Archimedes and the method of exhaustion

CEGEP JOHN ABBOTT COLLEGE

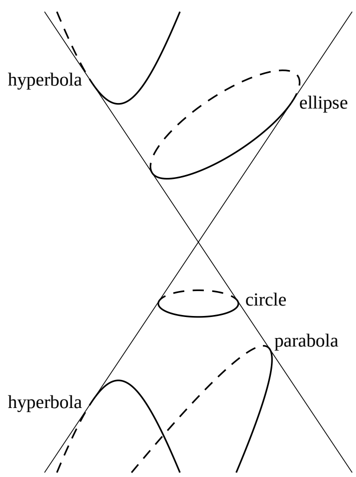
CALCULUS II (SCIENCE)

Winter 2007

Carefully study the text below and attempt the exercises at the end. You will be evaluated on this material by writing a 30 to 45 minute test (which may be part of a larger class test). This test will be worth 10% of your class mark and may include questions drawn from the exercises at the end.
This activity will contribute to your attainment of the Science Program competency: To put in context the emergence and development of scientific concepts.

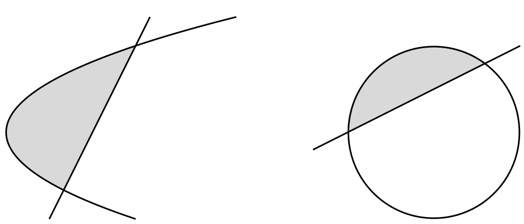
1. Conic Sections

The Greeks originally viewed parabolas (and also circles, ellipses and hyperbolas) as *conic sections*. Imagine rotating a straight line about a vertical line that intersects it, thus obtaining a circular (double) cone. A horizontal plane intersects the cone in a circle. When the plane is slightly inclined, the section becomes an ellipse. As the intersecting plane is inclined more towards the vertical, the ellipse becomes more elongated until, finally, the plane becomes parallel to a generating line of the cone, at which point the section becomes a parabola. If the intersecting plane is inclined still nearer to the vertical, it meets both branches of the cone (which it did not do in the previous cases); now the curve of intersection is a hyperbola.

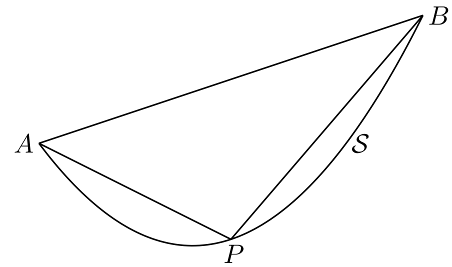


2. Archimedes' Theorem

A *segment* of a convex curve (such as a parabola, ellipse or hyperbola) is a region bounded by a straight line and a portion of the curve.



In his book *Quadrature of the Parabola*, Archimedes gives two methods for finding the area of a segment of a parabola (previous mathematicians had successfully attempted to find the area of a segment of a circle and of a hyperbola). The second of these methods, which we discuss below, is based on the so-called "method of exhaustion."



Given a parabolic segment $\mathcal{S}$ with *base* AB (see the above figure), the point P of the segment that is farthest from the base is called the *vertex* of the segment, and the (perpendicular) distance from P to AB is its *height*. (The vertex of a segment is not to be confused with the vertex of the parabola which, as you will recall, is the intersection point of the parabola with its axis of symmetry.) Archimedes shows that the area of the segment is four-thirds that of the inscribed triangle APB. That is,

> the area of a segment of a parabola is $4/3$ times the area of the triangle with the same base and height.

(Exercise 1 asks you to check Archimedes' result in a very simple case.)

