CAUTION

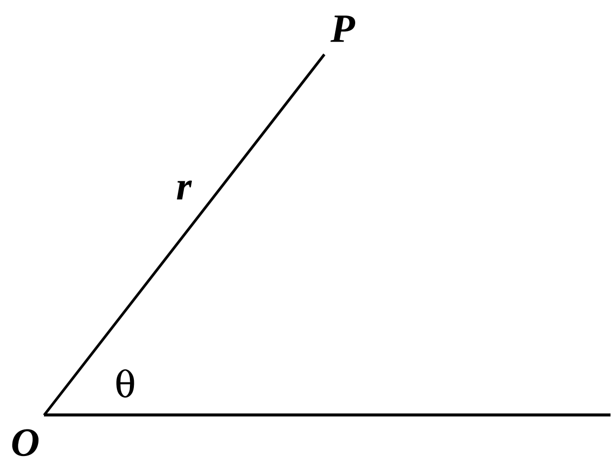
Printing this document can produce errors in the hard copy. For example, on some platforms, output to a postscript file converts plus-minus signs to plus signs. It is suggested that this manuscript be read from the monitor.

There are a few links in the text to other parts of the text. These function correctly only if all sections are expanded.

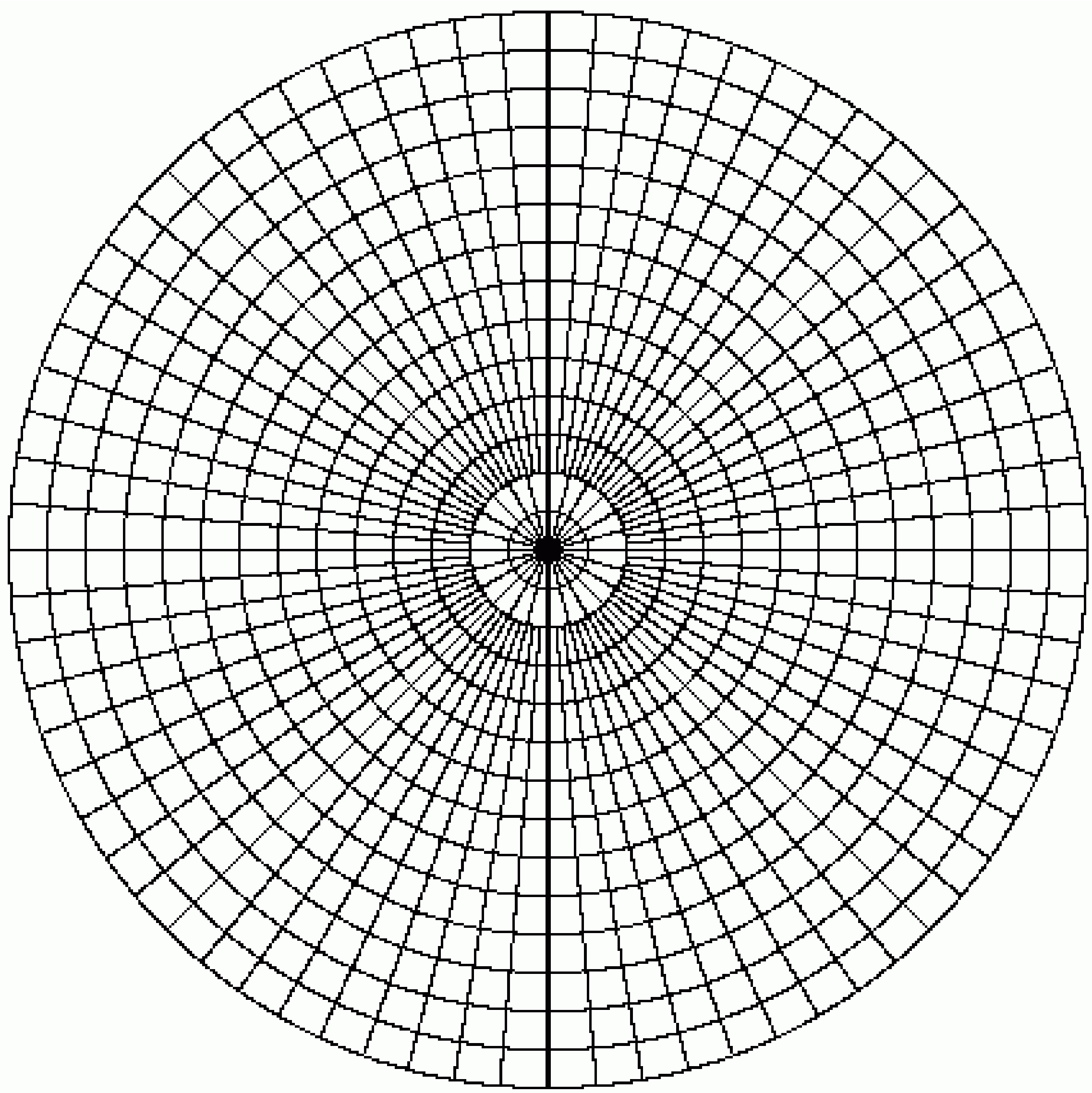
Introduction to Polar Coordinates

Definition of Polar Coordinates

A polar coordinate system is defined in the plane by selecting a point ***O*** called the *pole*,and a half-line emanating from the pole, called a *reference line* or *reference direction*. The reference line is usually drawn horizontally on the page, pointing towards the right (like the usual positive *x*-axis). Each point ***P*** in the plane is located by specifying the distance *r* from ***P*** to the pole ***O*** and the angle θ measured counter-clockwise from the reference line to the line through ***O*** and ***P***. The quantities *r* and θ are called *polar coordinates* of ***P***, and are listed in the order (r,θ).



Plotting points from polar coordinates is conveniently accomplished on polar graph paper, which consists of concentric circles about the pole, corresponding to different *r* values, and radial lines through the pole, corresponding to different θ values. For the type of sketching we shall do, it is not necessary to use polar graph paper, but a sample is displayed below for the student's information.



Non-uniqueness of Polar Coordinates

The above definition of polar coordinates is extended to allow for all real values of r and θ. For example, values of θ greater than 2π interpret as one or more complete counterclockwise rotations about ***O*** before arriving at ***P***. Such values of θ do not determine new points; they merely provide additional coordinate sets. The point located one unit directly above the pole would have polar coordinates $(1, \frac{\pi}{2})$, but the same point would also have polar coordinates $(1, \frac{5\pi}{2})$ and $(1, \frac{9\pi}{2})$ as well as infinitely many others. Negative values of θ interpret as clockwise rotations. The point $(1, \frac{\pi}{2})$ also has polar coordinates $(1, -\frac{3\pi}{2})$ as well as infinitely many others with negative θ.

Negative values of r interpret in terms of backwards (or reverse) motion in the direction determined by θ. Thus the point with polar coordinates $(1, \frac{\pi}{2})$ would also have polar coordinates

$(-1, \frac{3\pi}{2})$ and infinitely many others obtained from this one by adding or subtracting integer multiples of 2π to the angle $\frac{3\pi}{2}$.

The non-uniqueness of polar coordinates makes it necessary to be able to determine all polar coordinates for a given point. For example, suppose one wishes to determine whether or not a particular point is on the graph of a certain polar equation. In Cartesian coordinates, such a determination is simple. If the coordinates of the given point satisfy the Cartesian equation then the point is on its graph; otherwise it is not. But even when one set of polar coordinates of a given point fails to satisfy a polar equation, the point has infinitely many other sets of polar coordinates that still might satisfy the equation.

If a point has one set of polar coordinates (r, θ), then the others are $(r, \theta + 2k\pi)$ and $(-r, \theta + (2k+1)\pi)$, $k = 0, 1, 2, \ldots$.

If r is required to be nonnegative and θ is required to lie in the half open interval $[0, 2\pi)$ then there is no ambiguity in the polar coordinates (except that θ is arbitrary for $r = 0$). The student may reasonably ask why these restrictions are not imposed in order to simplify the treatment. The answer is that many applications and contexts require using values of r and θ outside of these ranges, so that these restrictions would unduly limit the usefulness of the polar coordinate system.

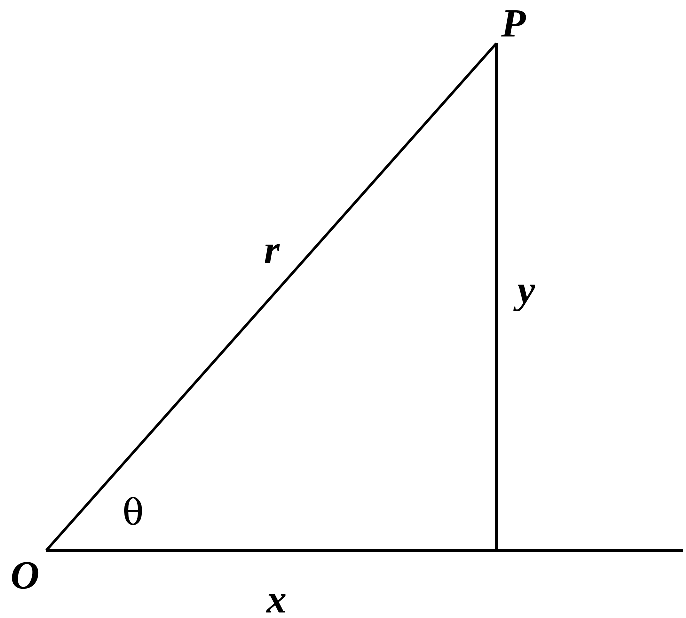
Associated Cartesian Coordinates and Transformation Equations

Definition of the Associated Cartesian System

It is very useful (and conventional) to associate, in a natural way, a Cartesian coordinate system with each polar coordinate system. The origin of the associated Cartesian system is taken as the pole of the polar system, and the positive x-axis is taken as the reference line in the polar system.

Transforming Coordinates

In this way, every point in the plane has two kinds of coordinates - Cartesian and polar. It is natural to ask how these are related. Given coordinates in one of the coordinate systems, how can coordinates be obtained in the other? This determination is referred to as *transformation of coordinates*, and the equations which effect the transformation are referred to as *transformation equations*. These equations can be obtained from the following diagram, in which the geometrical significance of the coordinates r, θ, x and y has been shown. The diagram assumes that the point in question is situated in the first quadrant, but the results can be verified for other quadrants, when necessary.



From Polar to Cartesian

In the right triangle above, the hypotenuse is r, the side opposite θ is y and the side adjacent to θ is x. Thus $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$ whence

$$x = r\cos(\theta),$$
$$y = r\sin(\theta).$$

These are the transformation equations from polar to Cartesian. They accept values of r and θ for input and give values for x and y as output. Satisfy yourself that these equations are valid no matter what quadrant the point (x, y) is in and no matter what sign r has.

Example problem

Show that the foregoing equations produce the same Cartesian coordinates for each of the polar coordinate sets $(1, \frac{\pi}{2})$, $(-1, -\frac{\pi}{2})$ and $(-1, \frac{3\pi}{2})$.

Solution: For the first set $x = (1)\cos\left(\frac{\pi}{2}\right) = 0$ and $y = (1)\sin\left(\frac{\pi}{2}\right) = 1$.

For the second set $x = (-1)\cos\left(-\frac{\pi}{2}\right) = 0$, $y = (-1)\sin\left(-\frac{\pi}{2}\right) = (-1)(-1) = 1$.

For the third set $x = (-1)\cos\left(\frac{3\pi}{2}\right) = 0$, $y = (-1)\sin\left(\frac{3\pi}{2}\right) = (-1)(-1) = 1$.

From Cartesian to Polar

Transforming Equations

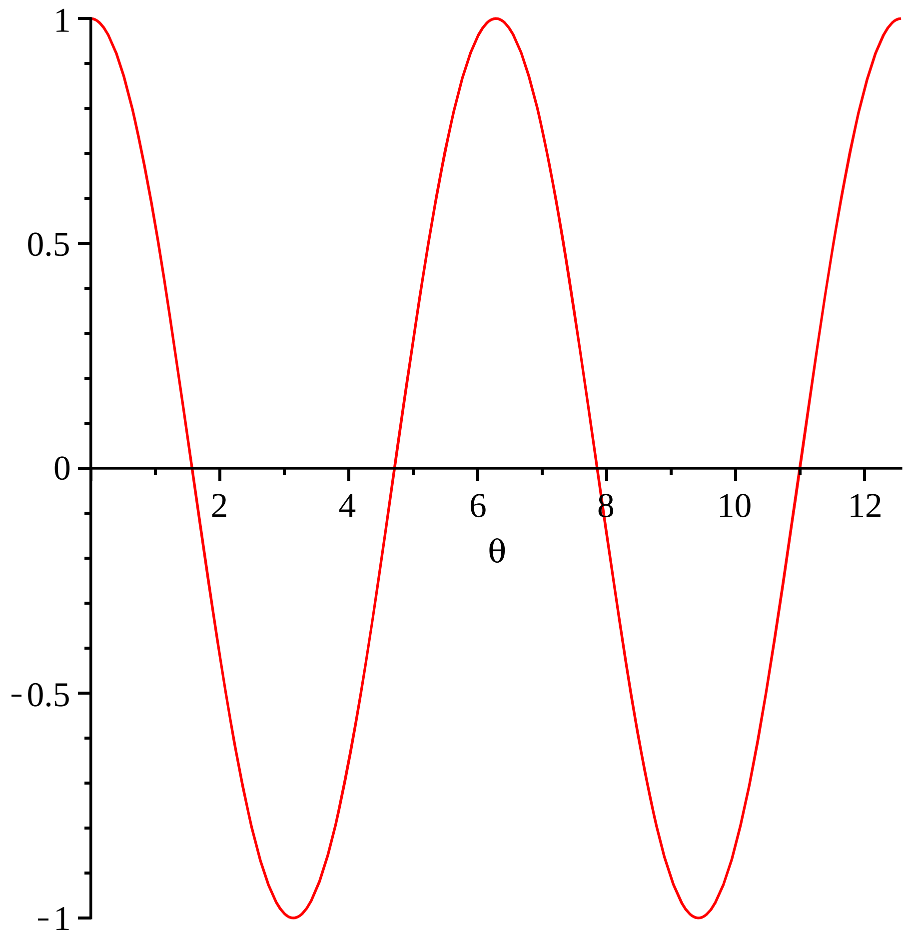
Recall what is meant by the graph of an equation. Given and equation in the variables x and y, the *graph of the equation* is the set of all points in the xy-plane whose coordinates (x, y) satisfy the given equation. Going the other way, starting with a figure in the plane which is the graph of an equation in the variables x and y, we refer to the equation as the *equation of the figure.* For example, we would say that the graph of $x^2 + y^2 = 25$ was the circle of radius 5 centered at the origin, and we would also say that the equation of the circle of radius 5 centered at the origin was

$x^2 + y^2 = 25.$

When both polar and Cartesian coordinate systems are under discussion, we must distinguish between the Cartesian graph and the polar graph of an equation, and between the Cartesian equation and the polar equation of a figure. In general, the variables x and y will be associated with Cartesian coordinates and the variables r and θ will be associated with polar coordinates. Any deviation from this convention must be made very clear.

