphi \to \pi_Y'^* \Sigma_{\pi_Y} \varphi$, where now $\pi'_Y \colon Z \times Y \to Y$.) $\forall$ is handled similarly.

The equality rules follow as immediate consequences of our definition of E_X as $\Sigma_{\Delta_X} \top_X$: (R) asserts the existence of a morphism $\top_X \to \Delta_X^* \Sigma_{\Delta_X} \top_X$, which is the unit of the adjunction $\Sigma_{\Delta_X} \dashv \Delta_X^*$ at $\top_X$. For (sub), note that for any $\varphi \in |\boldsymbol{P}(X)|$, there is a morphism $s\colon \pi_2^* \varphi \wedge \Sigma_{\Delta_X} \top_X \to \pi_1^* \varphi$: essentially this is just the counit ε for the adjunction $\Sigma_{\Delta_X} \dashv \Delta_X^*$, by Frobenius Reciprocity. (Consider

$$\pi_2^*\varphi \wedge \Sigma_{\Delta_X} \top_X \xrightarrow{\sim} \Sigma_{\Delta_X}(\Delta_X^* \pi_2^* \varphi \wedge \top_X) \xrightarrow{\sim} \Sigma_{\Delta_X} \varphi \xrightarrow{\sim} \Sigma_{\Delta_X} \Delta_X^* \pi_1^* \varphi \xrightarrow[\varepsilon_{\pi_1^*\varphi}]{} \pi_1^* \varphi .)$$

The actual form of (sub) can be easily derived from this. Note first that by a theorem of LAWVERE [10], $\Sigma_{\Delta_{X \times Y}} \top_{X \times Y} \cong \pi_{13}^* \Sigma_{\Delta_X} \top_X \wedge \pi_{24}^* \Sigma_{\Delta_X} \top_Y$, for projections $X \times X \xleftarrow[\pi_{13}]{} X \times Y \times X \times Y \xrightarrow[\pi_{24}]{} Y \times Y$. We will show how to derive the morphism corresponding to

$$\frac{s = s' \quad t = t' \quad \varphi(s, t)}{\varphi(s', t')}$$

for φ over $X \times Y$, and terms $s, s' \colon X' \to X$, $t, t' \colon Y' \to Y$. This morphism is

$$\pi_{12}'^*(s \times t)^* \varphi \wedge \pi_{13}'^*(s \times s')^* E_X \wedge \pi_{24}'^*(t \times t')^* E_Y \to \pi_{34}'^*(s' \times t')^* \varphi;$$

it is obtained by applying $(s \times t \times s' \times t')^*$ to

$$\pi_{12}^* \varphi \wedge \pi_{13}^* E_X \wedge \pi_{24}^* E_Y \to \pi_{34}^* \varphi,$$

which exists because $\pi_{12}^* \varphi \wedge E_{X \times Y} \to \pi_{34}^* \varphi$ does, by LAWVERE's result.

$$
\begin{tikzcd}[column sep=small]
X' \times Y' \times X' \times Y' \arrow[rrrr, "s \times t \times s' \times t'"] \arrow[d, "\pi'_{12}" near start, shift right=24] \arrow[d, "\pi'_{13}", shift right=8] \arrow[d, "\pi'_{24}", shift left=8] \arrow[d, "\pi'_{34}", shift left=24] & & & & X \times Y \times X \times Y \arrow[d, "\pi_{12}", shift right=24] \arrow[d, "\pi_{13}", shift right=8] \arrow[d, "\pi_{24}", shift left=8] \arrow[d, "\pi_{34}", shift left=24] \\
X' \times Y' \quad X' \times X' \quad Y' \times Y' \quad X' \times Y' & & & & X \times Y \quad X \times X \quad Y \times Y \quad X \times Y
\end{tikzcd}
$$

(Note the four evident commutative squares.)

We have shown that T_P allows all the derivations we would expect of a theory in **LPCE**; we can do more, though, for it also has all the equivalences we would expect from the operations of § 2. As examples, we shall prove the only two that might pose any problems: (= Red) and (= Exp).

(= Red) To see this equivalence holds in T_P, it is sufficient to show (for $\varphi \in |P(X)|$)

$$\varphi \xrightarrow[\langle 1_\varphi, !_\varphi \rangle]{} \varphi \wedge \top_X \xrightarrow[1_\varphi \wedge \eta_{\top_X}]{} \varphi \wedge \Delta_X^* E_X \xrightarrow[\Delta_X^* s]{} \varphi$$

is the identity, where s is the morphism defined above (essentially (sub)), and $E_X = \Sigma_{\Delta_X} \top_X$. Using the definition of s and the triangle equality $\Delta_X^* \varepsilon \cdot \eta \Delta_X^* = \mathrm{id}$, this reduces to showing the square (*) commutes:

$$
\begin{tikzcd}
 & \varphi \wedge \top_X \arrow[r, "1_\varphi \wedge \eta_{\top_X}"] \arrow[dr, phantom, "(*)_3"] & \varphi \wedge \Delta_X^* \Sigma_{\Delta_X} \top_X \arrow[d, "\eta_{\Delta_X^*}(\pi^*\varphi \wedge E_X)"] \arrow[dd, bend left=70, "\mathrm{id}"] \\
\varphi \arrow[ur, "\langle 1_\varphi, !_\varphi \rangle"] \arrow[r, "\eta_\varphi"] \arrow[d, "\langle 1_\varphi, !_\varphi \rangle"'] \arrow[dr, phantom, "(*)"] & \Delta_X^* \Sigma_{\Delta_X} \varphi \arrow[r, "\Delta_X^* \Sigma_{\Delta_X} \langle 1_\varphi, \eta_{\top_X} \rangle"] \arrow[d, "\Delta_X^* f"] \arrow[dr, phantom, "(*)_2"] & \Delta_X^* \Sigma_{\Delta_X} \Delta_X^* (\pi^* \varphi \wedge E_X) \arrow[dl, "\Delta_X^* \varepsilon_{\pi^*\varphi \wedge E_X}"] \arrow[d, phantom, "(*)_4"] \\
\varphi \wedge \top_X \arrow[r, "1_\varphi \wedge \eta_{\top_X}"'] & \varphi \wedge \Delta_X^* \Sigma_{\Delta_X} \top_X & {}
\end{tikzcd}
$$

f is the canonical map, so that $(*)_2$ commutes by definition; $(*)_3$ commutes by the naturality of η, and $(*)_4$ by the triangle equality. □

(= Exp) We must show

$$E_X \xrightarrow[\langle !_{E_X}, !_{E_X}, 1_{E_X} \rangle]{} \top_{X^2} \wedge \top_{X^2} \wedge E_X \xrightarrow[\pi_1^* \eta_{\top_X} \wedge \pi_1^* \eta_{\top_X} \wedge 1_{E_X}]{}$$
$$\to \pi_1^* \Delta_X^* E_X \wedge \pi_1^* \Delta_X^* E_X \wedge E_X \xrightarrow[\nabla^* s]{} E_X$$

is the identity, where $\nabla\colon X \times X \to X \times X \times X \times X$ is $\langle 1_X, 1_X, 1_X \rangle \times 1_X$ (i.e. "$\langle x_0, x_1 \rangle \mapsto \langle x_0, x_0, x_0, x_1 \rangle$".) In fact, we will show the corresponding morphism $\top_X \to \Delta_X^* E_X$ is the unit $\eta_{\top_X}$. By a process similar to that used for (= Red), we can reduce this to showing

$$
\begin{tikzcd}
\top_X \arrow[rr, "\langle \eta^1_{\top_X}, \eta^1_{\top_X} \rangle"] \arrow[d, "\Delta_X^* \eta^2_{\top_{X^2}}"'] & & \Delta_X^* E_X \wedge \Delta_X^* E_X \arrow[d, "\Delta_X^* \Delta_{X^2}^* \varepsilon_{\pi_{13}^* E_X \wedge \pi_{24}^* E_X}"] \\
\Delta_X^* \Delta_{X^2}^* E_X{}^2 \arrow[rr, "\Delta_X^* \Delta_{X^2}^* \Sigma_{\Delta_{X^2}} (\pi_1^* \eta^1_{\top_X}, \pi_2^* \eta^1_{\top_X})"'] & & \Delta_X^* \Delta_{X^2}^* \Sigma_{\Delta_{X^2}} \Delta_{X^2}^* (\pi_{13}^* E_X \wedge \pi_{24}^* E_X)
\end{tikzcd}
$$

commutes, where η^i is the unit of $\Sigma_{\Delta_{X^i}} \vdash \Delta^*_{X^i}$ $(i = 1, 2)$ $(X^2 = X \times X$, of course!). (The rest of the notation is standard.) By considering the various components $\top_X \to \Delta^*_X E_X$, and recalling that "the counit of a product is the product of the counits", this reduces in turn to considering

$$\begin{tikzcd}
\top_X \arrow[rr, "\eta^1_{\top_X}"] \arrow[d, "\Delta^*_X \eta^2_{\top_X}"'] & & \Delta^*_X E_X \\
\Delta^*_X \Delta^*_{X^2} E_{X^2} \arrow[rr, "\Delta^*_X \Delta^*_{X^2} \Sigma_{\Delta_{X^2}} \pi^*_1 \eta^1_{\top_X}"'] & & \Delta^*_X \Delta^*_{X^2} \Sigma_{\Delta_{X^2}} \Delta^*_{X^2} \pi^*_{13} E_X \arrow[u, "\Delta^*_X \Delta^*_{X^2} \varepsilon^2_{\pi_{13}^* E_X}"']
\end{tikzcd}$$