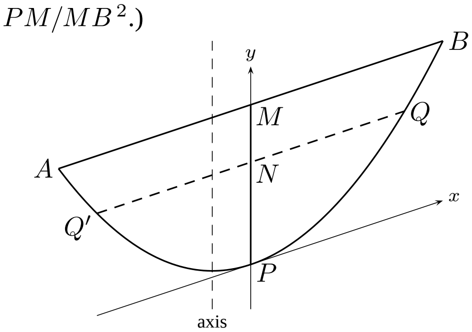
3. Preliminaries on Parabolic Segments

By the time of Archimedes, the following facts were known concerning an arbitrary parabolic segment $\mathcal{S}$ with base AB and vertex P.

P1. The tangent line at P is parallel to AB.
P2. The straight line through P parallel to the axis of the parabola intersects AB at its midpoint M.
P3. Every chord QQ' parallel to AB is bisected by PM.
P4. With the notation in the figure below,

$$\frac{PN}{PM} = \frac{NQ^2}{MB^2}$$

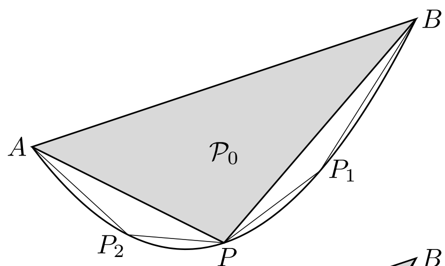
(Equivalently, $PN = (PM/MB^2) \cdot NQ^2$. In modern terms, this says that, in the pictured oblique xy-coordinate system, the equation of the parabola is $y = \lambda x^2$, where $\lambda = PM/MB^2$.)

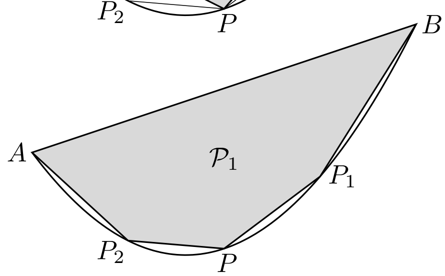


Archimedes quotes these facts without proof, referring to earlier treatises on the conics by Euclid and Aristaeus. (You are asked to prove the first three properties, mostly by modern methods, in exercises 2, 3 and 4. Using property P2 to find the vertex, you will then, in exercise 5, verify Archimedes' theorem in another special case. Exercise 6 will then guide you through a modern proof of the general case.)

4. Part 1: The Method of Exhaustion

To find the area of a given parabolic segment $\mathcal{S}$, Archimedes begins by constructing a sequence of inscribed polygons $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2, \ldots,$ that fill up or "exhaust" $\mathcal{S}$. The first polygon $\mathcal{P}_0$ is the inscribed triangle APB with AB the base of segment $\mathcal{S}$ and P its vertex. To construct the next polygon $\mathcal{P}_1$, consider the two smaller parabolic segments with bases PB and AP; let their vertices be P_1 and P_2, respectively, and let $\mathcal{P}_1$ be the polygon AP_2PP_1B.





We continue in this way, adding at each step the triangles inscribed in the parabolic segments remaining from the previous step. As seems clear from the above figures, the resulting polygons $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2, \ldots,$ exhaust the area of the original parabolic segment $\mathcal{S}$. In fact, Archimedes carefully *proves* this by showing that the difference between the area of $\mathcal{S}$ and the area of $\mathcal{P}_n$ can be made as small as one pleases by choosing n sufficiently large. In modern terms, this simply means that

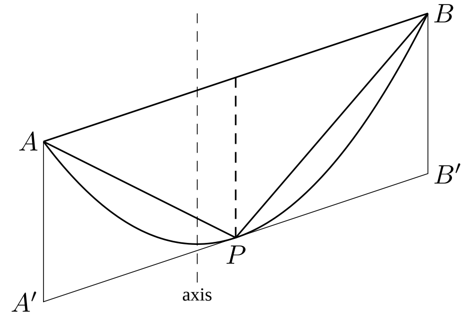
$$\lim_{n\to\infty} \text{area}(\mathcal{P}_n) = \text{area}(\mathcal{S}) \tag{1}$$

(but it is important to realize that Archimedes, like all the ancient Greek mathematicians, had no limit concept).