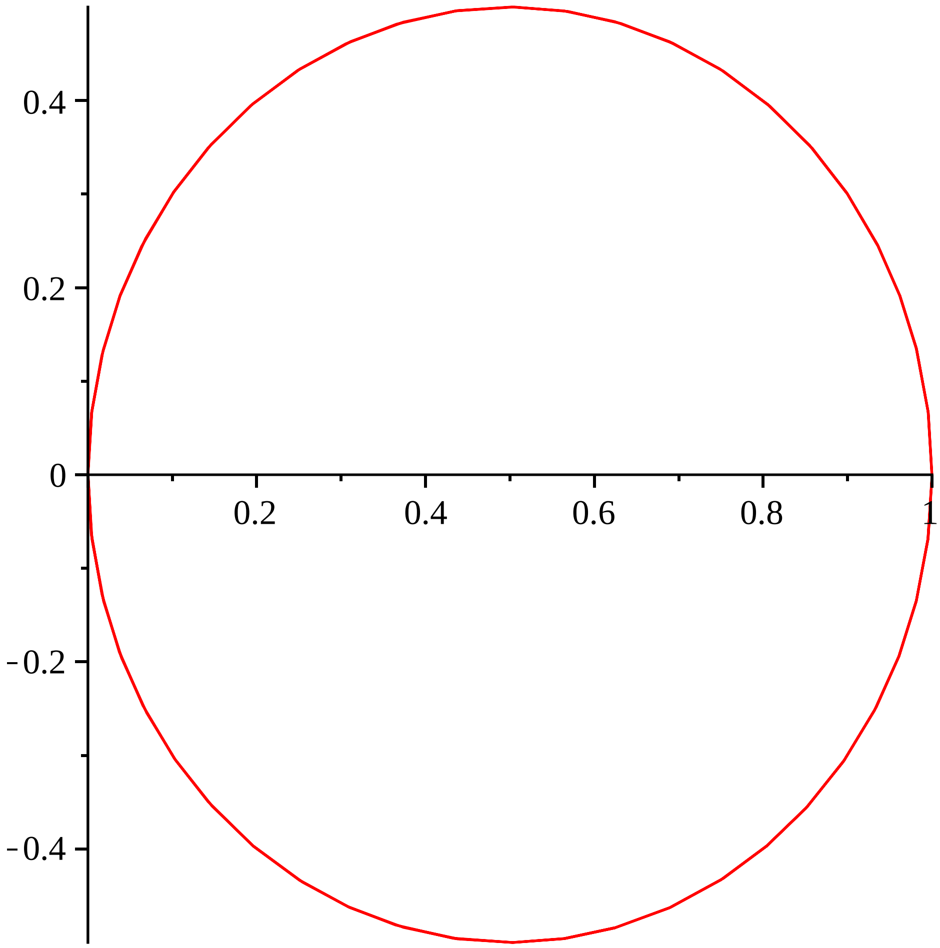
Consider the equation $r = \cos(\theta)$. One way to graph this equation would be to introduce a θ axis and an r axis and draw the cosine function.

> $plot(\cos(\theta), \theta = 0 ..4\,\pi)$



In this plot, r and θ have been plotted as Cartesian coordinates, not polar coordinates, so this is the Cartesian graph of the equation and not the polar graph. If this was what was wanted, it would have been referred to as the "Cartesian graph" of the equation. Following our convention above, we should interpret r and θ as polar coordinates and construct the polar graph. Maple will do that for us as follows:

> $plot([\cos(\theta), \theta, \theta = 0 ..2\,\pi], coords = \text{polar})$



We shall verify later that this is a circle.

One other point should be made regarding the definition of the polar graph of an equation. Because points in the plane have many different sets of polar coordinates, we must define the polar graph of an equation to consist of all points in the plane *having at least one set of polar coordinates* which satisfy the equation.

From Cartesian to Polar

The rule for transforming a given equation from one coordinate system to another is simple and natural: *always substitute equals for equals.* Suppose, for example, we want to find the polar equation of the vertical line $x = 3$. Since $x = r\cos(\theta)$, we obtain $r\cos(\theta) = 3$, which can also be written $r = 3\sec(\theta)$. As another example, consider the circle of radius 2 centered on the y-axis at the point (0, -2). The Cartesian equation of this circle is $x^2 + (y+2)^2 = 4$. To find the polar equation of the circle we substitute $x = r\cos(\theta)$ and $y = r\sin(\theta)$ to obtain $r^2\cos(\theta)^2 + (r\sin(\theta) + 2)^2 = 4$. Technically, this is the polar equation of the circle, but the equation can and should be simplified. In fact,

$r^2\cos(\theta)^2 + r^2\sin(\theta)^2 + 4\,r\sin(\theta) + 4 = 4$,

$r^2 + 4\,r\sin(\theta) = 0$,

$r = -4\sin(\theta)$.

From Polar to Cartesian

The simple and natural rule given above (*i.e.* substitute equals for equals) is "easier said than

done" when it comes to transforming polar equations to Cartesian equations. The reason is that we do not have simple expressions to substitute for r and θ in terms of x and y, as explained above. However we *do* have expressions for r^2 and for $\tan(\theta)$, and sometimes a polar equation can be manipulated to make use of these, as in the following example. Suppose we want to transform the polar equation $r = 2\sin(\theta) - 4\cos(\theta)$ to Cartesian form. Multiplying through the equation by r gives $r^2 = 2r\sin(\theta) - 4r\cos(\theta)$ wherein the substitutions $r^2 = x^2 + y^2$, $r\sin(\theta) = y$ and $r\cos(\theta) = x$ give
$x^2 + y^2 = 2y - 4x$. This in turn simplifies to $(x+2)^2 + (y-1)^2 = 5$.

Some Standard Polar Curves

In order to illustrate calculus topics associated with polar coordinates, it is necessary to deal with figures defined by polar equations. Such figures are more accessible when the student has a supply of familiar figures and equations to draw from. For that reason we introduce and describe a number of standard examples below. The student should commit this material to memory.

$r = c$

The graph of $r = c$ (where c denotes a constant) is a circle cenetered at the origin (pole) of radius $|c|$. As simple as this example is, it merits some elaboration. Note first that because the variable θ does not appear in the equation, it is unrestricted. That is, when we seek polar coordinates (r, θ) satisfying the equation, *any* value of θ satisfies the equation so long as the accompanying value of r is c. This is why the graph of the equation consists of an entire circle. Secondly, note that the equation has solutions even for negative values of c, since we have assigned a meaning to polar coordinates (r, θ) for all values of r.

