The purpose of this note is to illustrate a tabular notation that organizes partial integration effectively and efficiently. Remember that partial integration involves decomposing an integrand into two factors—one of which is to be integrated and one of which is to be differentiated—and is expressed by the equation

$$\int f(x)g(x)\,dx = f(x)g_1(x) - \int f'(x)g_1(x)\,dx, \tag{1}$$

where f' is the derivative of f and g_1 is an antiderivative of g. The tabular notation for Equation (1) is displayed below.

$+$	$f(x)$	$g(x)$
$-$	$f'(x)$	$g_1(x)$

(2)

In the tabular notation,

- the signs, each of which modifies the expression immediately to its right, are alternately positive and negative,
- immediately below an entry in the left column is its derivative, if anything, and
- immediately below an entry in the right column is an antiderivative of it, if anything.

Table (2) summarizes Equation (1) in this sense: The integral of the product of the functions in a row ($\int f(x)g(x)\,dx$) is the sum of the product of the functions in the downward diagonal beginning in that row ($f(x)g_1(x)$) and the integral of the product of the functions in the next row ($-\int f'(x)g_1(x)\,dx$). Likewise, the table

$+$	$f(x)$	$g(x)$
$-$	$f'(x)$	$g_1(x)$
$+$	$f''(x)$	$g_2(x)$

summarizes the calculation

$$\begin{aligned}\int f(x)g(x)\,dx &= f(x)g_1(x) - \int f'(x)g_1(x)\,dx\\ &= f(x)g_1(x) - \left\{f'(x)g_2(x) - \int f''(x)g_2(x)\,dx\right\}\\ &= f(x)g_1(x) - f'(x)g_2(x) + \int f''(x)g_2(x)\,dx,\end{aligned}$$

where f'' is the derivative of f' and g_2 is an antiderivative of g_1.

Just so, the tabular notation summarizes repeated partial integration; the sign alternations are handled automatically, and each factorization (inside or out of an integral) is written only once. Examples 1–4, the third and fourth of which involve refactoring the integrand between partial integrations, illustrate the tabular notation.

Example 1. The tabular notation is used to evaluate $\int x^4\cos 2x\,dx$. The last row of the table represents the integral of 0, which provides the constant of integration.

$$\begin{aligned}&\int x^4\cos 2x\,dx\\ &\quad= \tfrac{1}{2}x^4\sin 2x + x^3\cos 2x - \tfrac{3}{2}x^2\sin 2x - \tfrac{3}{2}x\cos 2x + \tfrac{3}{4}\sin 2x + C\\ &\quad= \tfrac{1}{4}(2x^4 - 6x^2 + 3)\sin 2x + \tfrac{1}{2}x(2x^2 - 3)\cos 2x + C\end{aligned}$$

$+$	x^4	$\cos 2x$
$-$	$4x^3$	$\frac{1}{2}\sin 2x$
$+$	$12x^2$	$-\frac{1}{4}\cos 2x$
$-$	$24x$	$-\frac{1}{8}\sin 2x$
$+$	24	$\frac{1}{16}\cos 2x$
$-$	0	$\frac{1}{32}\sin 2x$

Example 2. The tabular notation is used to evaluate $\int e^{5x}\sin 3x\,dx$. In this case the third row is a constant multiple of the first row, which yields an equation that determines the integral. If $\mathscr{I} = \int e^{5x}\sin 3x\,dx$, then

$$\begin{aligned}\mathscr{I} &= -\tfrac{1}{3}e^{5x}\cos 3x + \tfrac{5}{9}e^{5x}\sin 3x - \tfrac{25}{9}\mathscr{I};\\ \therefore\ \mathscr{I} &= \tfrac{1}{34}e^{5x}(5\sin 3x - 3\cos 3x) + C.\end{aligned}$$

$+$	e^{5x}	$\sin 3x$
$-$	$5e^{5x}$	$-\frac{1}{3}\cos 3x$
$+$	$25e^{5x}$	$-\frac{1}{9}\sin 3x$

Example 3. In the course of evaluating $\int x^2(\ln x)^3\,dx$, the integrand is refactored between partial integrations. The absence of a horizontal line between two rows of the table (and the unchanging sign to their left) indicates that the integrand has been refactored, and that no partial integration has been performed.