To prove (1), we let

$$M_n = \text{area}(\mathcal{S}) - \text{area}(\mathcal{P}_n) \quad \text{for } n = 0, 1, 2, \ldots$$

and show that $\lim_{n\to\infty} M_n = 0$. Consider the parallelogram $ABB'A'$ circumscribed about the segment $\mathcal{S}$, whose sides AA' and BB' are parallel to the axis of the parabola, and whose base $A'B'$ is tangent to the parabola at P (and therefore parallel to AB, by property P1).



Since the area of the inscribed triangle APB is half that of the circumscribed parallelogram (why?), it follows that the area of this triangle is more than half the area of the parabolic segment $\mathcal{S}$. The remaining area, which equals M_0 because $\mathcal{P}_0 = \triangle APB$, must therefore be less than half the area of $\mathcal{S}$:

$$M_0 < \tfrac{1}{2}\,\text{area}(\mathcal{S})$$

Now consider the two triangles ($\triangle AP_2P$ and $\triangle PP_1B$) that are added in the next step to form polygon $\mathcal{P}_1$. The above argument, applied to the two smaller parabolic segments with bases AP and PB, shows that the areas of these triangles are more than half the areas of the two segments. It therefore follows that

$$M_1 < \tfrac{1}{2}M_0$$

Continuing in this way, we see that

$$M_2 < \tfrac{1}{2}M_1, \quad M_3 < \tfrac{1}{2}M_2, \ldots,$$

and in general $M_n < \frac{1}{2}M_{n-1}$. It is now easy to prove that $\lim_{n\to\infty} M_n = 0$ (see exercise 7), and therefore (1) follows.

5. Part 2: Finding the area of $\mathcal{P}_n$

At each step in the construction of the polygons $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2, \ldots,$ we add triangles to the previous polygon: a single triangle ($\triangle APB$) begins the process, then two triangles ($\triangle AP_2P$ and $\triangle PP_1B$) are added in the next step, then four triangles are added, etc. Let a_0, a_1, $a_2, \ldots,$ be the total areas of the triangles added at each step. Thus

$$a_0 = \text{area}(\triangle APB),$$
$$a_1 = \text{area}(\triangle AP_2P) + \text{area}(\triangle PP_1B),$$

and so on. In the second part of his proof, Archimedes finds the area of the polygons $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2, \ldots,$ by evaluating the sum

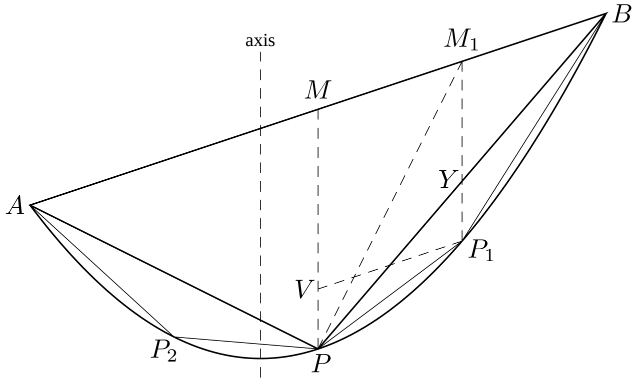
$$a_0 + a_1 + a_2 + \cdots + a_n = \text{area}(\mathcal{P}_n) \tag{2}$$

The key step is to show that the total area of the triangles added at each step is equal to $1/4$ the total area of the triangles added at the previous step. In other words,

$$a_1 = \tfrac{1}{4}a_0, \quad a_2 = \tfrac{1}{4}a_1, \ldots, \tag{3}$$

and in general $a_n = \frac{1}{4}a_{n-1}$. We will describe Archimedes' proof that $a_1 = \frac{1}{4}a_0$, leaving the general case to exercise 8.