The following Maple command will plot the graph of $r = -3$.

```
> plot([-3, θ, θ = 0..2π], coords = polar)
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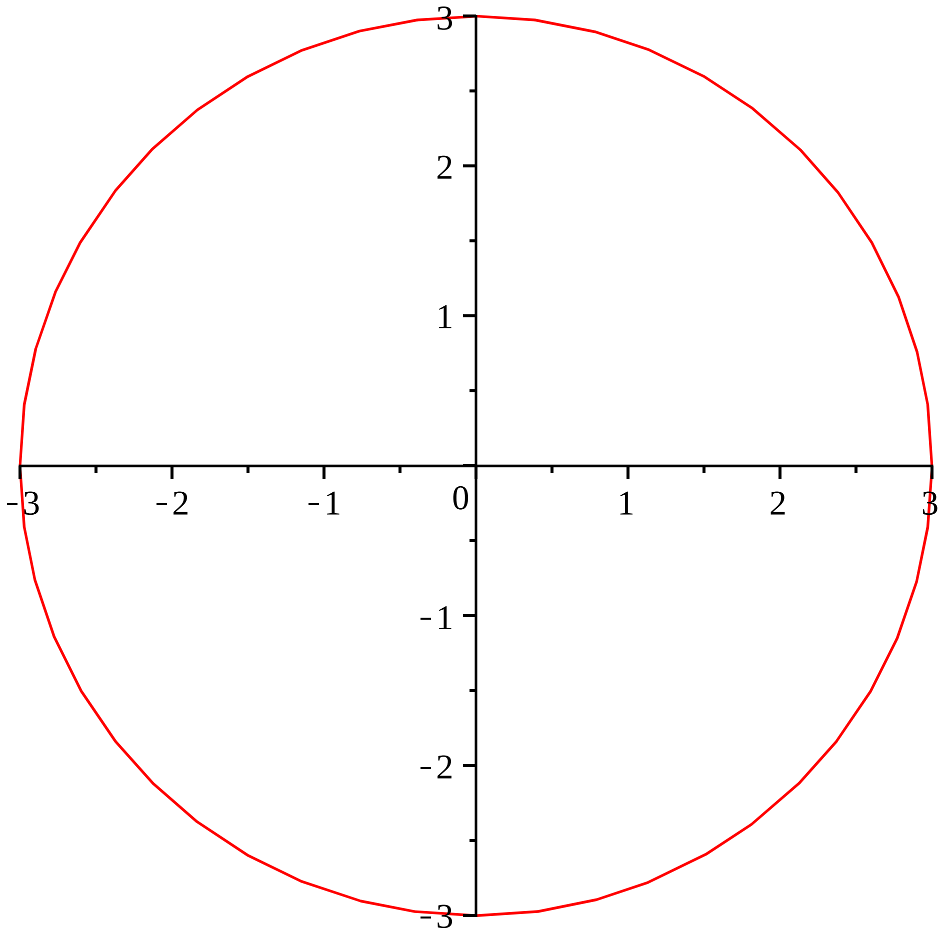
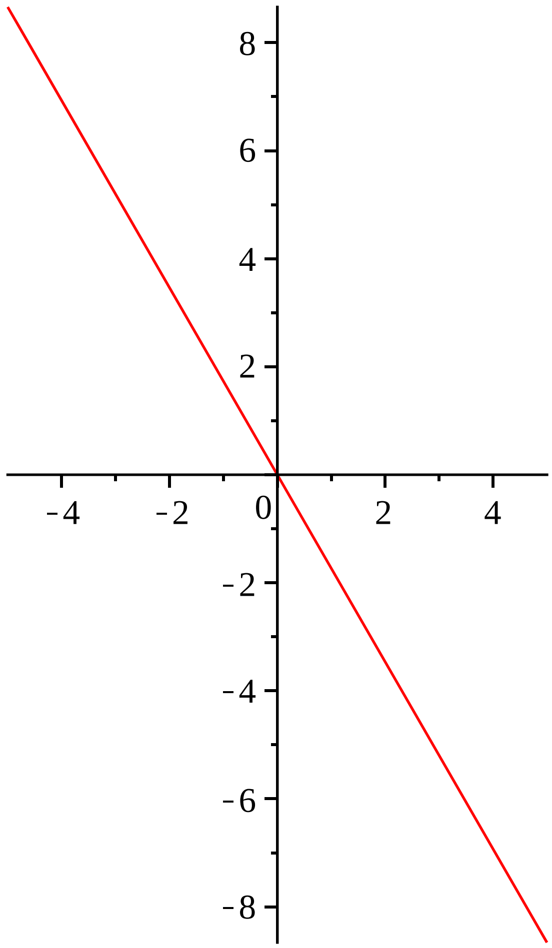


$\theta = c$

The graph of $\theta = c$ is a line through the origin (pole) which makes an angle of θ with the positive x-axis (reference direction). Because r does not appear in the equation, its value is unrestricted (just as the value of θ is unrestricted in the previous example) and this is what causes the graph to be an entire line. Note that negative values of r determine points on the line lying on the opposite side of the origin from points determined by positive r.

The following Maple command will plot a portion of the line $\theta = \frac{2\pi}{3}$.

> $plot\left(\left[r, \frac{2\pi}{3}, r = -10..10\right], coords = \mathrm{polar}, scaling = CONSTRAINED\right)$



$r = 2\,a\,\cos(\theta)$

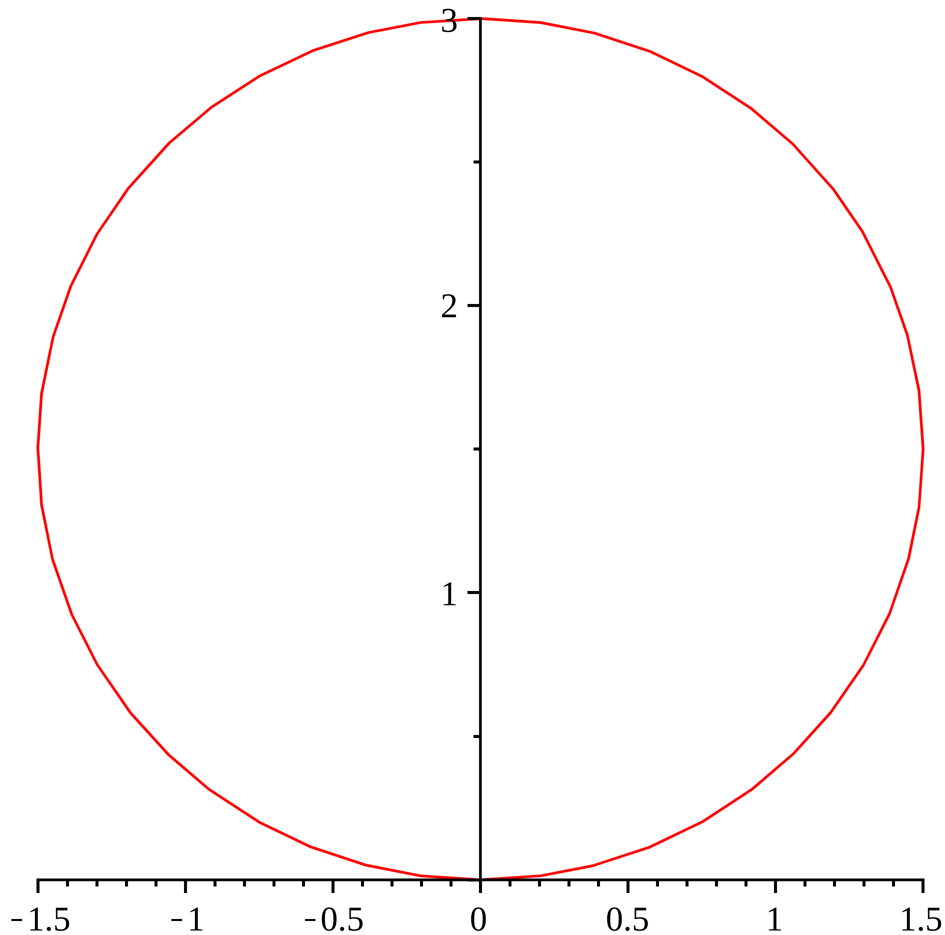
This is a spendid example of a polar equation that can be effectively analyzed by transforming to Cartesian coordinates. Multiplying both sides by r gives $r^2 = 2\,a\,r\cos(\theta)$ and substituting for each side gives $x^2 + y^2 = 2\,ax$. Bringing the $2\,ax$ to the left side and completing the square gives $(x - a)^2 + y^2 = a^2$. Thus we see that **the polar graph of the equation** $r = 2\,a\,\cos(\theta)$ **is a circle of radius** $|a|$ **centered at (**a, 0**).** We used Maple to plot an example of this equation in an earlier discussion.

Here are some useful remarks about this example. First of all, the student should memorize the information just presented. This makes the sketch of the equation immediately available without having to go through all the steps of transforming the equation to Cartesian coordinates and then simplifying the result. Secondly, notice that the description of the graph holds even for negative values of a, which is the reason the the radius needs to be described in general as $|a|$ instead of just a. Third, the appearance of the 2 in $r = 2\,a\,\cos(\theta)$ is wholly a matter of convenience. The numerical coefficient of $r\cos(\theta)$ could just as well be denoted by a, instead of by $2\,a$, but in that case the radius would be described by $\left|\frac{a}{2}\right|$ and the center would be situated at the point $(\frac{a}{2}, 0)$. So using the 2 in naming the coefficient makes for a "cleaner" description of the graph. Finally, the point determined by the coordinates (a, 0) is the same whether the coordinates are interpreted as polar or as Cartesian. In the next example, the coordinates of the center are *not* the same in either system, so we will have to be careful to specify the system when giving the coordinates.

$r = 2\,a\,\sin(\theta)$

Mutiplying through this equation by r and substituting gives $x^2 + y^2 = 2\,a\,y$ which simplifies to $x^2 + (y - a)^2 = a^2$. Thus the graph is again a circle of radius $|a|$, this time centered at polar $(a, \frac{\pi}{2})$ (*i.e.* Cartesian $(0,a)$).

> $plot([3\sin(\theta), \theta, \theta = 0..\pi], coords = \text{polar})$

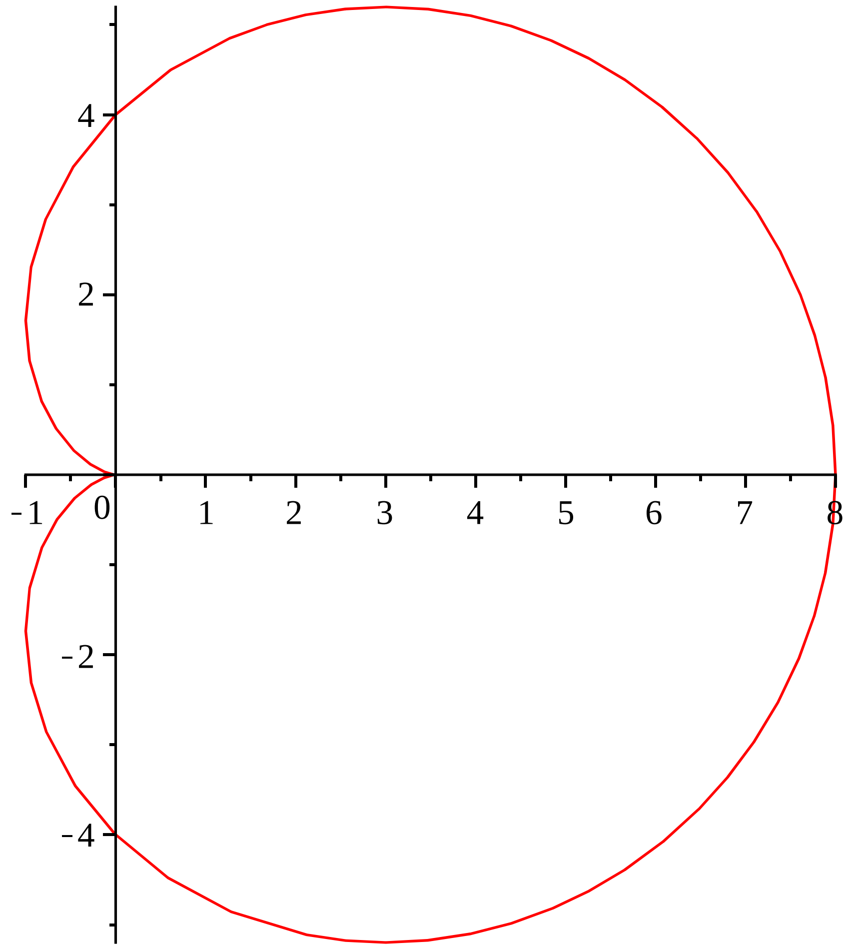


Notice that the entire circle is traced out as θ varies from 0 to π.

$r = a\,(1 + \cos(\theta))$

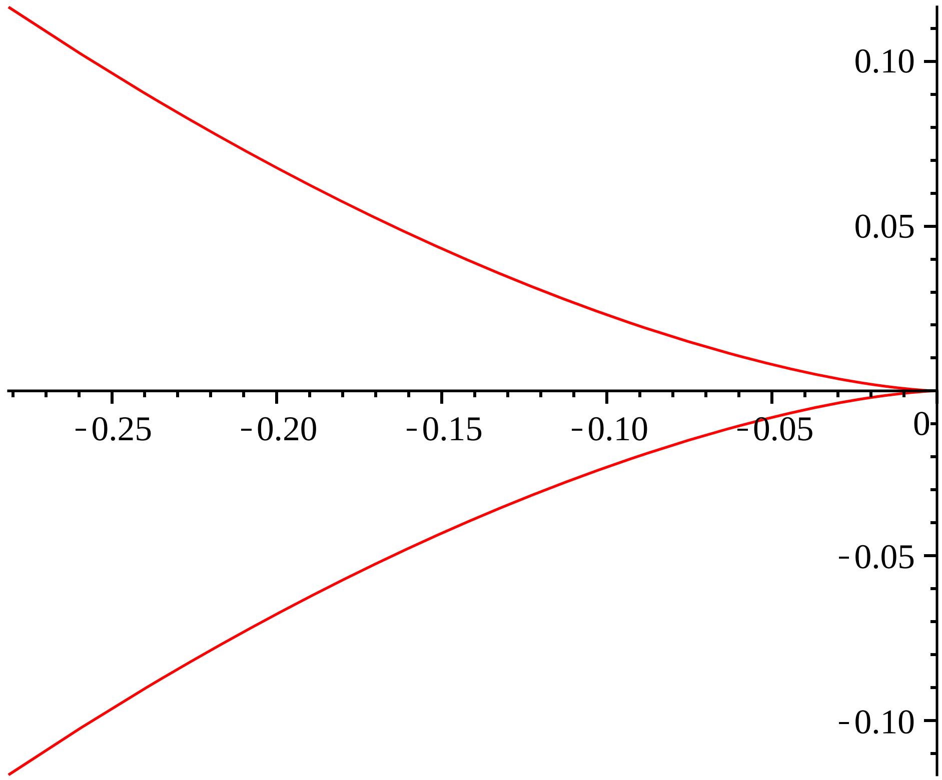
The graph of this equation is shown below for $a = 4$. Because the curve is heart shaped, it is called a *cardioid.*

> $plot([4\,(1 + \cos(\theta)), \theta, \theta = 0..2\pi], coords = \text{polar}, scaling = constrained)$



Once the student knows the general shape of the graph, it is fairly easy to sketch the figure from the four points determined by $\theta = 0$, $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$. Notice that the graph is symmetric about the *x*-axis. An interesting phenomenon occurs at the pole. Let's re-plot the curve around the pole for a closer look.

$$> plot\left(\left[4\,(1+\cos(\theta)),\,\theta,\,\theta=\frac{7\pi}{8}\,..\,\frac{9\pi}{8}\right],\,coords=\mathrm{polar},\,scaling=constrained\right)$$

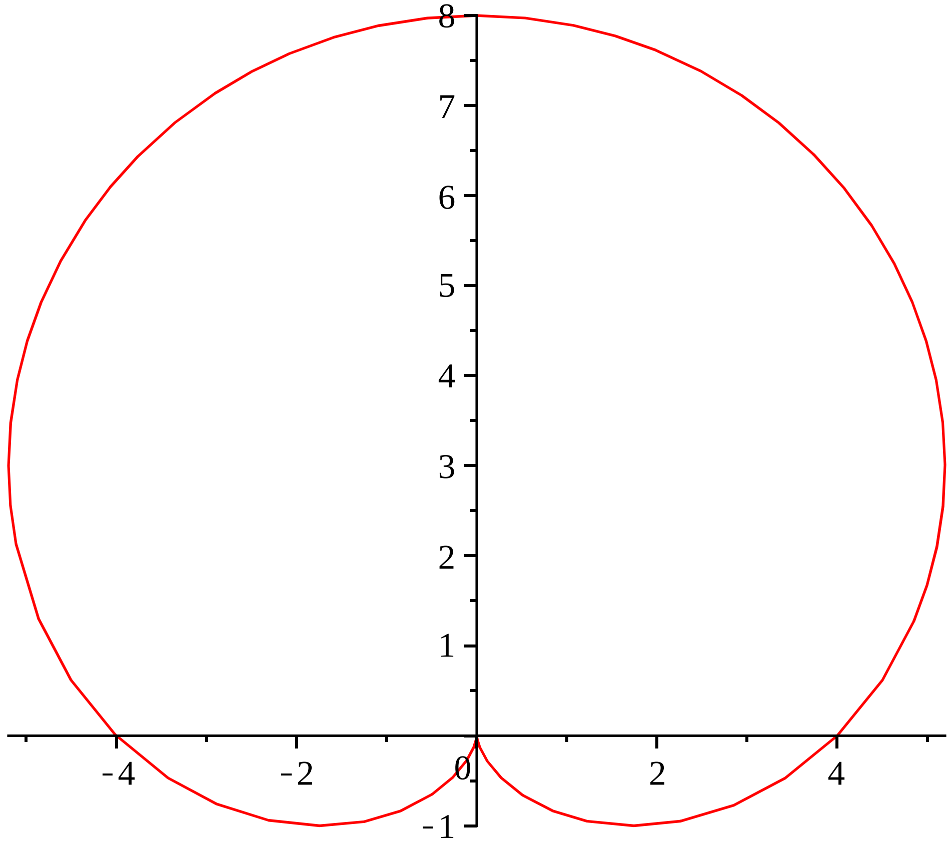


The graph is seen to approach the pole from above, coming in tangent to the *x*-axis and then do a 180 turn around and leave the pole turning downwards. A "kink" on a curve is usually referred to as a *corner*, but this is stronger than a mere corner in that there is a complete reversal of direction. Such a phenomenon is referred to as a *cusp*.

$r = a\,(1 + \sin(\theta))$

This is also a cardioid. It is oriented differently than the foregoing cardioid.

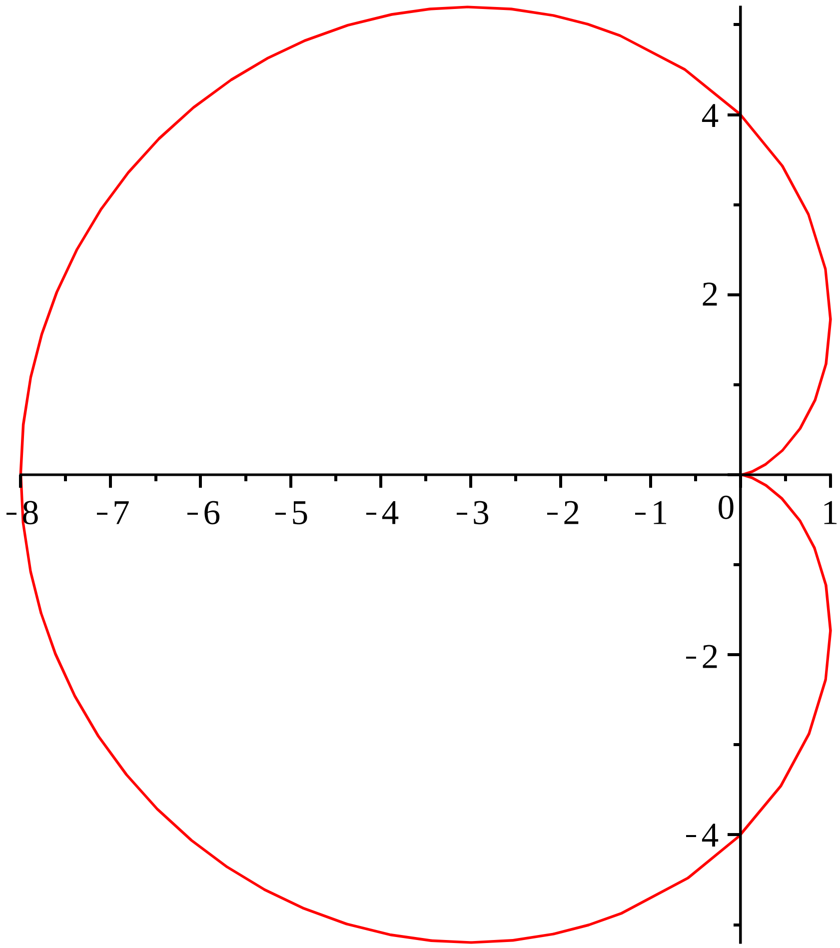
> $plot([4\,(1+\sin(\theta)), \theta, \theta = 0..2\,\pi], coords = \text{polar}, scaling = constrained)$



$r = a\ (1 - \cos(\theta))$

Ditto.

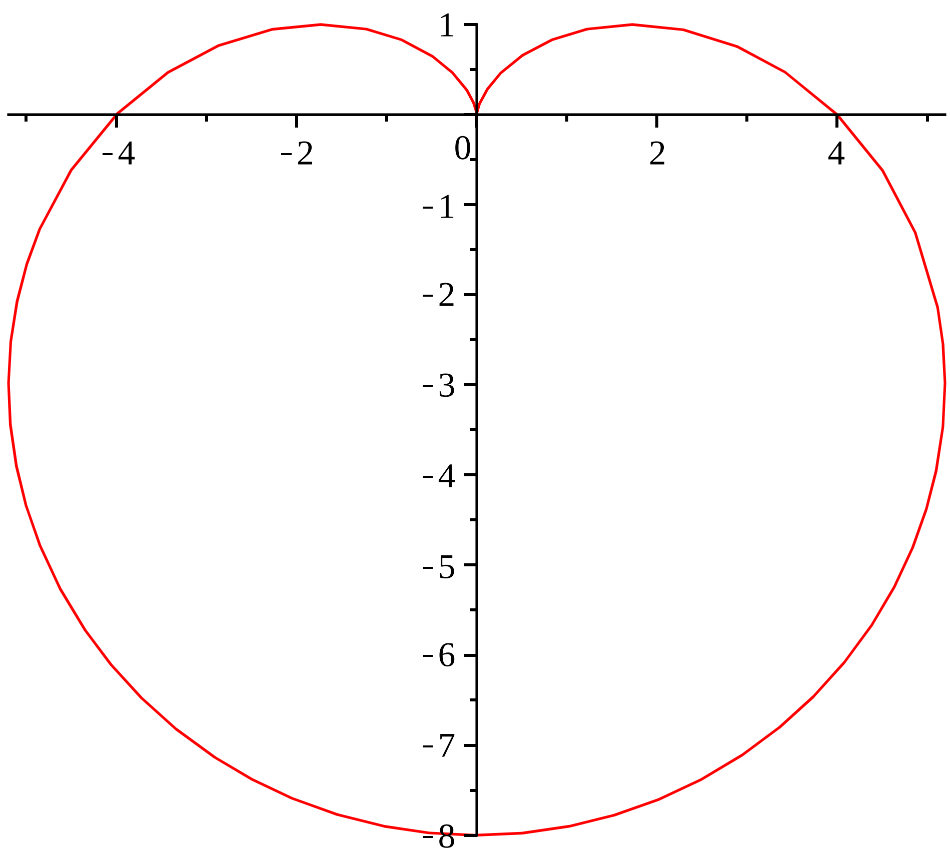
$> plot([4\ (1 - \cos(\theta)), \theta, \theta = 0\ ..2\ \pi], coords = \text{polar}, scaling = constrained)$



$r = a\ (1 - \sin(\theta))$

You guessed it!

> $plot([4\ (1 - \sin(\theta)), \theta, \theta = 0\,..2\,\pi], coords = \mathrm{polar}, scaling = constrained)$



Any of these four figures is readily sketched from the four points determined by the values $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ once the figure is recognized from its equation as a cardioid.

$r = f(\theta - \theta_0)$

In the foregoing examples we have seen similar equations which produce identical figures in differing orientations. In this section we explain the reason for this and open up possibilities for even more orientations.

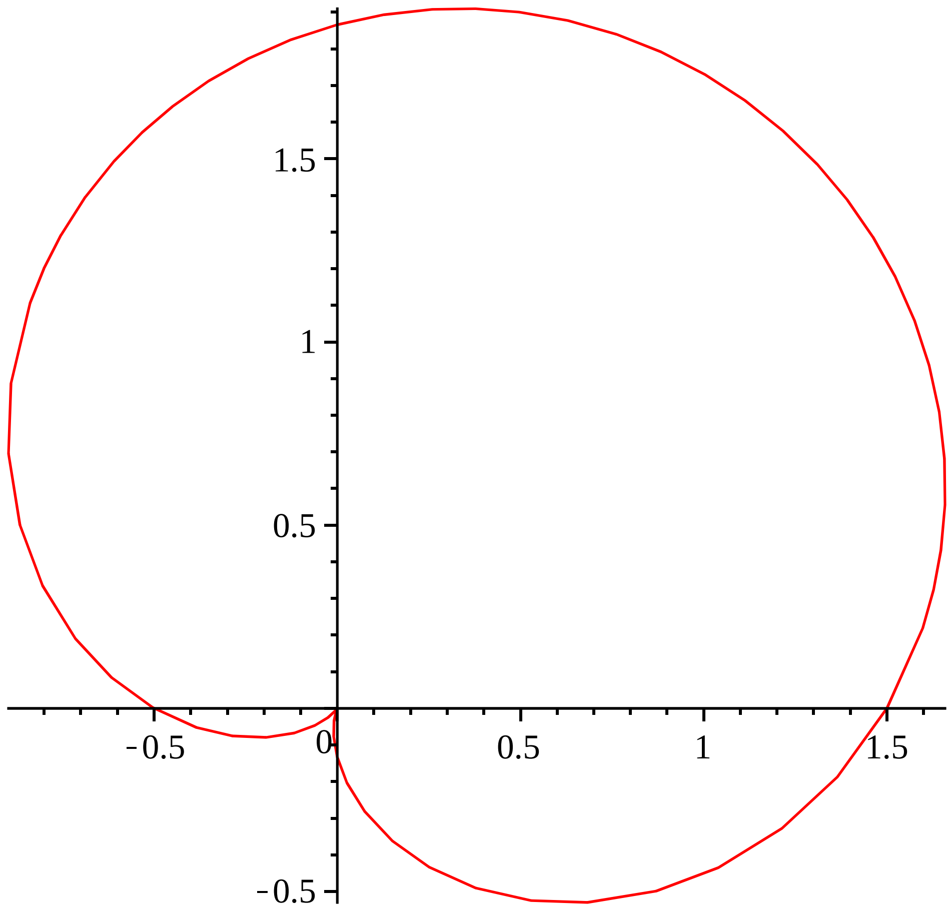
The student should have learned at some point that for a given function $f(x)$ and a given constant x_0, the Cartesian graphs of $y = f(x)$ and $y = f(x - x_0)$ are simply related by a horizontal shift, or *translation*. The reason for this is as follows. If the point (a, b) is on the graph of $y = f(x)$, then $b = f(a)$ which means that $b = f(a + x_0 - x_0)$, and this places the point $(a + x_0, b)$ on the graph of $y = f(x - x_0)$. The converse is clearly also true.

A very similar reasoning applies when comparing the graphs of $r = f(\theta)$ and $r = f(\theta - \theta_0)$. The point with polar coordinates (a, b) satisfies $r = f(\theta)$ exactly when the point with polar coordinates $(a, b + \theta)$ satisfies $r = f(\theta - \theta_0)$. But θ is an angle, not a distance so the result of increasing each value of θ by a constant amount θ_0 produces a *rotation*. In summary, **the graph of $r = f(\theta - \theta_0)$ is a rotation of the graph of $r = f(\theta)$ through the angle θ_0.**

The first equation of the cardioid presented above was $r = a\ (1 + \cos(\theta))$. To rotate the figure counter-clockwise through 90 we write $r = a\ \left(1 + \cos\left(\theta - \frac{\pi}{2}\right)\right)$. From the trig identity (you should know this one!) $\cos(\alpha + \beta) = \cos(\alpha)\ \cos(\beta) - \sin(\alpha)\ \sin(\beta)$ we have $\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)$ so that $r = a\ (1 + \sin(\theta))$. The other two cases of the cardioid can be obtained in similar fashion.

Let's have Maple draw a cardioid at a 60 angle with the *x*-axis.

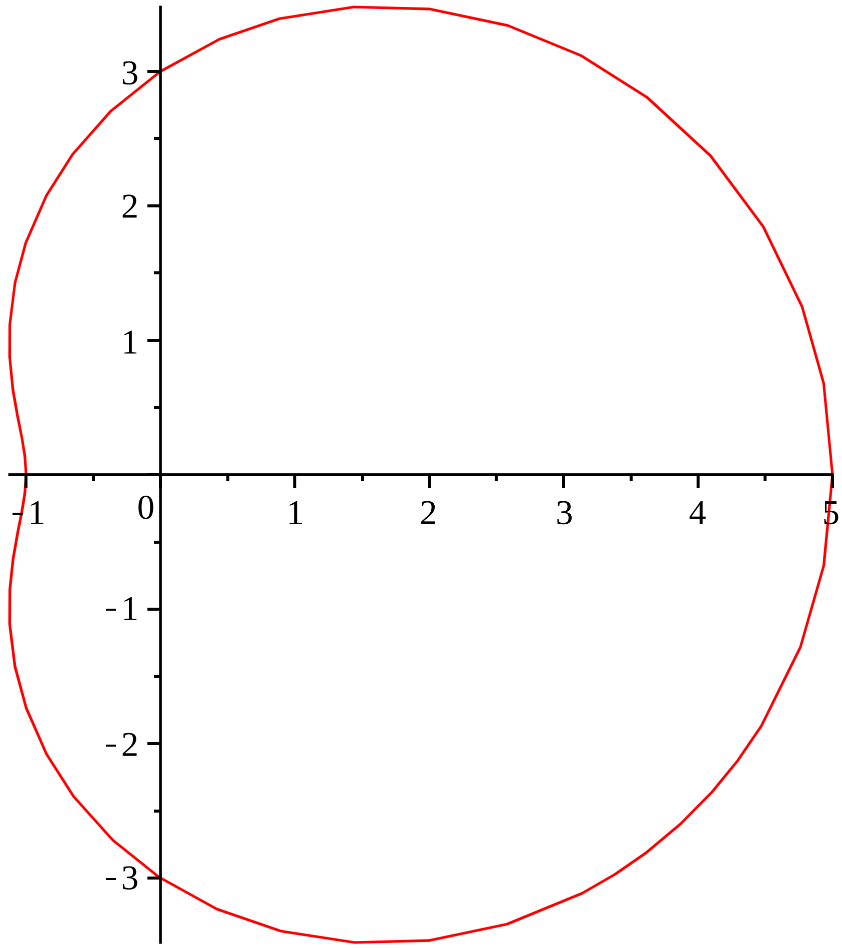
$> plot\left(\left[1 + \cos\left(\theta - \frac{\pi}{3}\right), \theta, \theta = 0 .. 2\,\pi\right], coords = \text{polar}, scaling = constrained\right)$



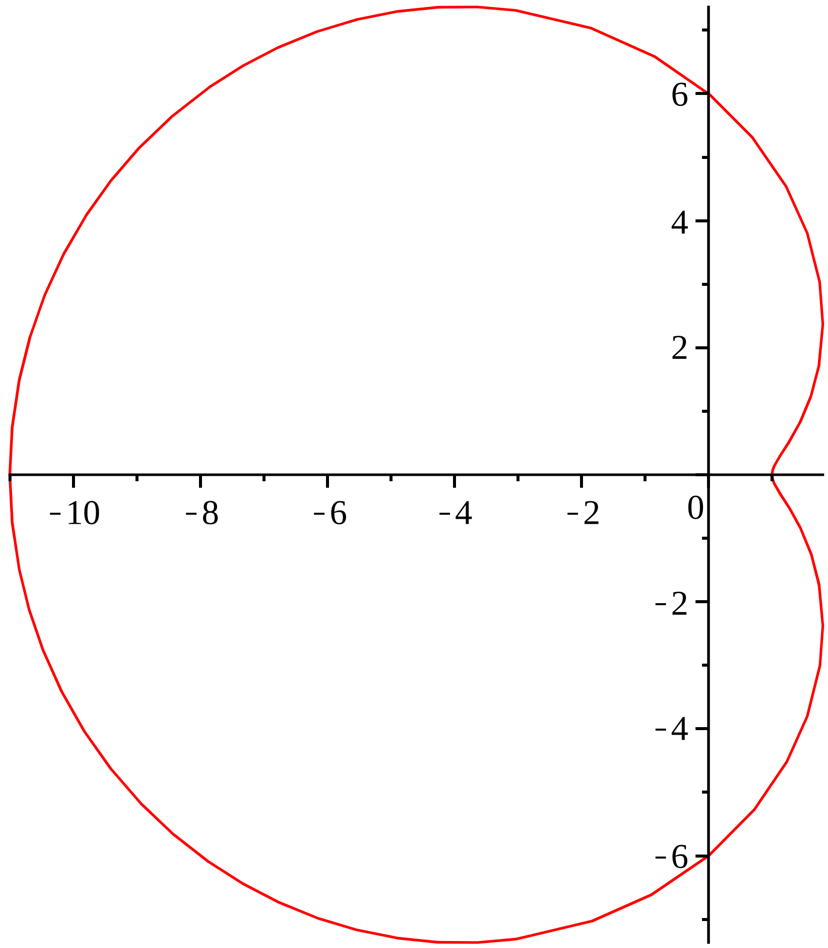
$r = a + b\cos(\theta)$

The graph of this equation is called a *limaon*. Limaons come in three varieties, depending on the relation of $|a|$ to $|b|$. The simplest case occurs when $|a| = |b|$. In that case either $a = b$ or $a = -b$ so the limaon is a cardioid. In case $|b| < |a|$ the value of r can never be 0 and either is always positive or else is always negative. The graphs looks like a cardioid that "changes it mind" just before reaching the pole. The following two Maple plots illustrate this.

$> plot([3 + 2\cos(\theta), \theta, \theta = 0 .. 2\,\pi], coords = \text{polar}, scaling = constrained)$

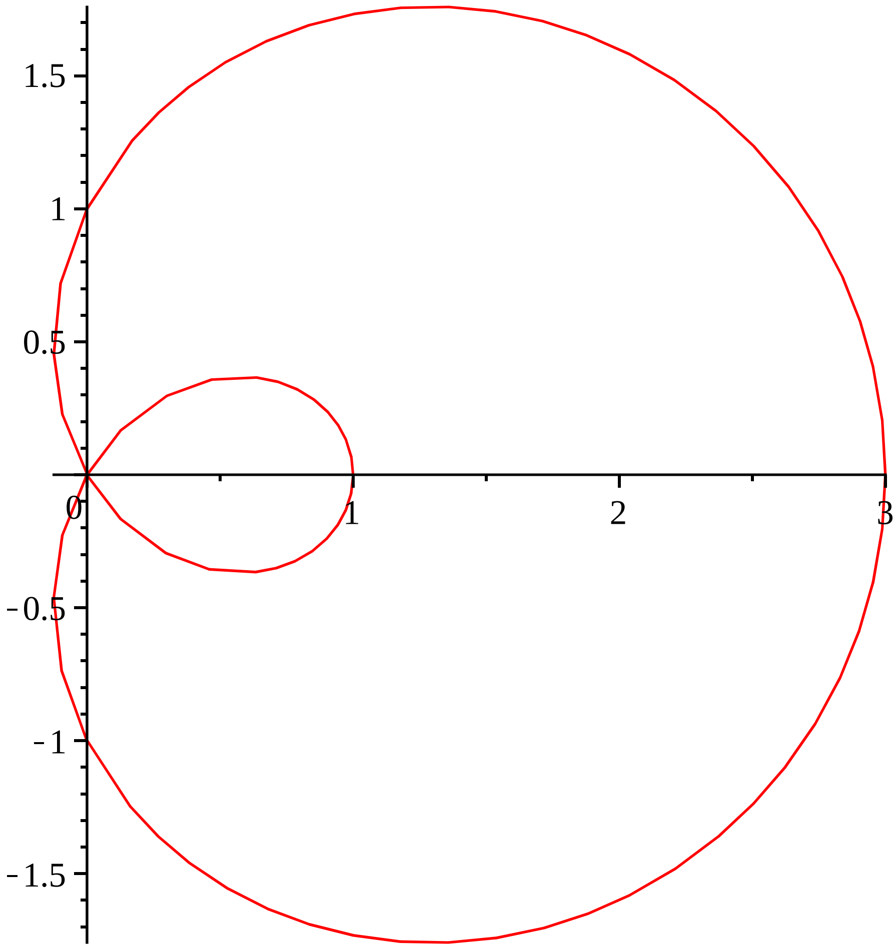


> $plot([-6 - 5\cos(\theta), \theta, \theta = 0..2\pi], coords = \mathrm{polar}, scaling = constrained)$



In case $|a| < |b|$ the value of r changes sign; that is, r is sometimes positive and sometimes negative. In this case the limaon has a *loop*. Consider, for example, the graph of $r = 1 + 2\cos(\theta)$.

> $plot([1+2\cos(\theta), \theta, \theta = 0..2\pi], coords = \mathrm{polar}, scaling = constrained)$



Notice that the graph enters and exits the loop at the pole. If we set $r = 0$ in the equation $r = 1 + 2\cos(\theta)$ we get $\cos(\theta) = -\frac{1}{2}$ which will occur for $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. For θ in the interval $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$ the value of r is negative. This causes the points along the loop to fall in the first and fourth quadrants, even though the associated values of θ fall in the second and third quadrants.

Given an equation of a limaon, the easiest way to determine which of the three types of limaons its graph will be is to consider the largest and smallest values of r. If these two values are either both positive or both negative, then the value of r is never zero and the limaon is the first type considered above. If either the maximum or minimum value of r is zero, then the limaon is in fact a cardioid. If the maximum value of r is positive and the minimum value of r is negative, then the limaon has a loop.

$r = a + b\sin(\theta)$

Based on the discussion above concerning rotations we see that the graph of $r = a + b\sin(\theta)$ is a rotation of the foregoing. Consider, for example, the graph of $r = 4 - 5\sin(\theta)$. The largest and smallest values of r in this equation are 9 and -1, corresponding to the extreme values -1 and +1 of θ, so this is a limaon with a loop.