However, this commutes (using the triangle equality $\Delta^*_{X^2}\varepsilon^2 \cdot \eta^2 \Delta^*_{X^2} = \mathrm{id}$) by the naturality of η^2:

$$\begin{tikzcd}
\top_{X^2} \arrow[rr, "\pi^*_1 \eta^1_{\top_X}"] \arrow[d, "\eta^2_{\top_{X^2}}"'] & & \pi^*_1 \Delta^*_X E_X \arrow[d, "\eta^2_{\pi_1^* \Delta_X^* E_X}"] \\
\Delta^*_{X^2} \Sigma_{\Delta_{X^2}} \top_{X^2} \arrow[rr, "\Delta^*_{X^2} \Sigma_{\Delta_{X^2}} \pi^*_1 \eta^1_{\top_X}"'] & & \Delta^*_{X^2} \Sigma_{\Delta_{X^2}} \pi^*_1 \Delta^*_X E_X
\end{tikzcd}$$

So we have the following:

Theorem. *If $\boldsymbol{P}$ is a hyperdoctrine, then $T_{\boldsymbol{P}}$, as constructed above, is a theory in* LPCE, *in that every deduction rule of* LPCE *is derivable in $T_{\boldsymbol{P}}$, and all the equivalences generated by the operations of* § 2, *are satisfied (and in fact, are identities).*

§ 6. Equivalences

What happens if we iterate these constructions? Nothing new:

Theorem. *If T is a theory in* LPCE, *then $T_{\boldsymbol{P}_T}$ is equivalent to T in the following way: for every formula φ of T, there is an (atomic) formula $\bar{\varphi}$ of $T_{\boldsymbol{P}_T}$ canonically induced by it. Also, for every derivation $P\colon \varphi \vdash \psi$ of T, there is a derivation (rule of inference, in fact) $\bar{P}\colon \bar{\varphi} \vdash \bar{\psi}$ of $T_{\boldsymbol{P}_T}$ canonically induced by it. Moreover, every formula φ' of $T_{\boldsymbol{P}_T}$ is logically equivalent to $\bar{\varphi}$ for some φ of T (in fact, isomorphic, in the sense of the categorical structure on $T_{\boldsymbol{P}_T}$), and for every derivation $P'\colon \varphi' \vdash \psi'$ of $T_{\boldsymbol{P}_T}$ there is a derivation $\bar{P}\colon \bar{\varphi} \vdash \psi$ induced by some $P\colon \varphi \vdash \psi$ in T, where $\bar{\varphi}$ (respectively $\bar{\psi}$) is logically equivalent (in fact isomorphic) to φ' (respectively ψ'). (P' and $\bar{P}$ are also "isomorphic".)*

(What we shall actually prove is: T and $T_{\boldsymbol{P}_T}$ are equivalent as categorical structures. This implies they are logically equivalent theories.)

Proof. Given φ in T, by construction $\varphi \in |\boldsymbol{P}_T|$ and so is a predicate symbol of $T_{\boldsymbol{P}_T}$; $\bar{\varphi}$ is just this predicate symbol with the suitable free variables to make it a well-formed (atomic) formula. Given $P\colon \varphi \vdash \psi$ in T, P is (or rather, its equivalence class is) a morphism of $\boldsymbol{P}$, and so is a rule of inference of $T_{\boldsymbol{P}_T}$; $\bar{P}$ is the corresponding (trivial) derivation.

In defining $T_{\boldsymbol{P}}$ for any hyperdoctrine $\boldsymbol{P}$, we added enough rules to enable us to conclude that every formula is equivalent to an atomic formula. Moreover, by considering the adjointness property of equality, we see that $x_0 = x_1$ is isomorphic to

$\exists\xi(\xi = x_0 \wedge \xi = x_1 \wedge \top_X)$ (i.e., $\Sigma_{\Delta_X}\top_X$), so "new" and "old" equality agree. Finally, in any hyperdoctrine the Beck condition for type (b) pullbacks implies that the "equality" meta-notion we defined was correct: equality over the *sort* $X \times Y$ of T_{P_T} and "meta-equality" over the *type* $X \times Y$ of T agree. (See SEELY [18] for details.)

So we have proved the theorem, and even more, once we note that the operation (functor) $\varphi \mapsto \bar{\varphi}$, $P \mapsto \bar{P}$ is full and faithful (up to equivalence of derivations). (This justifies the assertion that T and T_{P_T} are equivalent as categories.)

Theorem. *If $\boldsymbol{P}$ is a hyperdoctrine, then $\boldsymbol{P}_{T_{\boldsymbol{P}}}$ is an equivalent hyperdoctrine.*

Proof. First, note that there are canonical functors $\boldsymbol{P} \xrightarrow[j]{} \boldsymbol{P}_{T_{\boldsymbol{P}}}$, $\boldsymbol{T} \xrightarrow[i]{} \boldsymbol{T}_{T_{\boldsymbol{P}}}$, which are fully faithful. These functors are also essentially surjective—the proof of this fact also shows that $\boldsymbol{P}$ and $\boldsymbol{P}_{T_{\boldsymbol{P}}}$ are equivalent as hyperdoctrines, since $(\ )^*$ and its adjoints Σ, Π are preserved up to isomorphism. That i is essentially surjective is seen easily: the only "new" objects of $\boldsymbol{T}_{T_{\boldsymbol{P}}}$ correspond to the types of $T_{\boldsymbol{P}}$ which are not sorts, and these are isomorphic to obvious sorts (or objects of $\boldsymbol{T}_{T_{\boldsymbol{P}}}$); morphisms of $\boldsymbol{T}_{T_{\boldsymbol{P}}}$ ("terms") are treated similarly. The "propositional" part of the proof that j is essentially surjective is analagous. Since we added rules to $T_{\boldsymbol{P}}$ to make $t^*\varphi$ and $\varphi(t)$ isomorphic, and treated the quantifiers similarly, j must be essentially surjective. But even more can be asserted: the adjoints agree up to isomorphism. By a theorem of LAWVERE in [10] $\Sigma_t\varphi \cong \Sigma_{\pi_Y}(E_t \wedge \pi_X^*\varphi)$ for any hyperdoctrine with the Beck condition for pullbacks of type (a), where $\varphi \in |\boldsymbol{P}(X)|$ and $\pi_X\colon X \times Y \to X$, $\pi_Y^*\colon X \times Y \to X$, $t\colon X \to Y$ in $\boldsymbol{T}$, and $E_t =_{\mathrm{df}} (t \times Y)^* E_Y$. One can also show (under the same assumptions) $\Pi_t\varphi \cong \Pi_{\pi_Y}(E_t \supset \pi_X^*\varphi)$. Using this, it is easy to see that the adjoints to t^* agree —the left hand side (of these isomorphisms) giving the adjoint in $\boldsymbol{P}$, and the right hand side giving that in $\boldsymbol{P}_{T_{\boldsymbol{P}}}$. This completes the proof.

Remark. We have not yet defined the notion of a "morphism of hyperdoctrines", but it is clear that what we would want is the following. Suppose $\boldsymbol{P}_0$ (over $\boldsymbol{T}_0$), $\boldsymbol{P}_1$ (over $\boldsymbol{T}_1$) are hyperdoctrines. Then a morphism F from $\boldsymbol{P}_0$ to $\boldsymbol{P}_1$ is given by a functor $\boldsymbol{T}_0 \xrightarrow{F_0} \boldsymbol{T}_1$, and for each $X \in |\boldsymbol{T}_0|$, a functor $F_1(X)\colon \boldsymbol{P}_0(X) \to \boldsymbol{P}_1(F_0X)$, with the evident preservation properties: F_0 preserves finite products; for each X, $F_1(X)$ preserves the locally closed structure of $\boldsymbol{P}_0(X)$; and for any $X \xrightarrow{t} Y$ in $\boldsymbol{T}_0$, the following commute (up to isomorphism):

(i)
$$\begin{tikzcd} \boldsymbol{P}_0(X) \arrow[r, "\Sigma_t"] \arrow[d, "F_1(X)"'] & \boldsymbol{P}_0(Y) \arrow[d, "F_1(Y)"] \\ \boldsymbol{P}_1(F_0X) \arrow[r, "\Sigma_{F_0t}"'] & \boldsymbol{P}_1(F_0Y) \end{tikzcd}$$