We want to show that the sum a_1 of the areas of triangles AP_2P and PP_1B is $1/4$ that of $\triangle APB$. Apply property P2 to both the original parabolic segment $\mathcal{S}$ and the smaller segment with base PB: we obtain two lines parallel to the axis of the parabola, one going through P and intersecting AB at its midpoint M, and one going through P_1 and intersecting PB at its midpoint Y. Let M_1 be the intersection point of this second parallel line with AB. Then M_1 is the midpoint of MB because the triangles YM_1B and PMB are similar. Finally, let V be the intersection with PM of the line through P_1 parallel to AB (so VMM_1P_1 is a parallelogram).



Applying property P4 (with $N = V$ and $Q = P_1$) and noting that $VP_1 = MM_1 = \frac{1}{2}MB$, we have

$$\frac{PV}{PM} = \frac{{VP_1}^2}{MB^2} = \frac{1}{4}$$

so $PM = 4PV$. Two consequences follow from this. First, since $P_1M_1 = VM$, we have

$$P_1M_1 = 3PV \tag{4}$$

Second, because $M_1B = \frac{1}{2}MB$, $YM_1 = \frac{1}{2}PM$ (by similar triangles again), so that

$$YM_1 = 2PV \tag{5}$$

Now (4) and (5) imply that $P_1Y = PV$, so in fact

$$YM_1 = 2P_1Y$$

Now consider the two triangles PM_1B and PP_1B. They have the same base PB and, because $YM_1 = 2P_1Y$, a simple argument with similar triangles shows that the height of $\triangle PM_1B$ is twice the height of $\triangle PP_1B$ (both heights being relative to the common base PB). Therefore

$$\text{area}(\triangle PM_1B) = 2\,\text{area}(\triangle PP_1B) \tag{6}$$

Similarly, triangles PMB and PM_1B have the same base PB and, since $MB = 2M_1B$, the height of $\triangle PMB$ is twice that of $\triangle PM_1B$, so

$$\text{area}(\triangle PMB) = 2\,\text{area}(\triangle PM_1B) \tag{7}$$

It follows from (6) and (7) that

$$\text{area}(\triangle PP_1B) = \tfrac{1}{4}\,\text{area}(\triangle PMB) \tag{8}$$

An argument similar to the one in the last paragraph shows that

$$\text{area}(\triangle AP_2P) = \tfrac{1}{4}\,\text{area}(\triangle APM)$$

Combining this with (8) then gives

$$\begin{aligned} &\text{area}(\triangle AP_2P) + \text{area}(\triangle PP_1B) \\ &= \tfrac{1}{4}\,\text{area}(\triangle APM) + \tfrac{1}{4}\,\text{area}(\triangle PMB) \\ &= \tfrac{1}{4}\,\text{area}(\triangle APB) \end{aligned}$$

so $a_1 = \frac{1}{4}a_0$, as desired (see exercise 9 for a shorter proof).

Returning now to the area of polygon $\mathcal{P}_n$ (see (2) above) and using (3), we have finally

$$\text{area}(\mathcal{P}_n) = a_0 + \frac{1}{4}a_0 + \frac{1}{4^2}a_0 + \cdots + \frac{1}{4^n}a_0$$

In other words, the areas a_0, a_1, $a_2, \ldots,$ added at each step in the construction of $\mathcal{P}_n$ form a geometric sequence with common ratio $1/4$, and $\text{area}(\mathcal{P}_n)$ is the sum of the first $n+1$ terms of this sequence.

6. Conclusion of Archimedes' Proof

Now that the area of $\mathcal{P}_n$ has been determined, it follows from the conclusion of Part 1 that the difference between the area of the parabolic segment $\mathcal{S}$ and the sum

$$a_0 + \frac{1}{4}a_0 + \frac{1}{4^2}a_0 + \cdots + \frac{1}{4^n}a_0$$

can be made as small as one pleases by choosing n sufficiently large. In modern terms,

$$\text{area}(\mathcal{S}) = \lim_{n\to\infty}\left(a_0 + \frac{1}{4}a_0 + \frac{1}{4^2}a_0 + \cdots + \frac{1}{4^n}a_0\right) \tag{9}$$

and Archimedes now seeks to determine this limit.