```
> plot([4 - 5 sin(θ), θ, θ = 0 ..2 π], coords = polar, scaling = constrained)
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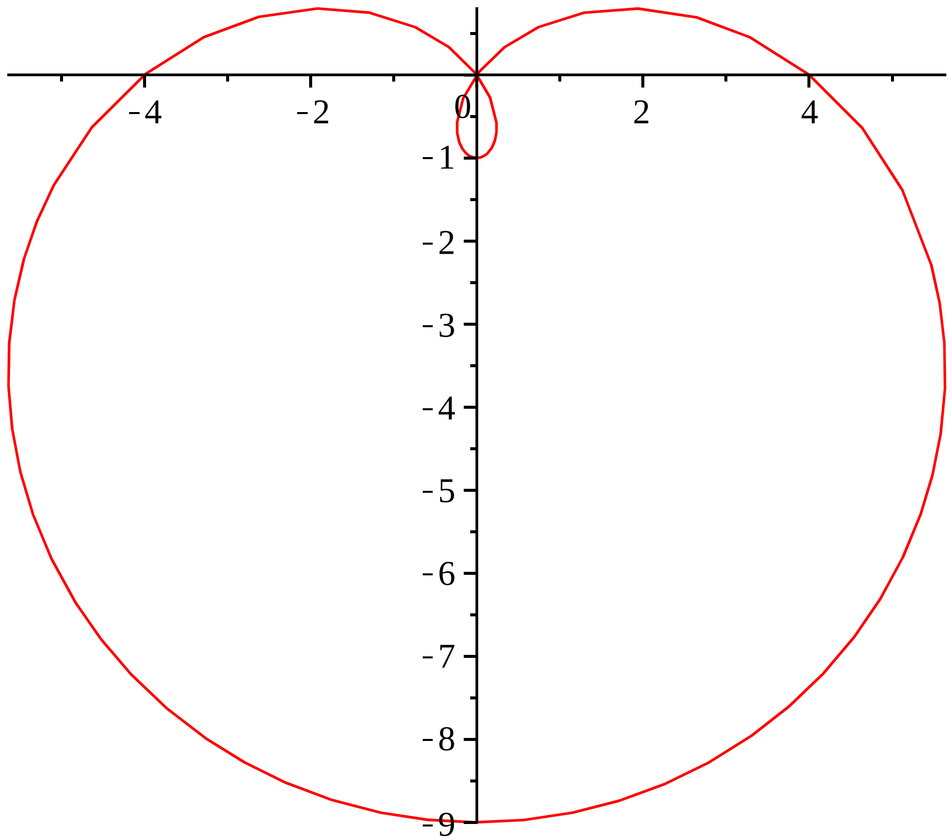