$$\begin{aligned}\int x^2(\ln x)^3\,dx &= \tfrac{1}{3}x^3(\ln x)^3 - \tfrac{1}{3}x^3(\ln x)^2 + \tfrac{2}{9}x^3\ln x - \tfrac{2}{27}x^3 + C\\ &= \tfrac{1}{27}x^3\bigl\{9(\ln x)^3 - 9(\ln x)^2 + 6\ln x - 2\bigr\} + C\end{aligned}$$

$+$	$(\ln x)^3$	x^2
$-$	$3(\ln x)^2/x$	$\frac{1}{3}x^3$
$-$	$(\ln x)^2$	x^2
$+$	$2(\ln x)/x$	$\frac{1}{3}x^3$
$+$	$\ln x$	$\frac{2}{3}x^2$
$-$	$1/x$	$\frac{2}{9}x^3$
$-$	1	$\frac{2}{9}x^2$
$+$	0	$\frac{2}{27}x^3$

The reader is encouraged to compare this calculation with one that begins with the change of variable $t = \ln x$.

Example 4. Evaluating $\int x(\operatorname{arcsec} x)^2\,dx$ involves more judicious refactoring.

$$\int x(\operatorname{arcsec} x)^2\,dx = \tfrac{1}{2}(x\operatorname{arcsec} x)^2 - (\operatorname{arcsec} x)\sqrt{x^2-1} + \ln|x| + C$$

$+$	$(\operatorname{arcsec} x)^2$	x
$-$	$\dfrac{2\operatorname{arcsec} x}{x\sqrt{x^2-1}}$	$\frac{1}{2}x^2$
$-$	$\operatorname{arcsec} x$	$\dfrac{x}{\sqrt{x^2-1}}$
$+$	$\dfrac{1}{x\sqrt{x^2-1}}$	$\sqrt{x^2-1}$
$+$	1	$\dfrac{1}{x}$
$-$	0	$\ln\lvert x\rvert$

The reader is encouraged to compare this calculation with one that begins with the change of variable $\vartheta = \operatorname{arcsec} x$.

When refactoring is not involved, as in Example 1, it can be more efficient to apply the equation (which summarizes repeated partial integration)

$$\begin{aligned}\int f(x)g(x)\,dx &= f(x)g_1(x) - f'(x)g_2(x) + f''(x)g_3(x) - \cdots\\ &\qquad \cdots + (-1)^n\int f^{(n)}(x)g_n(x)\,dx,\end{aligned} \tag{3}$$

where $f', \ldots, f^{(n)}$ are the higher derivatives of f, g_1 is an antiderivative of g, g_2 is an antiderivative of g_1, and so on. If $f^{(n)}$ is identically zero then the last term provides the constant of integration. Equation (3) is illustrated by Example 5 below.

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Example 5. The following integral is evaluated using Equation (3). At each stage, the polynomial is differentiated and the power of $2x-7$ is integrated.

$$\begin{aligned}\int \frac{3x^3+x^2-1}{\sqrt[3]{(2x-7)^5}}\,dx &= -\tfrac{3}{4}(3x^3+x^2-1)(2x-7)^{-2/3} + \tfrac{9}{8}(9x^2+2x)(2x-7)^{1/3} - \tfrac{27}{32}(9x+1)(2x-7)^{4/3} + \tfrac{729}{448}(2x-7)^{7/3} + C\\ &= \tfrac{3}{448}(96x^3 + 812x^2 + 17052x - 89411)\,\sqrt[3]{(2x-7)^{-2}} + C\end{aligned}$$

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The integrals below exemplify the utility of the tabular notation (possibly after a change of variable). They appear on the third example sheet.

1. $\int (x^2+3x-2)\sqrt[5]{3-2x}\,dx$, 2. $\int e^{-2t}\sin(4t-3)\,dt$, 3. $\int (7x^4+x)(\ln x)^3\,dx$, 4. $\int (x^3-x+5)e^{-x}\,dx$, 5. $\int \sin\sqrt[5]{x}\,dx$,

6. $\int (\arcsin x)^3\,dx$, 7. $\int \cos(\ln y^3)\,dy$, 8. $\int \sin(3x-2)\cos(5x)\,dx$, 9. $\int w^{23}\cos(w^6)\,dw$, 10. $\int \dfrac{\arcsin x}{x^2}\,dx$.

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