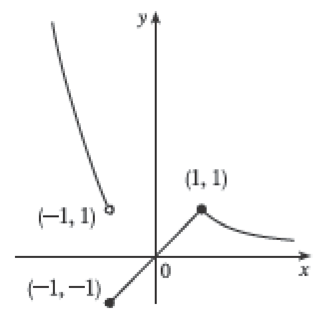
Instructor: Dr. R.A.G. Seely Quiz 4
(Oct 2017)

Cal I (S) (Maths 201–NYA)

Answers

The questions and the answers are all taken from Stewart.

1. $f(x) = \begin{cases} x^4 & \text{if } x < -1 \\ x & \text{if } -1 \le x < 1 \\ 1/x & \text{if } x \ge 1 \end{cases}$



f is continuous on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$, where it is a polynomial, a polynomial, and a rational function, respectively.

Now $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^4 = 1$ and $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x = -1$,

so f is discontinuous at -1. Since $f(-1) = -1$, f is continuous from the right at -1. Also, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x} = 1 = f(1)$, so f is continuous at 1.

2. $f(x) = \begin{cases} kx^2 + 2x & \text{if } x < 2 \\ x^3 - kx & \text{if } x \ge 2 \end{cases}$

f is continuous on $(-\infty, 2)$ and $(2, \infty)$. Now $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (kx^2 + 2x) = 4k + 4$ and

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^3 - kx) = 8 - 2k$. So f is continuous $\Leftrightarrow$ $4k + 4 = 8 - 2k$ $\Leftrightarrow$ $6k = 4$ $\Leftrightarrow$ $k = \frac{2}{3}$. Thus, for f to be continuous on $(-\infty, \infty)$, $k = \frac{2}{3}$.

3. $f(x) = x^2 + 10\sin x$ is continuous on the interval $[31, 32]$, $f(31) \approx 957$, and $f(32) \approx 1030$. Since $957 < 1000 < 1030$, there is a number c in $(31, 32)$ such that $f(c) = 1000$ by the Intermediate Value Theorem. *Note:* There is also a number c in $(-32, -31)$ such that $f(c) = 1000$.

Note: any two x values will do, as long as 1000 lies between their y values; they may even be far apart, such as $x = 0$ and $x = 1000$. The answer given here makes them close enough that one has the value of c to within ± 1.

4. (a) We must first find the function f. Since f has a vertical asymptote $x = 4$ and x-intercept $x = 1$, $x - 4$ is a factor of the denominator and $x - 1$ is a factor of the numerator. There is a removable discontinuity at $x = -1$, so $x - (-1) = x + 1$ is a factor of both the numerator and denominator. Thus, f now looks like this: $f(x) = \dfrac{a(x-1)(x+1)}{(x-4)(x+1)}$, where a is still to be determined. Then $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1} \dfrac{a(x-1)(x+1)}{(x-4)(x+1)} = \lim\limits_{x\to -1} \dfrac{a(x-1)}{x-4} = \dfrac{a(-1-1)}{(-1-4)} = \dfrac{2}{5}a$, so $\dfrac{2}{5}a = 2$, and $a = 5$. Thus $f(x) = \dfrac{5(x-1)(x+1)}{(x-4)(x+1)}$ is a ratio of quadratic functions satisfying all the given conditions and $f(0) = \dfrac{5(-1)(1)}{(-4)(1)} = \dfrac{5}{4}$.

(b) $\lim\limits_{x\to\infty} f(x) = 5 \lim\limits_{x\to\infty} \dfrac{x^2-1}{x^2-3x-4} = 5 \lim\limits_{x\to\infty} \dfrac{(x^2/x^2) - (1/x^2)}{(x^2/x^2) - (3x/x^2) - (4/x^2)} = 5\dfrac{1-0}{1-0-0} = 5(1) = 5$

5. $y = f(x) = x^4 - x^6 = x^4(1 - x^2) = x^4(1+x)(1-x)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0, -1, and 1 [found by solving $f(x) = 0$ for x]. Since $x^4 > 0$ for $x \neq 0$, f doesn't change sign at $x = 0$. The function does change sign at $x = -1$ and $x = 1$. As $x \to \pm\infty$, $f(x) = x^4(1 - x^2)$ approaches $-\infty$ because $x^4 \to \infty$ and $(1 - x^2) \to -\infty$.