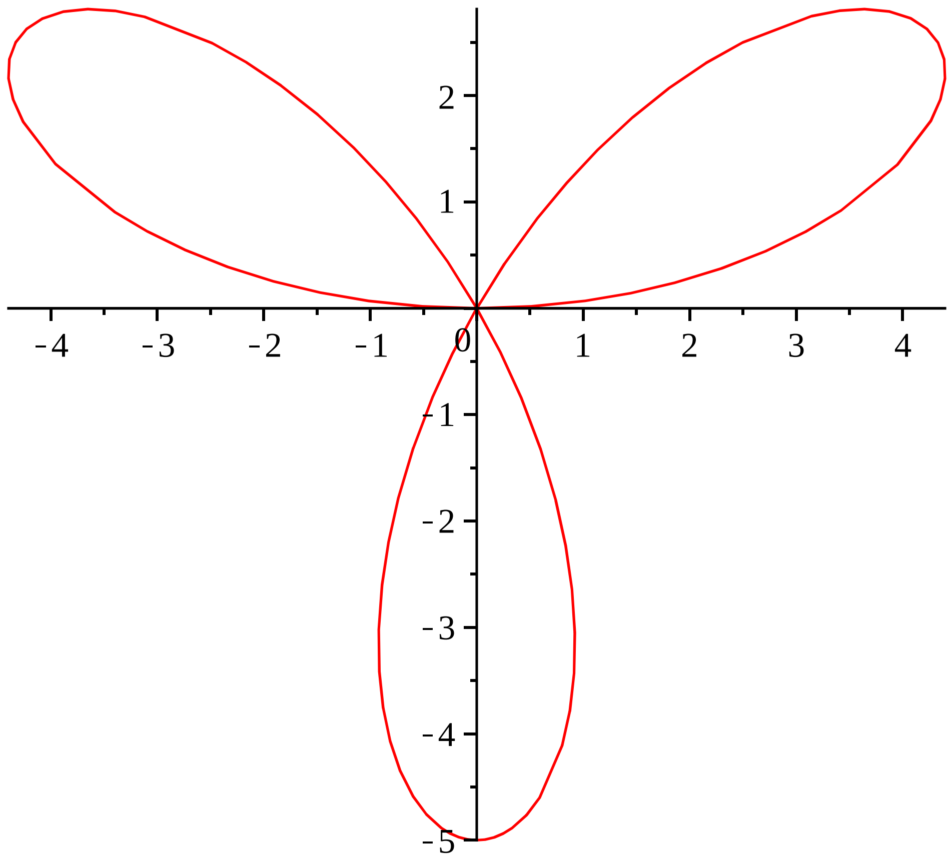
$r = a\cos(n\,\theta),\ n = 2, 3, 4$ **...**

$r = a\sin(n\,\theta),\ n = 2, 3, 4$ **...**

We will discuss these two curves together, since they are realted by a rotation. In fact, we will describe first how to sketch the second equation, then modify the procedure slightly to sketch the first. Our discussion will be easier to follow if we assign specific values to a **and** n**.**

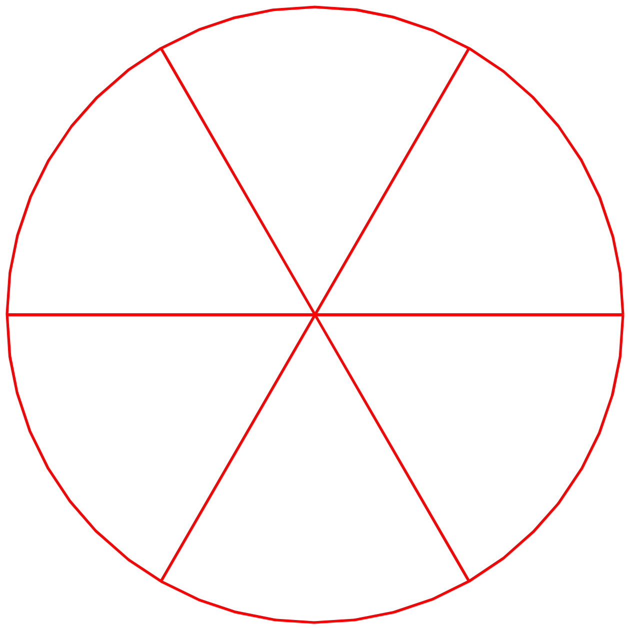
The value of n should be an integer. The case of $n = 1$ has been discussed above and is a circle, so that case is not included here. Suppose we take $a = 5$ and $n = 3$. The resulting figure is called a *3-leaved rose*, and is sketched below.