(ii)
$$\begin{tikzcd} \boldsymbol{P}_0(X) \arrow[d, "F_1(X)"'] & \boldsymbol{P}_0(Y) \arrow[l, "t^*"'] \arrow[d, "F_1(Y)"] \\ \boldsymbol{P}_1(F_0X) & \boldsymbol{P}_1(F_0Y) \arrow[l, "(F_0t)^*"] \end{tikzcd}$$

(iii)
$$\begin{tikzcd} \boldsymbol{P}_0(X) \arrow[r, "\Pi_t"] \arrow[d, "F_1(X)"'] & \boldsymbol{P}_0(Y) \arrow[d, "F_1(Y)"] \\ \boldsymbol{P}_1(F_0X) \arrow[r, "\Pi_{F_0t}"'] & \boldsymbol{P}_1(F_0Y) \end{tikzcd}$$

This just corresponds to the notion of an interpretation of $T_{\boldsymbol{P}_0}$ in $T_{\boldsymbol{P}_1}$, SHOENFIELD [20] (modified to suit our treatment of first order logic, of course.) (If we think of $\boldsymbol{P}$ as a (pseudo) functor $\boldsymbol{T}^{\mathrm{op}} \xrightarrow{\boldsymbol{P}} \mathscr{C}at$, then (ii) is just the condition that $F_1 : \boldsymbol{P}_0 \to \boldsymbol{P}_1 F_0$ is natural.)

If we then define the category $\mathscr{H}yp$ of hyperdoctrines and morphisms of hyperdoctrines, and the category $\mathscr{T}hy$ of theories and interpretations of theories, then we have shown

Theorem. *The categories $\mathscr{H}yp$ and $\mathscr{T}hy$ are equivalent.*

(In § 8 we will refine the equivalence to one dealing with hyperdoctrines satisfying the Beck condition locally.)

§ 7. Examples

We shall briefly describe two well known examples of hyperdoctrines, and a structure not quite a hyperdoctrine.

7.1 Definition. A *logos* is a category $\boldsymbol{L}$ with finite limits and stable image factorisations, such that every object has a minimal subject, every pair of subobjects has a union, and for every morphism $X \xrightarrow{t} Y$ and subobject $X' \rightarrowtail X$ there is a subobject $\Pi_t X' \rightarrowtail Y$ maximal among subobjects of Y whose pullbacks along t lie in X'.

There are many examples of logoi: for instance, any topos is a logos. (This is no more than the truism that first order logic is part of higher order logic, in view of the following.)

Any logos $\boldsymbol{L}$ induces an indexed category $\mathrm{Sub}_{\boldsymbol{L}}$ (over $\boldsymbol{L}$) in the obvious way by taking the poset of subobjects of X as the fibre over X, and defining t^* as pullback along t. The following is well known (see REYES [16]):

Proposition. *If $\boldsymbol{L}$ is a logos, then* $\mathrm{Sub}_{\boldsymbol{L}}$ *is a hyperdoctrine. Conversely, if $\boldsymbol{L}$ has finite limits and* $\mathrm{Sub}_{\boldsymbol{L}}$ *is a hyperdoctrine, then $\boldsymbol{L}$ is a logos.*

So we can identify logoi as certain hyperdoctrines, whose fibres are posets. In fact, if $\mathscr{L}og$ is the category of logoi (and structure preserving functors) and $\mathscr{P}ohyp$ the full subcategory of $\mathscr{H}yp$ of hyperdoctrines with poset fibres ("po-hyperdoctrines"), then:

Proposition. *$\mathscr{L}og$ is a reflective subcategory of $\mathscr{P}ohyp$.*

(The reflection is given by the construction of the "Lindenbaum-Tarski-category" of the theory corresponding to a po-hyperdoctrine. See SEELY [18] for details.)

7.2. Our first example of a hyperdoctrine dealt with the notion of "propositional function"; the next will concern that of "indexed family".

Definition. A *locally closed category* is a category $\boldsymbol{B}$ with finite limits and finite coproducts (including a 0), such that for any object X of $\boldsymbol{B}$, $\boldsymbol{B}/X$ is cartesian closed.

I believe the following observation is due independently to LAWVERE and BENABOU – a proof appears in FREYD [2] (Proposition 1.34).

Proposition. *If $\boldsymbol{B}$ is locally closed, then $\boldsymbol{B}$, as a $\boldsymbol{B}$-category, is a hyperdoctrine. Conversely, if $\boldsymbol{B}$ has finite limits, and $\boldsymbol{B}$, as a $\boldsymbol{B}$-category, is a hyperdoctrine, then $\boldsymbol{B}$ is locally closed. In fact, if $\boldsymbol{B}$ has finite limits and finite coproducts, and $\boldsymbol{B}$, as a $\boldsymbol{B}$-category, is locally small, then $\boldsymbol{B}$ is locally closed.*

Remark. $\boldsymbol{B}$, as a $\boldsymbol{B}$-category, is the indexed category with fibres $\boldsymbol{B}(X) = \boldsymbol{B}/X$, the comma category. This is locally small iff for each X, $\boldsymbol{B}/X$ is cartesian closed, iff for each t, t^* has a right adjoint Π_t. (See, e.g., PARÉ-SCHUMACHER [13].) (The presence of coproducts is irrelevant to all this – we include them only because our logic includes $\vee$.)

Remark. *If $\boldsymbol{B}$ is both a logos and locally closed, (e.g. a topos) then there is an indexed functor between these hyperdoctrines,* $\mathrm{Im}\colon \boldsymbol{B} \to \mathrm{Sub}_{\boldsymbol{B}}$, *taking a morphism to its image. This preserves $\wedge$, $\vee$, $\top$, $\bot$, $(\ )^*$, and Σ; it also preserves Π (equivalently $\supset$) and so is a hyperdoctrine morphism iff for any $t\colon X \to Y$, Π_t preserves regular epis. In a topos, this is a weak version of the axiom of choice.* (See JOHNSTONE [6] § 5.2; also SEELY [18] for details.)

7.3. Our third example just misses being a hyperdoctrine: The base category is $\mathscr{G}\!\mathit{pd}$, the category of all groupoids. (A groupoid is a category all of whose morphisms are isomorphisms.) Given a groupoid $\boldsymbol{X}$, the fibre $\boldsymbol{P}(\boldsymbol{X}) =_{\mathrm{df}} \mathscr{S}\!\mathit{ets}^{X}$ is the category of functors $\boldsymbol{X} \to \mathscr{S}\!\mathit{ets}$. (If $\boldsymbol{X}$ is a group – i.e., has only one object – then $\mathscr{S}\!\mathit{ets}^{X}$ is perhaps better known as the category of permutation representations of $\boldsymbol{X}$.) $\mathscr{S}\!\mathit{ets}^{X}$ is a topos, and so is certainly locally closed. Given a functor (groupoid homomorphism) $\boldsymbol{X} \xrightarrow{t} Y$, $t^*\colon \mathscr{S}\!\mathit{ets}^{Y} \to \mathscr{S}\!\mathit{ets}^{X}$ is defined by composition with t. Σ_t, Π_t are defined as left and right Kan extensions (c.f. MACLANE [11]). (For groups $\Sigma_t\varphi$ is the "induced representation" of Y; we may call $\Pi_t\varphi$ the "dual induced representation".)

For the benefit of the non-category theorist (and because we will need this later) I will explicitly calculate $\Sigma_t\varphi$, for $\varphi\colon \boldsymbol{X} \to \mathscr{S}\!\mathit{ets}$. To simplify notation, let us suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$ are groups: denote their unique objects by X and Y respectively, and write "$z \in Z$" for "z is a morphism of Z". In this case $\varphi\colon \boldsymbol{X} \to \mathscr{S}\!\mathit{ets}$ can be represented as $\langle \Phi, \varphi_x \rangle_{x \in X}$, where Φ is a set (it is $\varphi(X)$), and for each $x \in X$, φ_x is a permutation of Φ (satisfying the usual equation $\varphi_{x_0 x_1^{-1}} = \varphi_{x_0}\varphi_{x_1}^{-1}$, so that the φ_x define an X action on Φ).

Then $\Sigma_t\varphi$ is, as a set, the "orbit set" of $Y \times \Phi$, i.e. $Y \times \Phi/\!\sim$ for $(y_0, a_0) \sim (y_1, a_1)$ iff for some $x \in X$ $(y_1, a_1) = (y_0 x, x^{-1}a_0) =_{\mathrm{df}} x(y_0, a_0)$. (This defines the action of X on $\Sigma_t\varphi$ also.). For groupoids the idea is similar, (details in SEELY [18]).

Proposition. (i) *The groupoid representation structure satisfies Frobenius Reciprocity.* (ii) *It does satisfy the Beck condition for pullbacks of types* (a) *and* (b) (cf. § 3), *but not, in general, for pullbacks of type* (c). (iii) *The Beck condition is preserved under products, in the case where φ is E_Y* (the case used in § 8).