He begins by deriving the identity

$$1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^n} + \frac{1}{3}\cdot\frac{1}{4^n} = \frac{4}{3} \tag{10}$$

which is a restatement of the formula you learned for the partial sums of a geometric series (see exercise 10). As Archimedes shows, (10) follows from the observation that

$$\frac{1}{4^k} + \frac{1}{3}\cdot\frac{1}{4^k} = \frac{4}{3\cdot 4^k} = \frac{1}{3}\cdot\frac{1}{4^{k-1}}$$

for we can then sum the terms on the left side of (10) by repeatedly adding the last two terms:

$$\begin{aligned} &1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \left(\frac{1}{4^n} + \frac{1}{3}\cdot\frac{1}{4^n}\right) \\ &\quad = 1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \left(\frac{1}{4^{n-1}} + \frac{1}{3}\cdot\frac{1}{4^{n-1}}\right) \\ &\quad = \ldots \\ &\quad = 1 + \frac{1}{4} + \left(\frac{1}{4^2} + \frac{1}{3}\cdot\frac{1}{4^2}\right) \\ &\quad = 1 + \left(\frac{1}{4} + \frac{1}{3}\cdot\frac{1}{4}\right) \\ &\quad = 1 + \frac{1}{3} = \frac{4}{3} \end{aligned}$$

From a modern perspective, Archimedes' theorem is now a simple consequence of (9) and (10):

$$\begin{aligned} \text{area}(\mathcal{S}) &= a_0 \cdot \lim_{n\to\infty}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^n}\right) \\ &= a_0 \cdot \lim_{n\to\infty}\left(\frac{4}{3} - \frac{1}{3}\cdot\frac{1}{4^n}\right) \\ &= \tfrac{4}{3}a_0 - a_0 \cdot \lim_{n\to\infty}\frac{1}{3\cdot 4^n} \\ &= \tfrac{4}{3}a_0 - a_0 \cdot 0 \\ &= \tfrac{4}{3}\,\text{area}(\triangle APB) \end{aligned}$$

No doubt Archimedes intuitively obtained the answer $4/3$ in a similar way but, rather than taking limits explicitly, he completed the proof by showing that the two alternative conclusions

$$\text{area}(\mathcal{S}) < \tfrac{4}{3}\,\text{area}(\triangle APB)$$

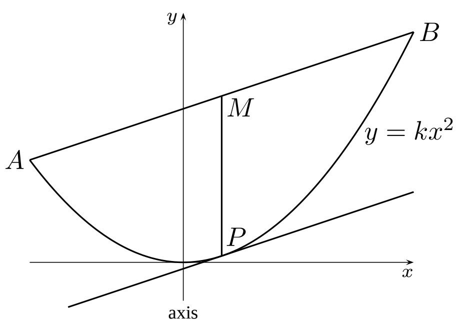
and

$$\text{area}(\mathcal{S}) > \tfrac{4}{3}\,\text{area}(\triangle APB)$$

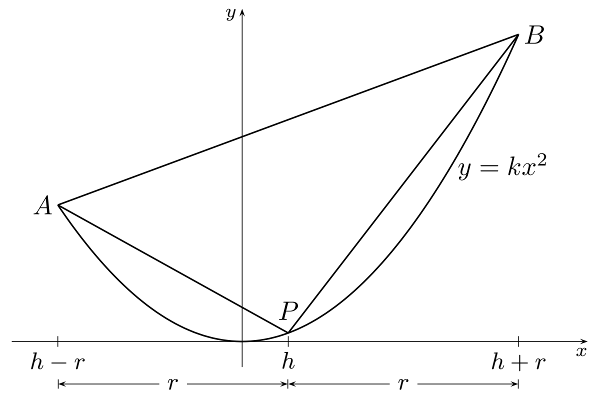
both lead to a *contradiction*, and so must be false. This approach (whose details we will omit) was in fact typical of Greek proofs by the method of exhaustion.