> $plot([5\sin(3\,\theta), \theta, \theta = 0\,..\pi], coords = \mathrm{polar}, scaling = constrained)$



From the equation $r = 5\sin(3\,\theta)$ we see that that the tips of the petals will occur where the factor $\sin(3\,\theta)$ is 1 or -1. Thus the rose is "enveloped" by the circle $r = 5$. A suggestion for sketching the graph of $r = a\sin(n\,\theta)$ is to begin by sketching the cirle $r = a$, even though only certain of the points on the circle will be on the final graph. Notice next that each single petal is generated in the graph as the factor $\sin(3\,\theta)$ varies either from 0 up to 1 and back down to 0, or else from 0 down to -1 and back up to 0. In order for either of these to happen, the angle $3\,\theta$ must increase by a total of π radians (*i.e.* 180). For this to occur, θ itself must increase by an amount $\frac{\pi}{3}$ (*i.e.* 60).

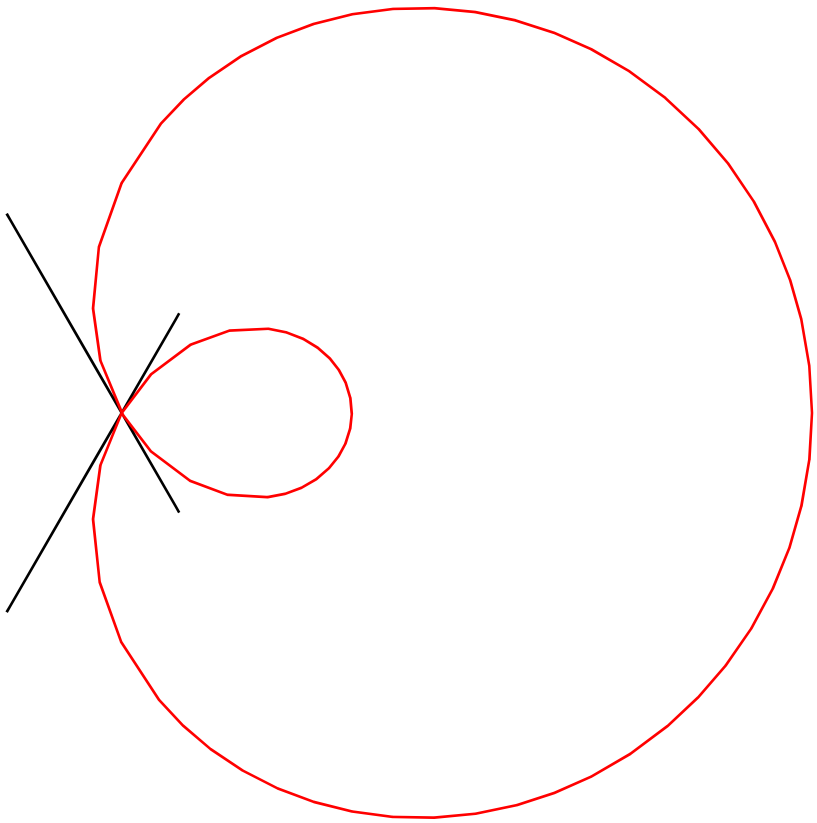
To help in sketching the equation $r = a\sin(n\,\theta)$, we define the quantity $\frac{\pi}{n}$ to be the *sector size* associated with the graph of the equation, and starting out at $\theta = 0$ we divide the circle $r = a$ into $2\,n$ sectors, like slicing a pie. In the case of the graph above, our sketch would look like this:



Each petal of the 3-leaved rose we are discussing will occupy one of these six sectors. As an additional aid in sketching the petals, we present the following useful fact concerning polar graphs.

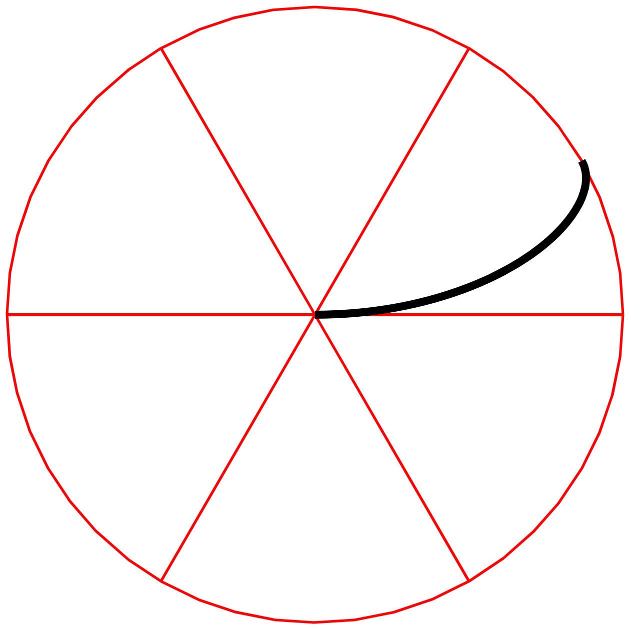
Suppose the polar graph of an equation $r = f(\theta)$ passes through the pole for $\theta = \theta_0$ (*i.e.* suppose $f(\theta_0) = 0$). Then the line $\theta = \theta_0$ is tangent to the graph at the pole. (We are assuming that the function $f(\theta)$ is differentiable at θ_0.)

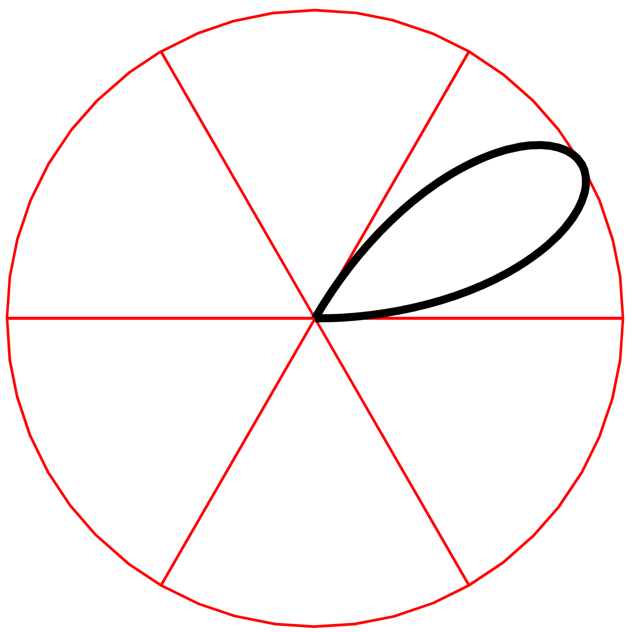
As an example of this consider the limaon $r = 1 + 2\cos(\theta)$ graphed above. We saw there that $r = 0$ for $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. Suppose we redraw the limaon and add the lines $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

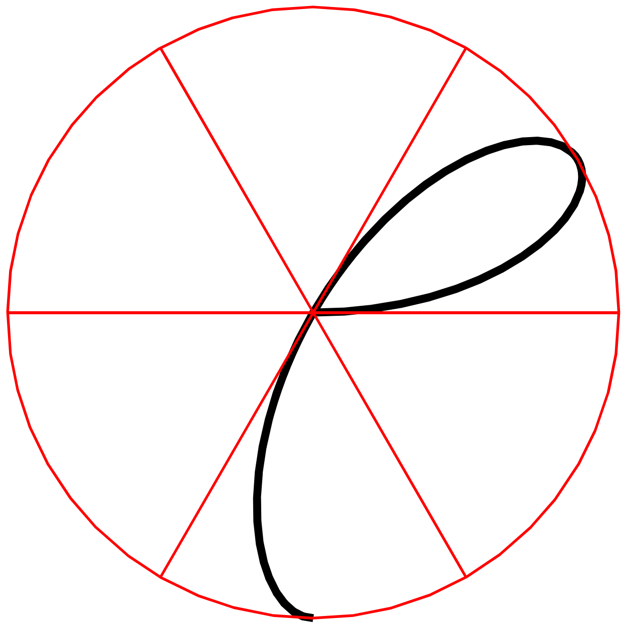


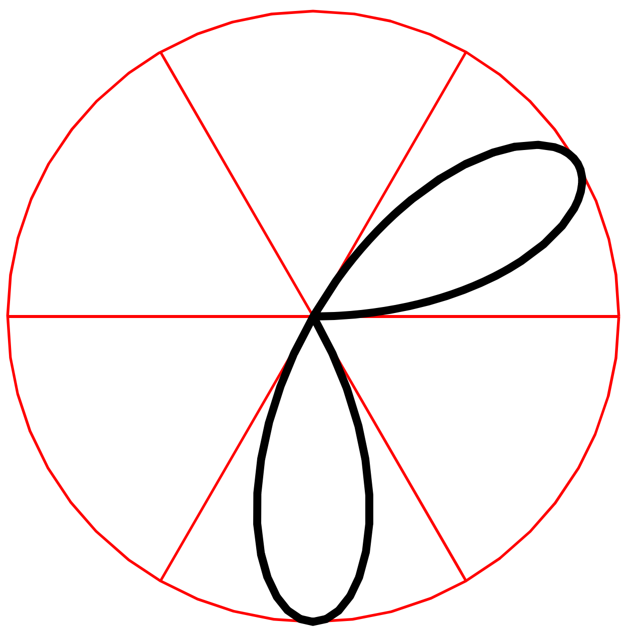
Now let's tie this observation in with our study of the 3-leaved rose. The crucial observation is that the tangent lines at the pole for the 3-leaved rose will coincide with sector lines that we have already drawn, since our sector lines were determined by setting the factor $\sin(3\,\theta)$ equal to zero.

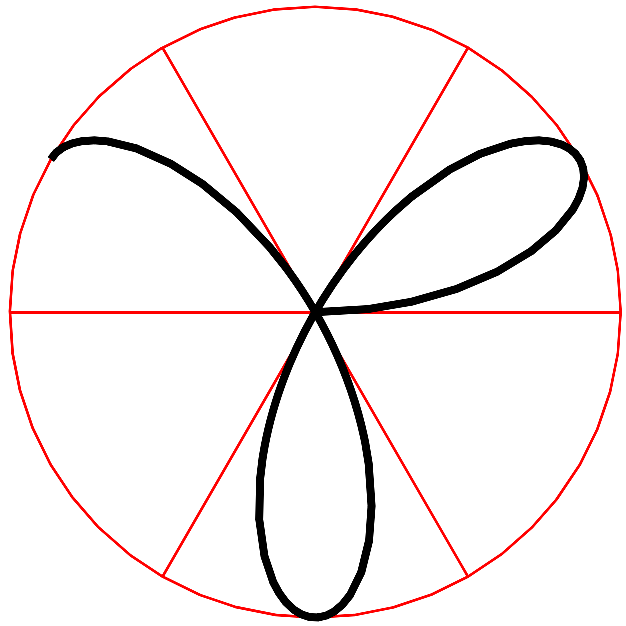
So here is a system for sketching the 3-leaved rose. Begin at the pole with $\theta = 0$. With increasing values of θ, enter the first sector tangent to the sector line $\theta = 0$. Gradually turn and approach the enveloping circle $r = 5$ tangentially at the middle of the sector and then turn and reenter the pole tangent to the sector line $\theta = \frac{\pi}{3}$. Now exit the pole entering the next sector with negative values of r and proceed in similar fashion tangential to the enveloping circle at the middle of the sector. Then turn around and reenter the pole tangent to the next sector line, and so on, and so on. By following the sector lines through the pole we are led to describe the three petals of the 3-leaved rose. The following plots illustration the steps described above:

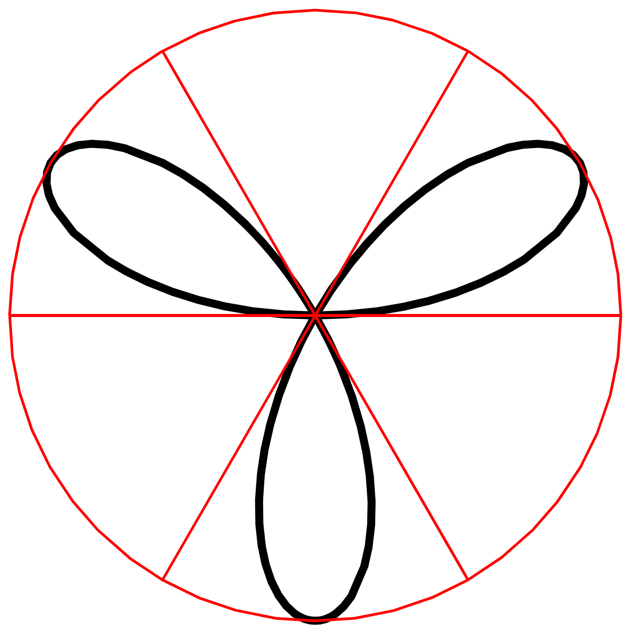






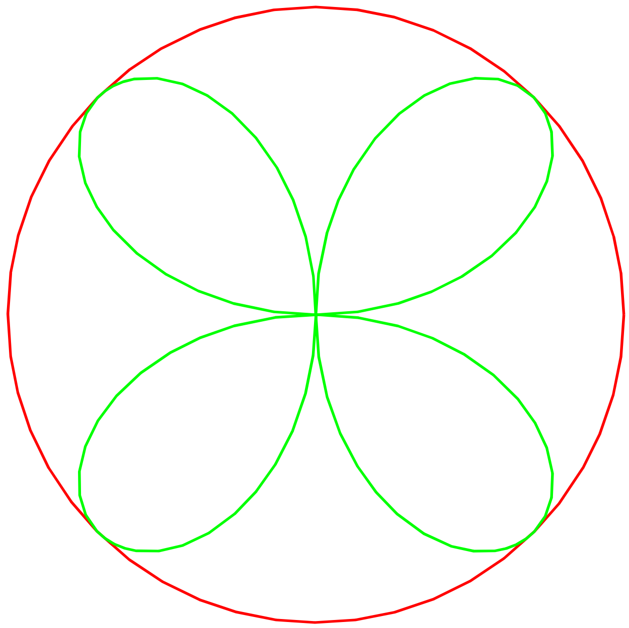






If the reader will trace out the above plots with increasing θ he will discover that the entire 3-leaved rose is generated as θ varies from 0 to π. If θ is allowed to increase from π to 2 π the rose is traced out a second time. This sort of thing always happens whenever the integer n is odd. Half of the 2 n sectors (each of size $\frac{\pi}{n}$) will be empty, and the n petals will be traced out as θ increases from 0 to π. However, when n is even, the petals will be traced out in <u>all</u> sectors. Once again n petals will be traced out as θ increases from 0 to π, but as θ increases from π to 2 π an additional n petals are traced out in the remaining n sectors, rather than a retracing of the first n petals. The neat part about the method outlined above (*i.e.* following the sector lines through the pole) is that the generation of 2 n petals for even values of n is automatic. Try it for the example below ($r = \sin(2\,\theta)$).

```
> plot( { [1, θ, θ = 0 ..2 π], [sin(2 θ), θ, θ = 0 ..2 π]}, coords = polar, scaling = constrained,
    axes = none)
```



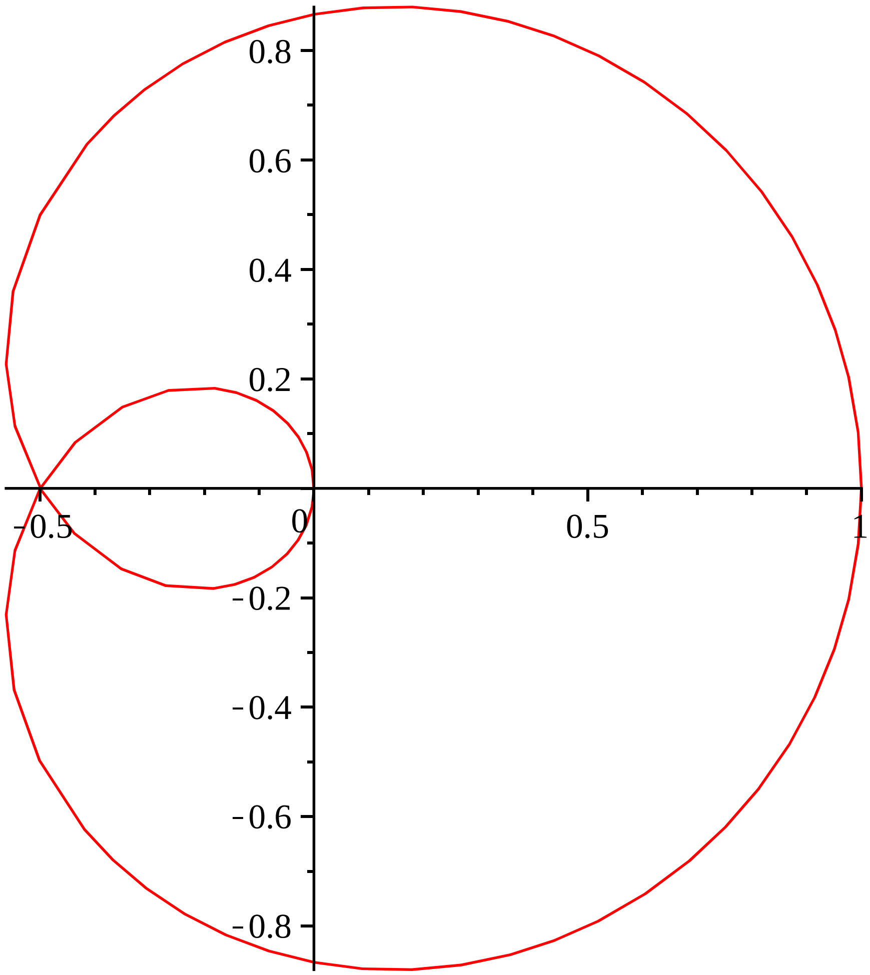
The graph of $r = a\cos(n\,\theta)$ is a rotation of the graph of $r = a\sin(n\,\theta)$ through half of a sector (through an angle of $\frac{\pi}{2\,n}$). This is intuitively clear since $\cos(0) = 1$, placing the graph at the tip of a petal when $\theta = 0$. It can also be seen from the discussion above on rotations and the fact that $\sin\left(n\left(\theta + \frac{\pi}{2\,n}\right)\right) = \cos(n\,\theta)$.

In summary, the graphs of $r = a\cos(n\,\theta)$ and $r = a\sin(n\,\theta)$ are *n*-leaved roses when n is an odd integer greater than 1 and are *2n*-leaved roses when n is an even integer.

Have Fun

What about polar curves that aren't standard? What if *n* isn't an integer in the "rose" equations?

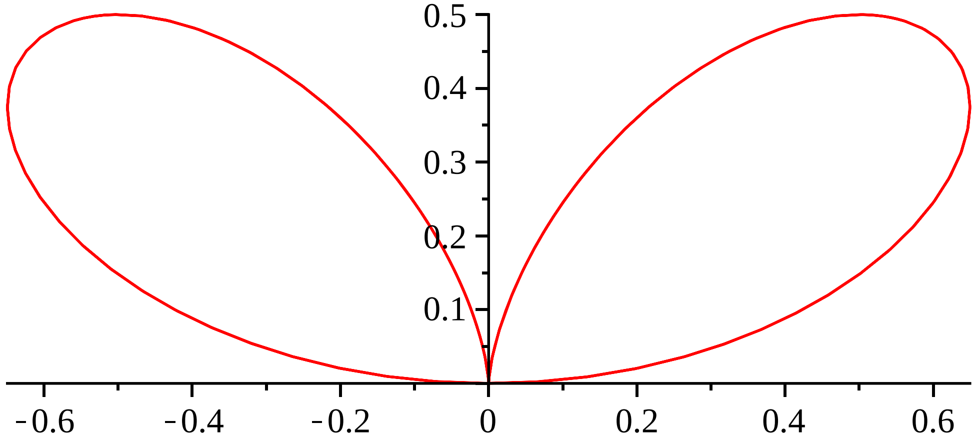
$> plot\left(\left[\cos\left(\frac{\theta}{3}\right), \theta, \theta = 0\,..3\,\pi\right], coords = \text{polar}, scaling = constrained\right)$



Wow! This looks like a limaon with a loop, except that the loop isn't at the origin. I wonder if it really is?

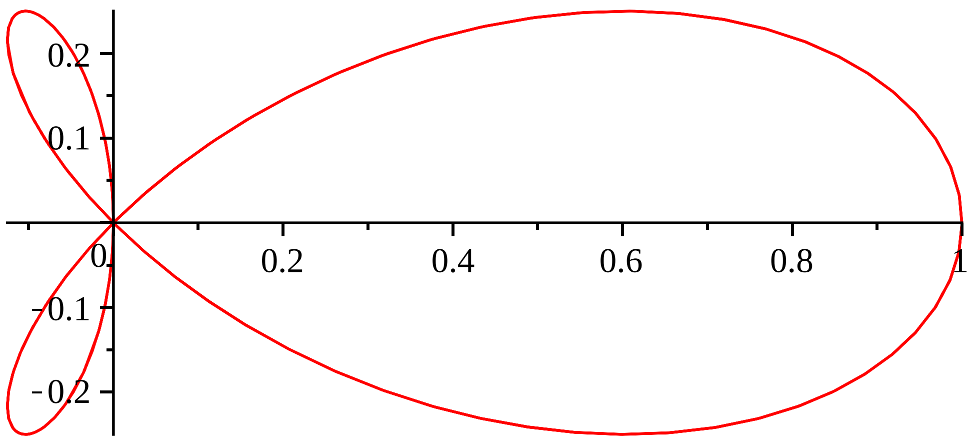
How about something crazy like $r = \cos(\theta)\ \sin(2\,\theta)$?

> *plot*$([\cos(\theta)\ \sin(2\,\theta), \theta, \theta = 0\,..2\,\pi]$, *coords* = polar, *scaling* = *constrained*)



Looks like a butterfly! Let's change the $\sin(2\,\theta)$ term to $\cos(2\,\theta)$ and see what happens.

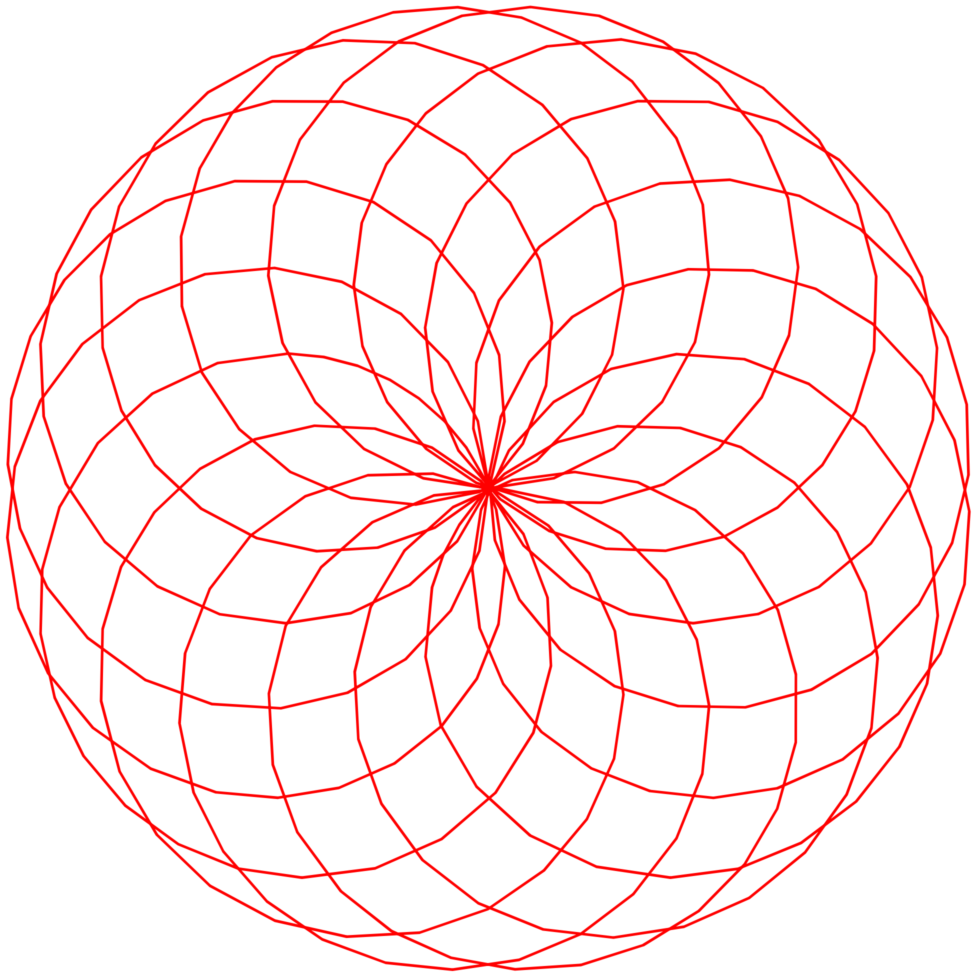
$> plot([\cos(\theta)\ \cos(2\,\theta), \theta, \theta = 0..2\,\pi], coords = \mathrm{polar}, scaling = constrained)$



Looks like a fish!

Check this one out.