Remarks. (i) is proven in LAWVERE [10], (where LAWVERE also seems to suggest that (ii) is false, in that he suggests that his "meagre theorems apparently do not hold in [this structure]", where these theorems depend only on the Beck condition for pullbacks of types (a) and (b)). (ii) and (iii) may in fact be proven by a direct "diagram chase": details are given in SEELY [18].

Among other things, this example shows that the operation (R Coh) is independent of the usual expansions, reductions, and permutations. Its logic is perfectly well behaved as far as the equality-free part is concerned, but the logic of equality does show some curiosities Clearly $t = t$ is quite different from $\top$, and so can be an important assumption in a derivation, for instance. Also $x = y$ and $\langle x, x\rangle = \langle y, y\rangle$ behave quite differently. However, some things remain: the rules (sub) and (R) (though (R) is not an isomorphism), and the equivalences (= Red), (= Exp) all hold in $\mathscr{G}\!\mathit{pd}$.

§ 8. The Beck Condition

Most of the structure in the definition of a hyperdoctrine is (from the logical point of view) more or less self-evident-that is, it is clear why the definition must impose such conditions. The only question one might pose is "why do we need the Beck condition?" Part of the answer is already given to us: to even begin interpreting the categorical structure logically, the Beck condition is needed for pullbacks of the three types given in § 3. Type (a) allows us to interpret $\Sigma_t\varphi$ as $\exists\xi(t\xi = x \wedge \varphi(\xi))$—otherwise it would just be some formula about which we knew little. Type (b) allows us to treat types as if they were sorts, especially with respect to equality. It also gives a result analogous to Frobenius Reciprocity, viz. that substitution is well-defined: for

$$[\exists\alpha\varphi(\alpha, y)]\,(t) = \exists\alpha\varphi(\alpha, tx)$$

becomes the isomorphism $t^*\Sigma_{\pi_Y}\varphi \cong \Sigma_{\pi_X}(A \times t)^*\,\varphi$, where

$$\begin{array}{ccc} A \times X & \xrightarrow{A \times t} & A \times Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{\quad t \quad} & Y \end{array}$$

and φ is over $A \times Y$. Type (c) gives the isomorphism $(x = x) \cong \top_X$ (and in practice allows us to treat Δ_X as the diagonal morphism it is). There is one further condition we need: the preservation of the Beck condition under products. This amounts to the following: if for all φ over Y, $\exists\xi(r\xi = x \wedge \varphi(t\xi)) \cong \exists\eta(s\eta = t'x \wedge \varphi(\eta))$ then for all ψ over $Y \times Z$,

$$\exists\xi\,\exists\zeta(r\xi = x \wedge \zeta = z \wedge \psi(t\xi, \zeta)) \cong \exists\eta\,\exists\zeta(s\eta = t'x \wedge \zeta = z \wedge \psi(\eta, \zeta)).$$

(Which says that we may suppress free variables in considering the Beck condition.) This would seem justified without further comment.

There is one remark we should perhaps make here: if we wished, we could do without the Beck condition for pullbacks of type (c). Of course, we would then have to drop (R Coh) from the operations of § 2, and pay very close attention to the assumptions of our derivations (not overlooking any of the form $t = t$) and to any use of the diagonal (not confusing the formulae $\langle t, t\rangle = \langle t', t'\rangle$ and $t = t'$). An example of a structure of this sort was seen in § 7. However, we shall continue to regard (R Coh) as a desirable equivalence, and so shall continue to suppose Beck for type (c).

For the rest of this section, let "hyperdoctrine" be understood in this new sense: we replace the full Beck condition with only those instances generated by diagrams of types (a), (b), (c) and by the "product" operation. (Of course, all t^* will still have both adjoints.) Then we can characterize the Beck condition for arbitrary pullbacks as follows:

Theorem. *Let $\boldsymbol{P}$ (over $\boldsymbol{T}$) be a hyperdoctrine (in the above sense) and let*

$$\text{(D)}\qquad \begin{array}{ccc} X & \xrightarrow{t} & Y \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} \\ X' & \xrightarrow{t'} & Y' \end{array}$$

be a pullback in $\boldsymbol{T}$. Then $\boldsymbol{T}$ has the Beck conditions for (D) (*viz. for any $\varphi \in |\boldsymbol{P}(Y)|$, $\Sigma_r t^*\varphi \cong t'^*\Sigma_s\varphi$ and for $\psi \in |\boldsymbol{P}(X')|$, $\Sigma_t r^*\psi \cong s^*\Sigma_{t'}\psi$*) *iff in $T_{\boldsymbol{P}}$ there are derivations*

$$P_I\colon t'x' = sy \vdash \exists\xi(r\xi = x' \wedge t\xi = y) \quad (\mathit{over}\ X' \times Y)$$

and

$$P_E\colon rx_0 = rx_1,\ tx_0 = tx_1 \vdash x_0 = x_1 \quad (\mathit{over}\ X \times Y).$$

Remarks. (1) In earlier versions of this paper (such as SEELY [18]) certain "coherence" conditions were attached to this theorem. For the benefit of readers of the earlier versions, I should remark that these conditions are consequences of Corollary 1 of § 2.

(2) The theorem can be given by the slogan: "a hyperdoctrine satisfies the Beck condition for a pullback diagram iff it knows that the diagram is a pullback", in view of the characterisation of pullbacks in *Sets*.

Proof of the theorem. We shall prove the theorem in two sections: in § 8.1 we construct morphisms in $\boldsymbol{P}$ which induce P_I, P_E in $\mathscr{T}_{\boldsymbol{P}}$, assuming that $\boldsymbol{P}$ has the Beck condition for (D). In § 8.2 we suppose $T_{\boldsymbol{P}}$ has P_I, P_E and show that $\boldsymbol{P}_{T_{\boldsymbol{P}}}$ (and so $\boldsymbol{P}$) satisfies the Beck condition for (D) by constructing an inverse to the suitable canonical proof. The difficulty comes in showing the two proofs inverse: to do this we write out each composite, reduce it to normal form, and then by a careful series of expansions and reductions (whose general purpose may be described as "replacing equals by equals") show that it is equivalent to the identity.

8.1. Suppose that $\boldsymbol{P}$ has the Beck conditions for (D). Then P_I is induced by the (iso)morphism $(t' \times Y)^* \Sigma_{s\times Y} E_Y \xrightarrow{\sim} \Sigma_{r\times Y}(t \times Y)^* E_Y$: this exists since we have assumed the Beck condition is preserved under products. It is easy to see (using LAWVERE's theorems, since we have the Beck condition for pullbacks of types (a) and (b), and $E_Y =_{\mathrm{df}} \Sigma_{\Delta_Y} \top_Y$) that $(t' \times Y)^* \Sigma_{s\times Y} E_Y \cong (t' \times s)^* E_{Y'}$, and that $\Sigma_{r\times Y}(t \times Y)^* E_Y \cong$ $\cong \Sigma_{X'Y}(\pi^*_{XX'}(r \times X')^* E_X \wedge \pi^*_{XY}(t \times Y)^* E_Y)$, so we have the required morphism.

P_E is induced by the (iso)morphism $(\langle r, t\rangle \times \langle r, t\rangle)^* E_{X'\times Y} \xrightarrow{\sim} E_X$. (As above, it is easy to show $(\langle r, t\rangle \times \langle r, t\rangle)^* E_{X'\times Y} \cong (r \times r)^* E_X \wedge (t \times t)^* E_Y$, so this does give the required morphism.) The existence of this isomorphism is shown by the following simple lemmas:

Lemma 1. *For any* $t\colon X \to Y$, $\langle t, t\rangle^* E_Y \cong \top_X$. In fact we can claim more: For any $t\colon X \to Y$, the Beck condition holds for

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "X"'] & Y \arrow[d, "\Delta_Y"] \\ X \arrow[r, "\langle t, t\rangle"'] & Y \times Y \end{tikzcd}$$

Proof. Consider
$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "X"'] & Y \arrow[r, "Y"] \arrow[d, "Y"'] & Y \arrow[d, "\Delta_Y"] \\ X \arrow[r, "t"'] & Y \arrow[r, "\Delta_Y"'] & Y \times Y. \end{tikzcd}$$

Lemma 2. *If* (D) *is a pullback, then so is*

(D′)
$$\begin{tikzcd} X \arrow[r, "t'r"] \arrow[d, "\langle r, t\rangle"'] & Y' \arrow[d, "\Delta_{Y'}"] \\ X' \times Y \arrow[r, "t' \times s"'] & Y' \times Y' \end{tikzcd}$$

In addition, if the Beck condition holds for (D), *then it does also for* (D′).