7. Exercises

1. Use integration to verify Archimedes' theorem for the segment bounded by $y = x^2$ and the line $y = 1$. (Determine the vertex of the segment and show that the inscribed triangle has area 1. Then integrate to verify that the segment has area $4/3$.)
2. Property P1 follows easily from an important theorem you learned in Calculus I. Which one? (Rotate the parabolic segment until its base AB is horizontal. What can you then say about the vertex P?)
3. Prove property P2 by modern methods as follows. Introduce a rectangular xy-coordinate system centred at the vertex of the parabola, as in the figure below.



In these coordinates, the equation of the parabola has the form $y = kx^2$ and its axis of symmetry is the y-axis. Assume first that $k = 1$ and write $A = (a, a^2)$, $B = (b, b^2)$ and $P = (x, x^2)$.
   (a) Show that $x = \frac{1}{2}(a + b)$ by computing the slope of the tangent line at P in two different ways: (i) using calculus and (ii) using property P1.
   (b) Explain why the result of part (a) proves property P2.
   (c) If $k \neq 1$, what changes do you need to make to the calculation in part (a)?
4. Explain why property P3 follows from P2. (What is the vertex of the parabolic segment $Q'PQ$?)
5. Consider the parabolic segment bounded by $y = x^2$ and the line $y = 2x + 3$.
   (a) Sketch the segment and find the points A, B and P. (Use property P2 to find the vertex P of the segment; see also exercise 3(a).)
   (b) Let M be the midpoint of AB. Find PM and use it to compute the area of triangle APB. (Add the areas of triangles APM and PBM, and note that these triangles have the same base PM and equal heights.)
   (c) Find the area of the segment by integrating and check Archimedes' theorem.
6. Prove Archimedes' theorem by modern methods as follows. Using property P2, we can label the x-coordinates of A, B and P as in the figure below.



Assume first that $k = 1$.
   (a) Use the method of exercise 5(b) to show that the area of triangle APB is r^3.
   (b) Show that the equation of line AB is $y = 2hx + r^2 - h^2$.
   (c) Find the area of the segment by integrating. (You should get a value of $\frac{4}{3}r^3$, thus proving Archimedes' theorem for $k = 1$.)
   (d) If $k \neq 1$, what changes do you need to make to the calculations in parts (a), (b) and (c)?
7. (a) Let M_0, M_1, $M_2, \ldots,$ be any sequence of positive numbers such that $M_n < \frac{1}{2}M_{n-1}$ for $n = 1, 2, \ldots$ Show that $\lim_{n\to\infty} M_n = 0$. This result, which is really the crux of the method of exhaustion, is originally due to Eudoxus, a predecessor of Euclid's. (For the proof, start by showing that $0 < M_n < M_0/2^n$.)
   (b) Given any positive number $r < 1$, show that the conclusion of part (a) still holds if the positive numbers M_0, M_1, $M_2, \ldots,$ satisfy $M_n < rM_{n-1}$ for $n = 1, 2, \ldots$
8. Show that $a_1 = \frac{1}{4}a_0$ implies $a_2 = \frac{1}{4}a_1$, and convince yourself of the general case $a_n = \frac{1}{4}a_{n-1}$.
9. In exercise 6, you showed that for the parabola $y = kx^2$, the area of triangle APB is kr^3. Use this to give a second proof that $a_1 = \frac{1}{4}a_0$.
10. Prove (10) using the formula for the partial sums of a geometric series:

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

11. Write a short summary of the three parts of Archimedes' proof.