1 Optimization

1. From a square piece of metal, whose side is 27 cm long, an open box with square base is to be made. What dimensions should the box have in order to contain the largest volume? [Form the box by cutting squares from the corners of the metal and folding up the sides.]

2. A farmer wishes to fence off a rectangular pasture along a straight river, the side along the river requiring no fence. He has wire to fence a distance of 1000 meters. What is the area of the largest pasture of the above description that can be fenced?

3. A rectangle has perimeter of 40 meters. What length and width yield the maximum area?

4. The sum of two positive numbers x and y is 120. Find the numbers such that the product x^2y is a maximum.

5. A wire 34 cm long is cut into 2 pieces. One piece is bent to form a square and the other is bent to form a rectangle that is twice as wide as it is long. How should the wire be cut in order to minimize the total area of the square and the rectangle?

6. A closed box having a square base is to be made from 300 cm^2 of a certain material. Find the dimensions of the box with maximum volume and the maximum volume.

7. A fisherman is in a rowboat on a lake and 3 km from shore. He wishes to reach a store 2 km down the (straight) shore. He can row at 5 km/h and run at 13 km/h. To what point down-shore should he row to get to the store as quickly as possible?

8. The cross-section of a tunnel has the form of a rectangle surmounted by a semi-circle. The perimeter of this cross section is 18 meters. For what radius of the semi-circle will the cross-section have maximum area?

9. Find the point on the graph of $y = \sqrt{x}$ which is closest to the point $(4, 0)$.

10. Find the point in the first quadrant on the graph of $y = x^2$ which is nearest to the point $(0, \frac{17}{16})$.

11. The function $f(x) = 3 + \sqrt[3]{(x-4)^2}$ is continuous for all x. Find its extrema on the closed interval $[3, 12]$.

12. A rectangular field is fenced and divided in half by another fence joining the midpoints of two opposite sides. Find the largest area of a field that can be fenced with 480 meters of fencing.

13. A rectangular beam is to be cut from a log with circular cross-section 10 cm in diameter. If the strength, S, of the beam is given by $S = 4xy^2$, where x is the width and y is the depth of the beam, find the values of x and y that give the strongest beam.

14. A slice of pizza, in the form of a sector of a circle, is to have perimeter of 60 cm. What should the radius of the pan be in order to make a slice of largest area? [The area of a sector of a circle is given by $A = \frac{1}{2}\theta r^2$, and the arc length of a sector is $s = r\theta$.]

15. Find the extreme values of $f(x) = \sqrt{x} + \dfrac{4}{x}$ on the open interval $(1, 9)$.

Answers:

1. 18 cm $\times$ 18 cm $\times$ $\frac{9}{2}$ cm.
2. 125,000 m^2.
3. 10 m $\times$ 10 m.
4. $x = 80$ and $y = 40$
5. 16 cm and 18 cm lengths.
6. $\sqrt{50}$ cm $\times$ $\sqrt{50}$ cm $\times$ $\sqrt{50}$ cm, $V = 50\sqrt{50}$
7. 0.75 km from the store.
8. $\dfrac{18}{4+\pi}$ meters.
9. $(x, y) = (\frac{7}{2}, \sqrt{\frac{7}{2}})$
10. $(x, y) = (\frac{3}{4}, \frac{9}{16})$
11. Minimum = 3; Maximum = 7.
12. 120 m $\times$ 80 m.
13. $x = \dfrac{10}{\sqrt{3}}$, $y = \dfrac{10\sqrt{2}}{\sqrt{3}}$
14. 15 cm.
15. Minimum = 3; No maximum.