Proof. Consider
$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d, "\langle r, X\rangle"'] & X' \arrow[r, "t'"] \arrow[d, "\Delta_{X'}"] & Y' \arrow[dd, "\Delta_{Y'}"] \\ X' \times X \arrow[r, "X' \times r"] \arrow[d, "X' \times t"'] & X' \times X' \arrow[d, "X' \times t'"] & \\ X' \times Y \arrow[r, "X' \times s"'] & X' \times Y' \arrow[r, "t' \times Y'"'] & Y' \times Y'. \end{tikzcd}$$

Lemma 3. *Consider*
$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "r"'] & Y \arrow[r, "t_1"] \arrow[d, "s"'] & Z \arrow[d, "p"'] \\ X' \arrow[r, "t'"'] & Y' \arrow[r, "t_1'"'] & Z' \end{tikzcd}$$
Suppose

$$\begin{tikzcd} X \arrow[r, "t_1 t"] \arrow[d, "r"'] & Z \arrow[d, "p"] \\ X' \arrow[r, "t_1' t'"'] & Z' \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} Y \arrow[r, "t_1"] \arrow[d, "s"'] & Z \arrow[d, "p"] \\ Y' \arrow[r, "t_1'"'] & Z \end{tikzcd}$$

are pullbacks and that the Beck condition holds for each. Then

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "r"'] & Y \arrow[d, "s"] \\ X' \arrow[r, "t'"'] & Y' \end{tikzcd}$$

is a pullback and the Beck condition holds for any φ *over* Y *of the form* $t_1^*\psi$ *for* ψ *over* Z (*viz.* $\Sigma_r t^*\varphi \cong t'^*\Sigma_s\varphi$ *if* $\varphi = t_1{}^*\psi$).

Proof trivial.

Corollary 1. *For* (D) *as in the theorem, the Beck condition for* $\top_X$ *holds for*

$$\begin{tikzcd} X \arrow[r, "X"] \arrow[d, "X"'] & X \arrow[d, "\langle r, t\rangle"] \\ X \arrow[r, "\langle r, t\rangle"'] & X' \times Y \end{tikzcd}$$

(*viz.* $\top_X \cong \langle r, t\rangle^*\Sigma_{\langle r,t\rangle}\top_X$).

Proof. Consider

$$\begin{tikzcd} X \arrow[r, "X"] \arrow[d, "X"'] & X \arrow[r, "t'r"] \arrow[d, "\langle r, t\rangle"'] & Y' \arrow[d, "\Delta_{Y'}"] \\ X \arrow[r, "\langle r, t\rangle"'] & X' \times Y \arrow[r, "t' \times s"'] & Y' \times Y' \end{tikzcd}$$

note that $(t' \times s)\langle r, t\rangle = \langle t'r, t'r\rangle$, and that $\top_X = (t'r)^*\top_{Y'}$.

Lemma 4. *For any* $t\colon X \to Y$, *if*

(D_0)

$$\begin{tikzcd} X \arrow[r, "X"] \arrow[d, "X"'] & X \arrow[d, "t"] \\ X \arrow[r, "t"'] & Y \end{tikzcd}$$

is a pullback, then so is

(D_1)

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "\Delta_X"'] & Y \arrow[d, "\Delta_Y"] \\ X \times X \arrow[r, "t \times t"'] & Y \times Y \end{tikzcd}$$

Moreover, if the Beck condition (*for* φ *over* X) *holds for* (D_0), *then it does also* (*respectively for any* ψ *over* Y *so that* $t^*\psi = \varphi$) *for* (D_1).

Proof. Immediate from Lemma 2.

Corollary 2. *For* (D) *as in the theorem,* $(\langle r,t\rangle \times \langle r,t\rangle)^* E_{X'\times Y} \cong E_X$.

Proof. $E_Z =_{\mathrm{df}} \Sigma_{\Delta_Z} \top_Z$.

8.2. Suppose T_P satisfies the condition of the theorem. We will show that from this it follows that $\Sigma_r t^*\varphi \cong t'^*\Sigma_s\varphi$ for any $\varphi \in |\boldsymbol{P}(Y)|$; the dual result will follow immediately.

Before we begin, however, we must mention an equivalence scheme we shall need:

Lemma 1. *If no assumption in* P_1 *is discharged by the use of* ($\exists$ E) *in question, the following derivations are equivalent:*

($\exists$ E Simp)
$$\frac{\begin{matrix}P & P_1\\ \exists\xi\varphi & \theta\end{matrix}}{\theta} \equiv \begin{matrix}P_1\\ \theta\end{matrix}$$

Remark. We could have included this in Corollary 2 of § 2, but as we have not yet needed it, and as the proof is simplified by the Theorems of § 6, we have left it instead until now.

Proof. First, we should clarify what is meant: P_1 is a derivation of θ from assumptions A_1, say, which do not include φ and do not have x as a nondummy free variable. P is a derivation of $\exists\xi\varphi$ from assumptions A, say. So the LHS is a derivation of θ from $A \wedge A_1$ (really the set $A \cup A_1$), and the RHS is supposed then to be the derivation P_1 with "dummy" assumption A — we can think of it as $A \wedge A_1 \xrightarrow{\text{proj}} A_1 \xrightarrow{P_1} \theta$. Suppose $\varphi \in |\boldsymbol{P}(X \times Y)|$, $A \xrightarrow{P_1} \theta$ is in $\boldsymbol{P}(Y)$, and $X \times Y \xrightarrow{\pi} Y$. To prove the lemma, it suffices to show that the canonical morphism $\Sigma_\pi\varphi \wedge A \xrightarrow{\bar{P}_1} \theta$, induced by $\varphi \wedge \pi^*A \xrightarrow{\text{proj}} \pi^*A \xrightarrow{\pi^*P_1} \pi^*\theta$ under $\Sigma_\pi \dashv \pi^*$, is in fact $\Sigma_\pi\varphi \wedge A \xrightarrow{\text{proj}} A \xrightarrow{P_1} \theta$. Now $\bar{P}_1$ is defined using the cartesian closedness of the fibres (and Frobenius Reciprocity):

$$\cfrac{\cfrac{\cfrac{\varphi \wedge \pi^*A \xrightarrow{\text{proj}} \pi^*A \xrightarrow{\pi^*P_1} \pi^*\theta}{\varphi \to (\pi^*A \supset \pi^*\theta) \xrightarrow{\sim} \pi^*(A \supset \theta)}}{\Sigma_\pi\varphi \to (A \supset \theta)}}{\underset{\bar{P}_1}{\Sigma_\pi\varphi \wedge A \to \theta}}$$

So the result is an immediate consequence of the fact that in any cartesian closed category with a terminal object 1, the bijection

$$\frac{A \wedge B \xrightarrow{f} C}{A \xrightarrow[\ulcorner f \urcorner]{} (B \supset C)}$$

restricts to a bijection

$$\frac{A \wedge B \xrightarrow{\text{proj}} B \xrightarrow{f} C}{A \xrightarrow[!_A]{} 1 \xrightarrow[\ulcorner f \urcorner]{} (B \supset C)} \quad \square$$

We may also remark that there is an analogous scheme for $\vee$:

($\vee$ E Simp)
$$\frac{\begin{matrix}P & P_1 & P_1\\ \varphi \vee \psi & \theta & \theta\end{matrix}}{\theta} \equiv \begin{matrix}P_1\\ \theta\end{matrix}$$

where neither φ nor ψ is an assumption of P_1. This is not as general as the form given by PRAWITZ [15]—however, his form is not valid in all hyperdoctrines. We can now return to the proof of the theorem.

For any formula φ of T_P over Y, there is a derivation $\exists\eta(s\eta = t'x' \wedge \varphi(\eta)) \vdash\!\!\!\dashv \exists\xi(r\xi = x' \wedge \varphi(t\xi))$ over X' as follows (we may suppose φ is atomic):

$$
\dfrac{\exists\eta(s\eta = t'x' \wedge \varphi(\eta)) \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{\overset{(1)}{sy = t'x' \wedge \varphi(y)}}{sy = t'x'}\,(\wedge\,\mathrm{E})}{[t'x' = sy]}\,(\mathrm{S})\\ P_I \\ \exists\xi(r\xi = x' \wedge t\xi = y)\end{array} \qquad \dfrac{\dfrac{\dfrac{(2)}{rx = x'}\,(\wedge\,\mathrm{E}) \quad \dfrac{\dfrac{\dfrac{\overset{(2)}{rx = x' \wedge tx = y}}{tx = y}}{y = tx}\,(\mathrm{S}) \quad \dfrac{(1)}{\varphi(y)}\,(\wedge\,\mathrm{E})}{\varphi(tx)}\,(\mathrm{sub})}{rx = x' \wedge \varphi(tx)}\,(\wedge\,\mathrm{I})}{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\,(\exists\,\mathrm{I})}{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\,(\exists\,\mathrm{E})\ (2)}{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\,(\exists\,\mathrm{E})\ (1)
$$

This derivation is inverse to that given in § 4 (condition (5)′ (ii)): $\exists\xi(r\xi = x' \wedge \varphi(t\xi)) \vdash\!\!\!\dashv \exists\eta(s\eta = t'x' \wedge \varphi(\eta))$ (or rather the corresponding morphisms are inverse). To prove this, we must show the two composite derivations equivalent to the identity derivation

(a) $\qquad \Sigma_r t^*\varphi \to t'^*\Sigma_s\varphi \to \Sigma_r t^*\varphi = \mathrm{id}.$

We begin with the composite derivation (IX) on p. 535.

