WeBWorK set G-MaxMin, Problem 4 – the graph

This problem was to graph the function $y = \dfrac{x^3}{x^2 - a^2}$ for various values of a. I'll sketch the solution for the case $a = 2$; you can make corresponding changes for the version you had.

If $y = \dfrac{x^3}{x^2 - a^2}$, then $y' = \dfrac{x^2(x^2 - 3a^2)}{(x^2 - a^2)^2}$ and $y'' = \dfrac{2a^2x(x^2 + 3a^2)}{(x^2 - a^2)^3}$. In particular, if $a = 2$ we get $y = \dfrac{x^3}{x^2 - 4}$, $y' = \dfrac{x^2(x^2 - 12)}{(x^2 - 4)^2}$, $y'' = \dfrac{8x(x^2 + 12)}{(x^2 - 4)^3}$. From this, the asymptotes, critical points, points of inflection, intervals where y is increasing or decreasing, where it is concave up or down, and so where local maxima, minima, all may be found. In general, the vertical asymptotes are at $x = \pm a$; the max and min are at $x = \pm a\sqrt{3}$, the concavity changes at $x = 0$ (and at the asymptotes).

The WeBWorK solutions for the $a = 2$ version of the question may be seen in this "mock-up":

(1 point) Answer the following questions for the function

$$f(x) = \frac{x^3}{x^2 - 4}.$$

Enter points, such as inflection points in ascending order, i.e. smallest x values first. Enter "INF" for ∞ and "MINF" for $-\infty$.

Enter intervals in ascending order also.

A. The function $f(x)$ has two vertical asympototes:
$x =$ -2 and $x =$ 2

B. $f(x)$ has one local maximum and one local minimum occuring at x values :
$x_{\text{max}} =$ $-\sqrt{12}$ and $x_{\text{min}} =$ $\sqrt{12}$

C. For each interval, tell whether $f(x)$ is increasing (type in INC) or decreasing (type in DEC).

$(-\infty, max)$ INC

$(max, -2)$ DEC

$(-2, 0)$ DEC

$(0, 2)$ DEC

$(2, min)$ DEC

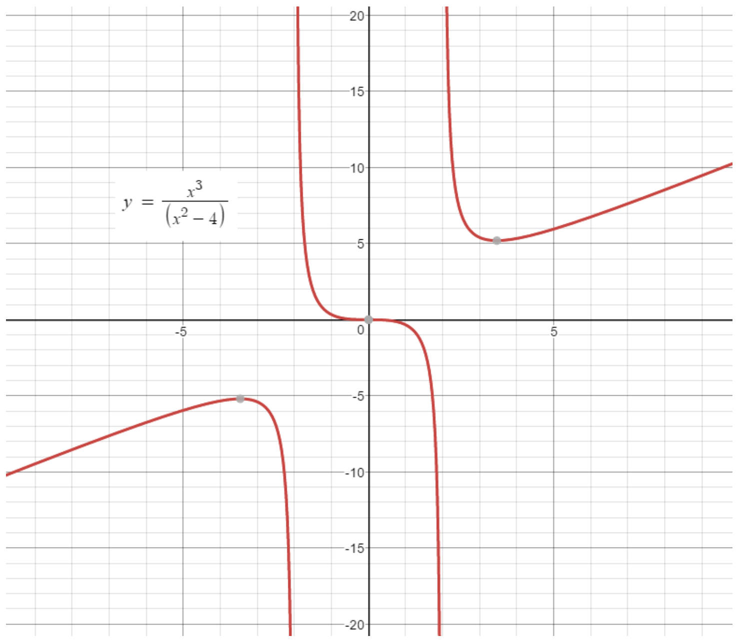
$(min, +\infty)$ INC

D. $f(x)$ is concave up on the interval (-2 , 0)
and on the inteval (2 , INF)

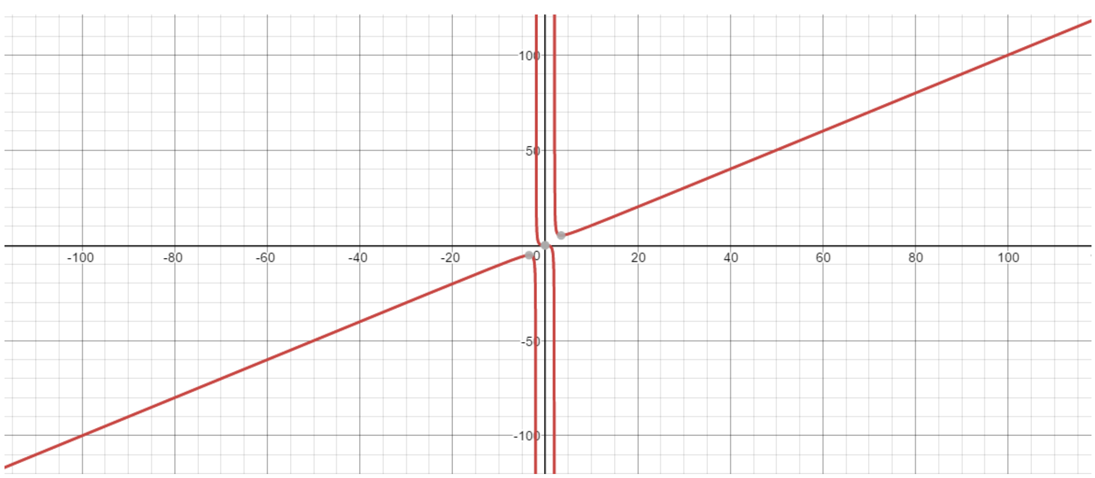
E. The inflection point for this function occurs at $x =$ 0

F. Sketch the graph of $f(x)$ and bring it to class.

From this data, we can sketch the function; for $a = 2$ the function is given below (the shape is the same for other values of a, but the x and y values will change appropriately).



A point of interest: with the "don't sweat the small stuff" idea, it seems likely that as $x \to \pm\infty$ the graph will look more and more like $y = x$ (we call this a "slant asymptote"). This is pretty clear if we change the scale of the picture, rather like this:[1]



[1]The graph looks remarkably like two vertical lines with a diagonal crossing them—the vertical and the slant asymptotes being the dominant features from "far away".