(x_1 is a free variable of type X, different from x. It is necessary so that the use of ($\exists$ Red) below is valid; "getting rid" of it will be the burden of this proof.) Using ($\exists$ Perm), ($\exists$ Red), ($\wedge$ Red), this is equivalent to the derivation:

$$
\dfrac{\exists\xi(r\xi = x' \wedge \varphi(t\xi)) \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{\overset{(1)}{rx = x' \wedge \varphi(tx)}}{rx = x'}\,(\wedge\,\mathrm{E})}{t'rx = t'x'}\,(\mathrm{ap})}{stx = t'x'}\,(\mathrm{id})'}{[t'x' = stx]}\,(\mathrm{S})\\ P_I \\ \exists\xi(r\xi = x' \wedge t\xi = tx)\end{array} \qquad \dfrac{\dfrac{\dfrac{(3)}{rx_1 = x'}\,(\wedge\,\mathrm{E}) \quad \dfrac{\dfrac{\dfrac{\overset{(3)}{rx_1 = x' \wedge tx_1 = tx}}{tx_1 = tx}\,(\wedge\,\mathrm{E})}{tx = tx_1}\,(\mathrm{S}) \quad \dfrac{(1)}{\varphi(tx)}\,(\wedge\,\mathrm{E})}{\varphi(tx_1)}\,(\mathrm{sub})}{rx_1 = x' \wedge \varphi(tx_1)}\,(\wedge\,\mathrm{I})}{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\,(\exists\,\mathrm{I})}{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\,(\exists\,\mathrm{E})\ (3)}{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\,(\exists\,\mathrm{E})\ (1)
$$

(IX)

$$
\dfrac{
\exists\xi(r\xi = x' \wedge \varphi(t\xi)) \qquad
\dfrac{
\exists\eta(s\eta = t'x' \wedge \varphi(\eta))
\qquad
\dfrac{
\dfrac{
\text{(id)}'\ \dfrac{\text{(ap)}\ \dfrac{(\wedge \mathrm{E})\ \dfrac{\overset{(1)}{rx = x' \wedge \varphi(tx)}}{rx = x'}}{t'rx = t'x'}}{stx = t'x'}
\qquad
\dfrac{rx = x' \wedge \varphi(tx)}{\varphi(tx)}\,(\wedge \mathrm{E})
}{stx = t'x' \wedge \varphi(tx)}\,(\wedge \mathrm{I})
}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\,(\exists \mathrm{I})
}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\,(\exists \mathrm{E})\ (1)
\qquad
\dfrac{
\begin{array}{c}
\dfrac{\dfrac{\overset{(2)}{sy = t'x' \wedge \varphi(y)}}{sy = t'x'}\,(\wedge \mathrm{E})}{[t'x' = sy]}\,(\mathrm{S}) \\
P_I \\
\exists\xi(r\xi = x' \wedge t\xi = y)
\end{array}
\qquad
\dfrac{
\dfrac{
\dfrac{(3)}{rx_1 = x'}\,(\wedge \mathrm{E})
\qquad
\dfrac{
\dfrac{\dfrac{\dfrac{\overset{(3)}{rx_1 = x' \wedge tx_1 = y}}{tx_1 = y}\,(\wedge \mathrm{E})}{y = tx_1}\,(\mathrm{S})
\qquad
\dfrac{(2)}{\varphi(y)}\,(\wedge \mathrm{E})
}{\varphi(tx_1)}\,(\mathrm{sub})
}{rx_1 = x' \wedge \varphi(tx_1)}\,(\wedge \mathrm{I})
}{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\,(\exists \mathrm{I})
}{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\,(\exists \mathrm{E})\ (3)
}{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\,(\exists \mathrm{E})\ (2)
$$

Call this derivation P_0. Notice that $rx_1 = x' \wedge tx_1 = tx$ could be replaced with

$$\dfrac{\dfrac{\overset{(1)}{rx = x' \wedge \varphi(tx)} \qquad \overset{(3)}{rx_1 = x' \wedge tx_1 = tx}}{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}\,(\wedge \mathrm{I})}{rx_1 = x' \wedge tx_1 = tx}\,(\wedge \mathrm{E})$$

since the latter reduces to the former by ($\wedge$ Red). We can expand this even further using the following:

Lemma 2. *Let Q_1 be the derivation*

$$\dfrac{\dfrac{rx = x' \wedge \varphi(tx) \wedge x_1 = x}{rx = x' \wedge \varphi(tx)}\,(\wedge \mathrm{E}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{rx = x' \wedge \varphi(tx) \wedge x_1 = x}{x_1 = x}\,(\wedge \mathrm{E})}{rx_1 = rx}\,(\mathrm{ap}) \qquad \dfrac{rx = x' \wedge \varphi(tx) \wedge x_1 = x}{rx = x'}\,(\wedge \mathrm{E})}{rx_1 = x'}\,(\mathrm{T}) \qquad \dfrac{\dfrac{rx = x' \wedge \varphi(tx) \wedge x_1 = x}{x_1 = x}\,(\wedge \mathrm{E})}{tx_1 = tx}\,(\mathrm{ap})}{rx_1 = x' \wedge tx_1 = tx}\,(\wedge \mathrm{I})}{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}\,(\wedge \mathrm{I})$$

and Q_2 the derivation

$$\dfrac{\dfrac{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}{rx = x' \wedge \varphi(tx)}\,(\wedge \mathrm{E}) \qquad \begin{matrix} \dfrac{\dfrac{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}{rx_1 = x'} \qquad \dfrac{\dfrac{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}{rx = x'}\,(\wedge \mathrm{E})}{x' = rx}\,(\mathrm{S})}{[rx_1 = rx}\,(\mathrm{T}) \qquad \dfrac{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}{tx_1 = tx]} \\ P_E \\ x_1 = x \end{matrix}}{rx = x' \wedge \varphi(tx) \wedge x_1 = x}\,(\wedge \mathrm{I})$$

Then the derivation

$$\begin{matrix} P \\ [rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx] \\ Q_2 \\ [rx = x' \wedge \varphi(tx) \wedge x_1 = x] \\ Q_1 \\ rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx \end{matrix}$$

is equivalent to

$$\begin{array}{c} P \\ rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx \end{array}$$

for any P.

Proof. The proof of this fact is fairly straightforward, and so we will not write out the details here. Note, however, that the proof is simplified if one notices the following:

$$\begin{array}{c} \begin{array}{cc} P & Q \\ [rx_0 = rx_1 & tx_0 = tx_1] \end{array} \\ \\ P_E \\ (\text{ap})\ \dfrac{x_0 = x_1}{rx_0 = rx_1} \end{array} \equiv \begin{array}{c} P \\ rx_0 = rx_1 \end{array}$$

and

$$\begin{array}{c} \begin{array}{cc} P & Q \\ [rx_0 = rx_1 & tx_0 = tx_1] \end{array} \\ \\ P_E \\ (\text{ap})\ \dfrac{x_0 = x_1}{tx_0 = tx_1} \end{array} \equiv \begin{array}{c} Q \\ tx_0 = tx_1 \end{array}$$

These equivalences are proved in the familiar way, using (= Exp) and (R Coh): replace x_1 with x_0, and note that $rx_0 = rx_0$ and $tx_0 = tx_0$ are isomorphic to $\top_X$ (so that the corresponding derivations must be equal). (Note also that there must be "dummy" assumptions on each RHS, as in ($\exists$ E Simp), to make sense of these equivalences.) □

Let P_1 be the derivation P_0 with the (discharged) assumption $rx_1 = x' \wedge tx_1 = tx$ replaced by the derivation

$$\begin{array}{c} \dfrac{\begin{array}{cc} (1) & (3) \\ rx = x' \wedge \varphi(x) & rx_1 = x' \wedge tx_1 = tx \end{array}}{rx = x' \wedge \varphi(tx) \wedge rx_1 = x' \wedge tx_1 = tx}\ (\wedge\, \mathrm{I}) \\ Q_2 \\ rx = x' \wedge \varphi(tx) \wedge x_1 = x \\ Q_1 \\ \dfrac{rx = x' \wedge \varphi(x) \wedge rx_1 = x' \wedge tx_1 = tx}{rx_1 = x' \wedge tx_1 = tx}\ (\wedge\, \mathrm{E}) \end{array}$$

Then P_1 is equivalent to P_0 (and so to the original composite derivation); we must show it equivalent to the identity.

First, consider that part of P_1 above $rx_1 = x' \wedge \varphi(tx_1)$; writing Q_1 in full, it is easy to see that this is equivalent to the derivation

$$
\dfrac{
 \dfrac{
  \dfrac{\dfrac{\dfrac{(1)\quad(3)}{Q_2}\,(\wedge\,\mathrm{I})}{x_1 = x}\,(\wedge\,\mathrm{E})}{rx_1 = rx}\,(\mathrm{ap})
  \qquad
  \dfrac{\dfrac{(1)\quad(3)}{Q_2}\,(\wedge\,\mathrm{E})}{rx = x'}\,(\wedge\,\mathrm{E})
 }{rx_1 = x'}\,(\mathrm{T})
 \qquad
 \dfrac{
  \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(1)\quad(3)}{Q_2}\,(\wedge\,\mathrm{I})}{rx = x' \wedge \varphi(tx) \wedge x_1 = x}\,(\wedge\,\mathrm{I})}{x_1 = x}\,(\wedge\,\mathrm{E})}{tx_1 = tx}\,(\mathrm{ap})}{tx = tx_1}\,(\mathrm{S})
  \qquad
  \dfrac{rx = x' \wedge \varphi(tx)}{\varphi(tx)}\,(\wedge\,\mathrm{E})
 }{\varphi(tx_1)}\,(\mathrm{sub})
}{rx_1 = x' \wedge \varphi(tx_1)}\,(\wedge\,\mathrm{I})
$$

Let P_2 be the derivation P_1 with this subderivation replacing the subderivation above $rx_1 = x' \wedge \varphi(tx_1)$. Use $(=\mathrm{Exp})$ and $(=\mathrm{Simp})$ to get the equivalent derivation

$$
\dfrac{
 \begin{array}{c}(1)\\ rx = x' \wedge \varphi(tx)\end{array}\ \to\ \exists\xi(r\xi = x' \wedge \varphi(t\xi))
 \qquad
 \dfrac{
  \begin{array}{c}
  \dfrac{\dfrac{\dfrac{\dfrac{(1)}{rx = x' \wedge \varphi(tx)}\,(\wedge\,\mathrm{E})}{rx = x'}\,(\mathrm{ap})}{t'rx = t'x'}\,(\mathrm{id})}{stx = t'x'}\,(\mathrm{S})\\
  [t'x' = stx]\\
  P_I\\
  \exists\xi(r\xi = x' \wedge t\xi = tx)
  \end{array}
  \qquad
  \dfrac{\dfrac{
   \dfrac{\dfrac{\dfrac{\top_X}{x = x}\,(\mathrm{R})}{rx = rx}\,(\mathrm{ap})\quad \dfrac{(1)}{rx = x'}\,(\wedge\,\mathrm{E})}{rx = x'}\,(\mathrm{T})
   \qquad
   \dfrac{\dfrac{\dfrac{\top_X}{x = x}\,(\mathrm{R})}{tx = tx}\,(\mathrm{ap})\quad \dfrac{(1)}{\varphi(tx)}\,(\wedge\mathrm{E})}{\varphi(tx)}\,(\mathrm{sub})
  }{rx = x' \wedge \varphi(tx)}\,(\wedge\,\mathrm{I})}{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\,(\exists\,\mathrm{I})
 }{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\,(\exists\,\mathrm{E})\ (2)
}{\exists\xi(r\xi = x' \wedge \varphi(t\xi))}\,(\exists\,\mathrm{E})\ (1)
$$

(We have used $(\wedge\,\mathrm{Red})$ to eliminate the derivation Q_2 above $rx = x'$.)

It is now easy to see that this is equivalent to the identity derivation, using $(=\mathrm{Red})$, $(=\mathrm{Exp})$, $(\wedge\,\mathrm{Exp})$, $(\exists\,\mathrm{E\ Simp})$, and $(\exists\,\mathrm{Exp})$.

(b) $\qquad t'^*\Sigma_s\varphi \to \Sigma_r t^*\varphi \to t'^*\Sigma_s\varphi = \mathrm{id}.$

We begin with the composite derivation (X) on p. 540.

(This time x_1 could be x—it will disappear of its own accord soon!) Using $(\exists\,\mathrm{Perm})$ (twice), $(\exists\,\mathrm{Red})$, $(\wedge\,\mathrm{Red})$ this is equivalent to the derivation

$$
\dfrac{\exists\eta(s\eta = t'x' \wedge \varphi(\eta)) \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{(2)}{sy = t'x'}\,(\wedge\,\mathrm{E})}{[t'x' = sy]}\,(\mathrm{S}) \\ P_I \\ \exists\xi(r\xi = x' \wedge t\xi = y)\end{array} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(3)}{rx = x'}\,(\wedge\,\exists)}{t'rx = t'x'}\,(\mathrm{ap})}{stx = t'x'}\,(\mathrm{id})' \qquad \dfrac{\dfrac{\dfrac{(3)}{tx = y}\,(\wedge\,\mathrm{E})}{y = tx}\,(\mathrm{S}) \quad \dfrac{(2)}{\varphi(y)}\,(\wedge\,\mathrm{E})}{\varphi(tx)}\,(\mathrm{sub})}{stx = t'x' \wedge \varphi(tx)}\,(\wedge\,\mathrm{I})}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\,(\exists\,\mathrm{I})}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\,(\exists\,\mathrm{E})\ (3)}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\,(\exists\,\mathrm{E})\ (2)
$$

By Corollary 1 of § 2 this is equivalent to

$$
\dfrac{\exists\eta(s\eta = t'x' \wedge \varphi(\eta)) \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{(2)}{sy = t'x'}\,(\wedge\,\mathrm{E})}{[t'x' = sy]}\,(\mathrm{S}) \\ P_I \\ \exists\xi(r\xi = x' \wedge t\xi = y)\end{array} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(3)}{tx = y}\,(\wedge\,\mathrm{E})}{y = tx}\,(\mathrm{S}) \quad \dfrac{\dfrac{\dfrac{\overset{(2)}{sy = t'x' \wedge \varphi(y)}}{sy = t'x'}\,(\wedge\,\mathrm{E})}{t'x' = sy}\,(\mathrm{S})}{sy = t'x'}\,(\mathrm{S})}{stx = t'x'}\,(\mathrm{sub}) \qquad \dfrac{\dfrac{\dfrac{(3)}{tx = y}\,(\wedge\,\mathrm{E})}{y = tx}\,(\mathrm{S}) \quad \dfrac{(2)}{\varphi(y)}\,(\wedge\,\mathrm{E})}{\varphi(tx)}\,(\mathrm{sub})}{stx = t'x' \wedge \varphi(tx)}\,(\wedge\,\mathrm{I})}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\,(\exists\,\mathrm{I})}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\,(\exists\,\mathrm{E})\ (3)}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\,(\exists\,\mathrm{E})\ (2)
$$

Using (= Exp) and (= Simp), this is equivalent to

$$
\dfrac{\exists\eta(s\eta = t'x' \wedge \varphi(\eta)) \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{(2)}{sy = t'x'}\,(\wedge\,\mathrm{E})}{[t'x' = sy]}\,(\mathrm{S}) \\ P_I \\ \exists\xi(r\xi = x' \wedge t\xi = y)\end{array} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\top_Y}{y = y}\,(\mathrm{R}) \quad \dfrac{\dfrac{\dfrac{\overset{(2)}{sy = t'x' \wedge \varphi(y)}}{sy = t'x'}\,(\wedge\,\mathrm{E})}{t'x' = sy}\,(\mathrm{S})}{sy = t'x'}\,(\mathrm{S})}{sy = t'x'}\,(\mathrm{sub}) \qquad \dfrac{\dfrac{\top_Y}{y = y}\,(\mathrm{R}) \quad \dfrac{(2)}{\varphi(y)}\,(\wedge\,\mathrm{E})}{\varphi(y)}\,(\mathrm{sub})}{sy = t'x' \wedge \varphi(y)}\,(\wedge\,\mathrm{I})}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\,(\exists\,\mathrm{I})}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\,(\exists\,\mathrm{E})\ (3)}{\exists\eta(s\eta = t'x' \wedge \varphi(\eta))}\,(\exists\,\mathrm{E})\ (2)
$$

(X)

$$
\begin{prooftree}
\AxiomC{(2)}
\noLine
\UnaryInfC{$sy = t'x' \wedge \varphi(y)$}
\RightLabel{($\wedge$ E)}
\UnaryInfC{$sy = t'x'$}
\RightLabel{(S)}
\UnaryInfC{$[t'x' = sy]$}
\noLine
\UnaryInfC{P_I}
\noLine
\UnaryInfC{$\exists\xi(r\xi = x' \wedge t\xi = y)$}
\AxiomC{(3)}
\RightLabel{($\wedge$ E)}
\UnaryInfC{$rx = x'$}
\AxiomC{(3)}
\noLine
\UnaryInfC{$rx = x' \wedge tx = y$}
\RightLabel{($\wedge$ E)}
\UnaryInfC{$tx = y$}
\RightLabel{(S)}
\UnaryInfC{$y = tx$}
\AxiomC{(2)}
\RightLabel{($\wedge$ E)}
\UnaryInfC{$\varphi(y)$}
\RightLabel{(sub)}
\BinaryInfC{$\varphi(tx)$}
\RightLabel{($\wedge$ I)}
\BinaryInfC{$rx = x' \wedge \varphi(tx)$}
\RightLabel{($\exists$ I)}
\UnaryInfC{$\exists\xi(r\xi = x' \wedge \varphi(t\xi))$}
\RightLabel{($\exists$ E) (3)}
\BinaryInfC{$\exists\xi(r\xi = x' \wedge \varphi(t\xi))$}
\AxiomC{$\exists\eta(s\eta = t'x' \wedge \varphi(\eta))$}
\RightLabel{($\exists$ E) (2)}
\BinaryInfC{$\exists\xi(r\xi = x' \wedge \varphi(t\xi))$}
\AxiomC{(1)}
\noLine
\UnaryInfC{$rx_1 = x' \wedge \varphi(tx_1)$}
\RightLabel{($\wedge$ E)}
\UnaryInfC{$rx_1 = x'$}
\RightLabel{(ap)}
\UnaryInfC{$t'rx_1 = t'x'$}
\RightLabel{(id)$'$}
\UnaryInfC{$stx_1 = t'x'$}
\AxiomC{(1)}
\RightLabel{($\wedge$ E)}
\UnaryInfC{$\varphi(tx_1)$}
\RightLabel{($\wedge$ I)}
\BinaryInfC{$stx_1 = t'x' \wedge \varphi(tx_1)$}
\RightLabel{($\exists$ I)}
\UnaryInfC{$\exists\eta(s\eta = t'x' \wedge \varphi(\eta))$}
\RightLabel{($\exists$ E) (1)}
\BinaryInfC{$\exists\eta(s\eta = t'x' \wedge \varphi(\eta))$}
\end{prooftree}
$$

It is easy to see this is equivalent to the identity (use (= Red), (S Coh), ($\wedge$ Exp), ($\exists$ E Simp), ($\exists$ Exp)). This completes the proof. $\square$

As an immediate corollary to the theorem, we can characterise all the "left" structure of the base category $\boldsymbol{T}$ of a hyperdoctrine $\boldsymbol{P}$ over $\boldsymbol{T}$ with the full Beck condition. As examples, we give the following. Suppose $\boldsymbol{P}$ has the full Beck condition.

Corollary 1. *$X \xrightarrow{t} Y$ is a monomorphism of $\boldsymbol{T}$ iff there is a derivation in $T_{\boldsymbol{P}}$:*

$$tx_0 = tx_1 \vdash x_0 = x_1 .$$

Corollary 2. *$E \xrightarrow{r} X \overset{s}{\underset{t}{\rightrightarrows}} Y$ is an equaliser diagram of $\boldsymbol{T}$ only if there are derivations in $T_{\boldsymbol{P}}$:*

$$sx = tx \vdash \exists\varepsilon(r\varepsilon = x), \qquad re_0 = re_1 \vdash e_0 = e_1 .$$

Corollary 3. *If u_0, u_1 are free variables of type 1, then there is a derivation*

$$\vdash u_0 = u_1$$

(which is why we don't need many free variables of type 1).

Proofs. The first corollary is immediate from the theorem and the fact that $t\colon X \to Y$ is a monomorphism iff

$$\begin{tikzcd} X \arrow[r, "X"] \arrow[d, "X"'] & X \arrow[d, "t"] \\ X \arrow[r, "t"'] & Y \end{tikzcd}$$

is a pullback. (So the Beck condition for this diagram is equivalent to the existence of the above derivation (and of $x_0 = x_1, x_0 = x_1 \vdash x_0 = x_1$!))

The second corollary is almost as immediate from the theorem and the fact that $E \xrightarrow{r} X \overset{s}{\underset{t}{\rightrightarrows}} Y$ is an equaliser iff

$$\begin{tikzcd} E \arrow[r, "tr"] \arrow[d, "r"'] & Y \arrow[d, "\Delta_Y"] \\ X \arrow[r, "\langle s, t\rangle"'] & Y \times Y \end{tikzcd}$$

is a pullback. The Beck condition for this is equivalent to the existence of derivations $sx = y \wedge tx = y \vdash \exists\varepsilon(r\varepsilon = x \wedge tr\varepsilon = y)$ and $re_0 = re_1,\ tre_0 = tre_1 \vdash e_0 = e_1$. From this to the derivations of the statement of the corollary is an easy step.

And finally, since 1 is a terminal object,

$$\begin{tikzcd} 1 \arrow[r, "\Delta_1"] \arrow[d, "\Delta_1"'] & 1 \times 1 \arrow[d, "1 \times 1"] \\ 1 \times 1 \arrow[r, "1 \times 1"'] & 1 \times 1 \end{tikzcd}$$

is a pullback, and moreover, since Δ_1 is an isomorphism, the Beck condition holds for this pullback. So there is a derivation $u_0 = u_0, \; u_1 = u_1 \vdash \exists v(v = u_0 \wedge v = u_1)$; this is just $\vdash u_0 = u_1$ as claimed.

References

[1] FOURMAN, M. P., The Logic of Topoi. In: J. BARWISE (ed.), Handbook of Mathematical Logic, North-Holland Publ. Comp., Amsterdam 1977, pp. 1053–1090.
[2] FREYD, P., Aspects of topoi. Bull. Australian Math. Soc. 7 (1972), 1–76.
[3] GIRARD, J.-Y., Une extension de l'interprétation de Gödel à l'analyse, et son application à l'élimination des coupures dans l'analyse et la théorie des types. In: J. E. FENSTAD (ed.), Proceedings of the Second Scandinavian Logic Symposium, North-Holland Publ. Comp., Amsterdam 1971, pp. 63–92.
[4] GOLDBLATT, R., Topoi, the categorial analysis of logic. North-Holland Publ. Comp., Amsterdam 1979.
[5] HYLAND, J. M. E., JOHNSTONE, P. T., and A. M. PITTS, Tripos theory. Math. Proc. Camb. Phil. Soc. 88 (1980), 205–232.
[6] JOHNSTONE, P. T., Topos Theory. Academic Press, London 1977.
[7] KOCK, A., and G. E. REYES, Doctrines in categorical logic. In: J. BARWISE (ed.), Handbook of Mathematical Logic, North-Holland Publ. Comp., Amsterdam 1977, pp. 283–313.
[8] LAMBEK, J., Deductive Systems and Categories III. In: F. W. LAWVERE (ed.), Toposes, Algebraic Geometry and Logic, Springer Lecture Notes in Math. 274 (1972), pp. 57–82.
[9] LAWVERE, F. W., Adjointness in foundations. Dialectica 23 (1969), 281–296.
[10] LAWVERE, F. W.. Equality in hyperdoctrines and the comprehension schema as an adjouint functor. In: A. HELLER (ed.), Proc. New York Symposium on Applications of Categorical Logic, Amer. Math. Soc. 1970, pp. 1–14.
[11] MAC LANE, S., Categories for the Working Mathematician. Springer-Verlag, Berlin–Heidelberg–New York 1971.
[12] MAKKAI, M., and G. E. REYES, First order categorical logic. Springer-Verlag, Berlin–Heidelberg–New York 1977.
[13] PARÉ, R., and G. SCHUMACHER, Abstract families and the adjoint functor theorems. In: P. T. JOHNSTONE and R. PARÉ (eds.), Indexed Categories and their Applications, Springer Lecture Notes in Math. 661 (1978), pp. 1–125.
[14] PRAWITZ, D., Natural Deduction: a proof-theoretical study. Almqvist and Wiksell, Stockholm 1965.
[15] PRAWITZ, D., Ideas and results in proof theory. In: J. E. FENSTAD (ed.), Proc. of the Second Scandinavian Logic Symposium, North-Holland Publ. Comp., Amsterdam 1971, pp. 235–307.
[16] REYES, G. E., From sheaves to logic. In: A. DAIGNEAULT (ed.), Studies in Algebraic Logic, M.A.A. Studies in Math. 9 (1974), pp. 143–204.
[17] SCOTT, P. J., The "Dialectica" interpretation and categories. This Zeitschr. 24 (1978), 553–575.
[18] SEELY, R. A. G., Hyperdoctrines and Natural Deduction. Ph. D. Thesis, University of Cambridge 1977.
[19] SEELY, R. A. G., Weak adjointness in proof theory. In: M. P. FOURMAN, C. J. MULVEY, and D. S. SCOTT (eds.), Applications of Sheaves, Springer Lecture Notes in Math. 753 (1979), pp. 697–701.
[20] SHOENFIELD, J. R., Mathematical Logic, Addison-Wesley, Reading (Mass.) 1967.
[21] SZABO, M. E., Algebra of Proofs. North-Holland Publ. Comp., Amsterdam 1978.

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