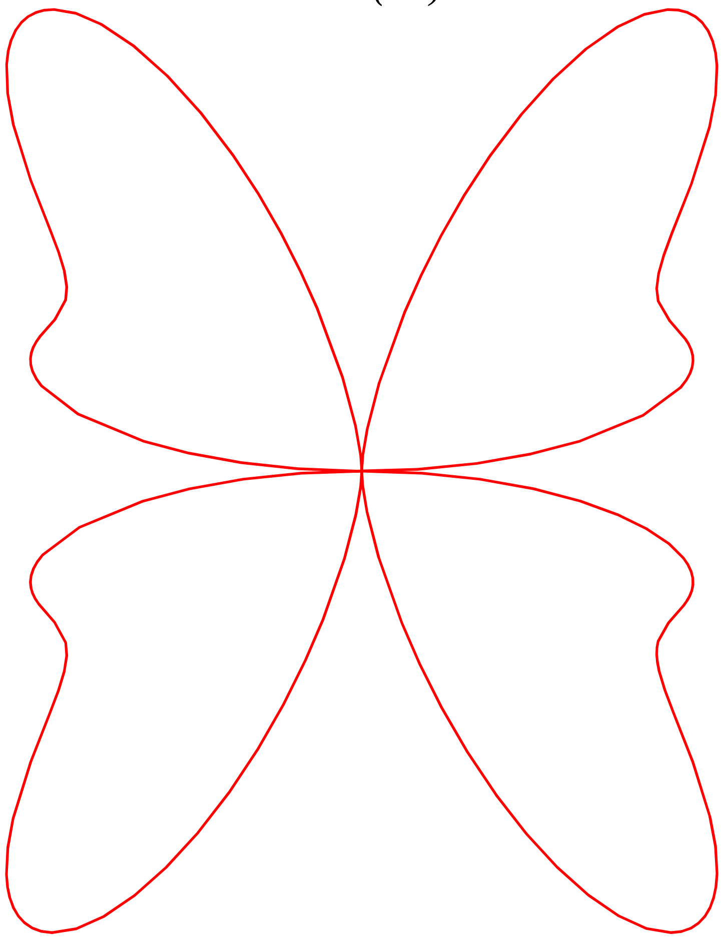
$> plot\left(\left[\sin\left(\frac{8\,\theta}{9}\right), \theta, \theta = 0\,..18\,\pi\right], coords = \mathrm{polar}, scaling = constrained, axes = none\right)$



Here are some pretty plots that we get by using an alternate command for plotting in polar coordinates. You can find an explanation with the Maple help engine.

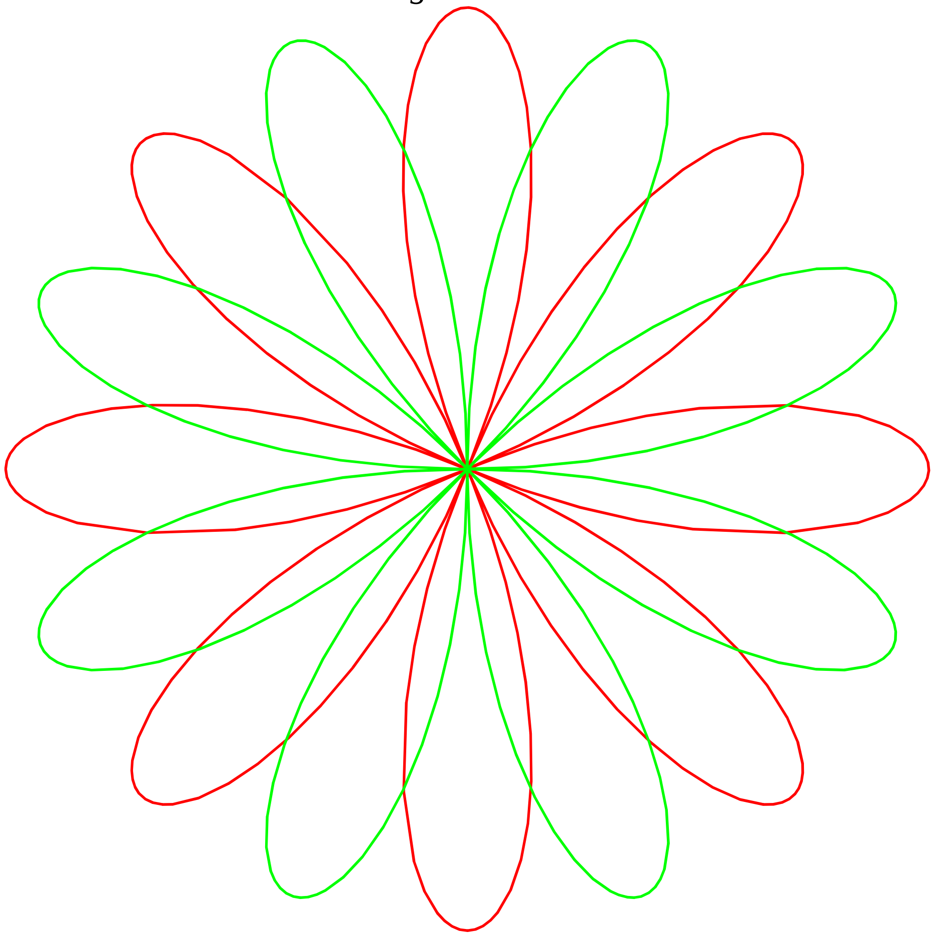
```
> with(plots):
  polarplot(sin(2*t)+0.2*sin(8*t),scaling=CONSTRAINED,axes=NONE,
  title=`r = sin2t+(1/5)sin8t`);
```

$r = sin2t + (1/5)sin8t$



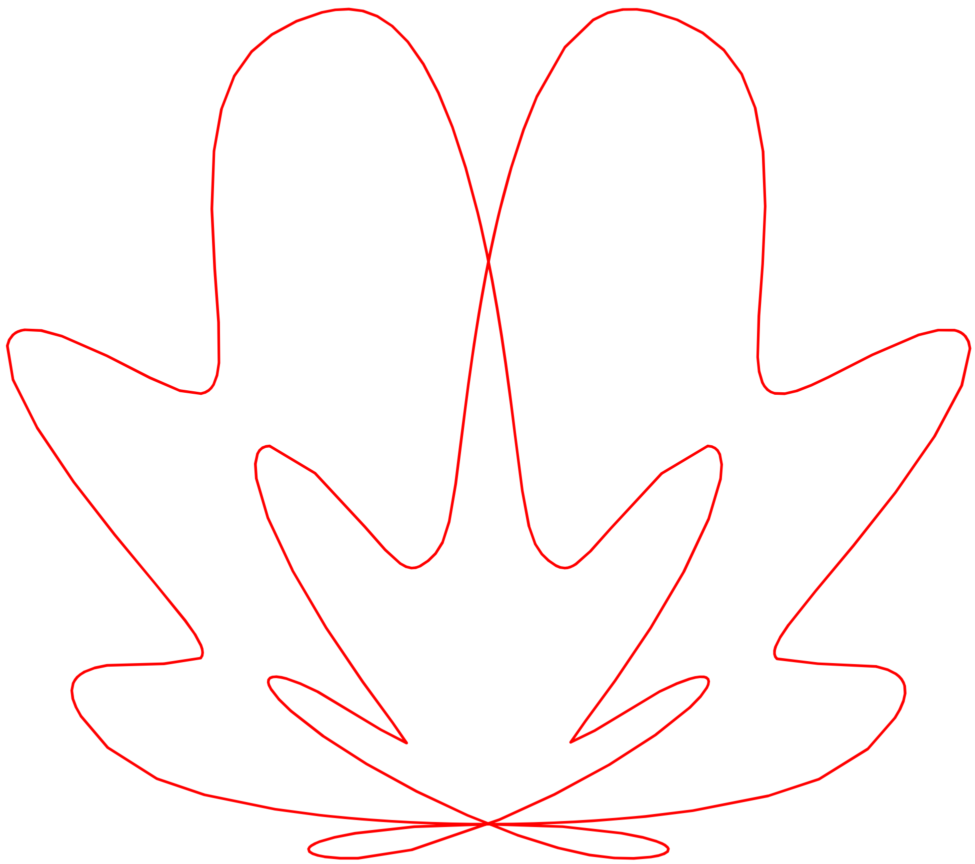
```
> plot({[cos(4*t),t,t=-Pi..Pi],[sin(4*t),t,t=-Pi..Pi]},coords=
  polar,axes=NONE,scaling=CONSTRAINED,title=`r = cos4t together
  with r = sin4t`);
```

r = cos4t together with r = sin4t



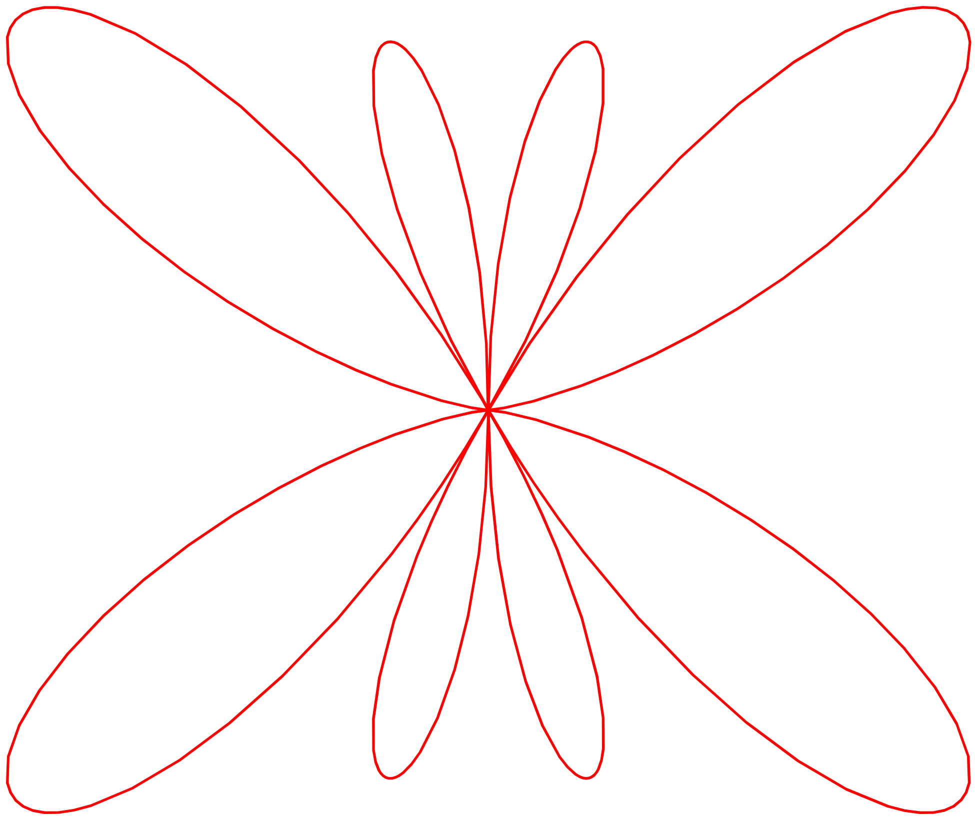
```
> polarplot(2*sin(t)+sin(2*t+sin(8*t)),scaling=CONSTRAINED,axes=
  NONE,title=`r = 2sint + sin(2t+sin8t)`);
```

$r = 2sint + sin(2t + sin8t)$



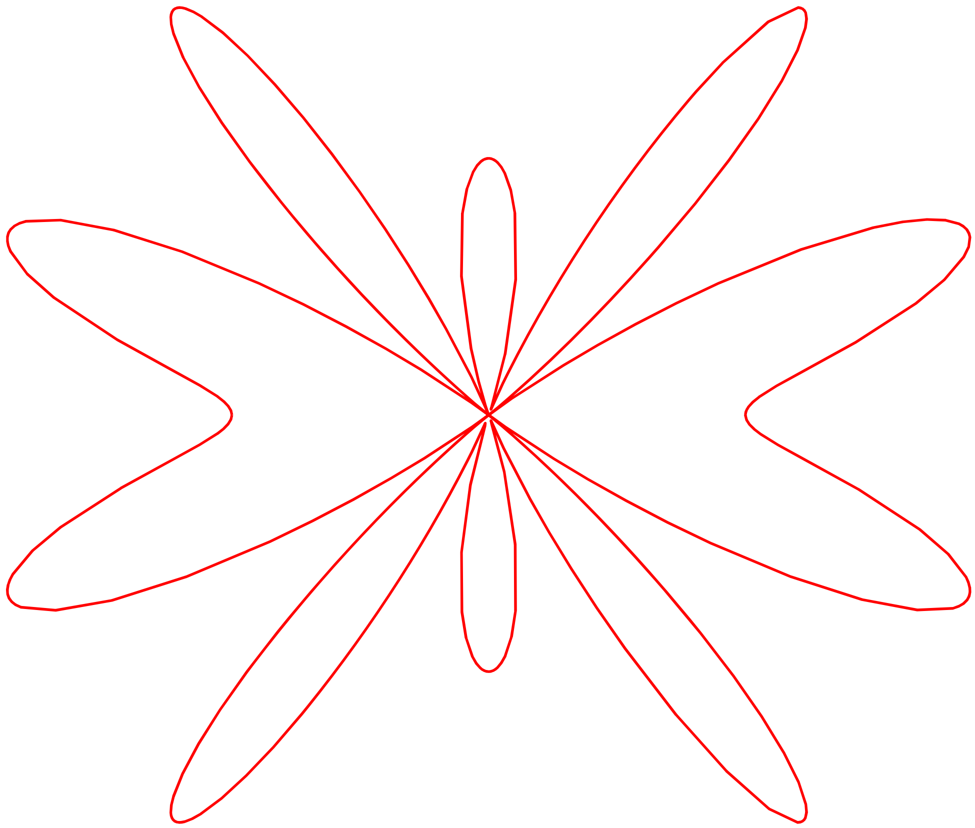
```
> polarplot(sin(t)*sin(2*t)*sin(3*t),t=-Pi..Pi,scaling=
  CONSTRAINED,axes=NONE,title=`r = sint sin2t sin3t`);
```

r = sint sin2t sin3t



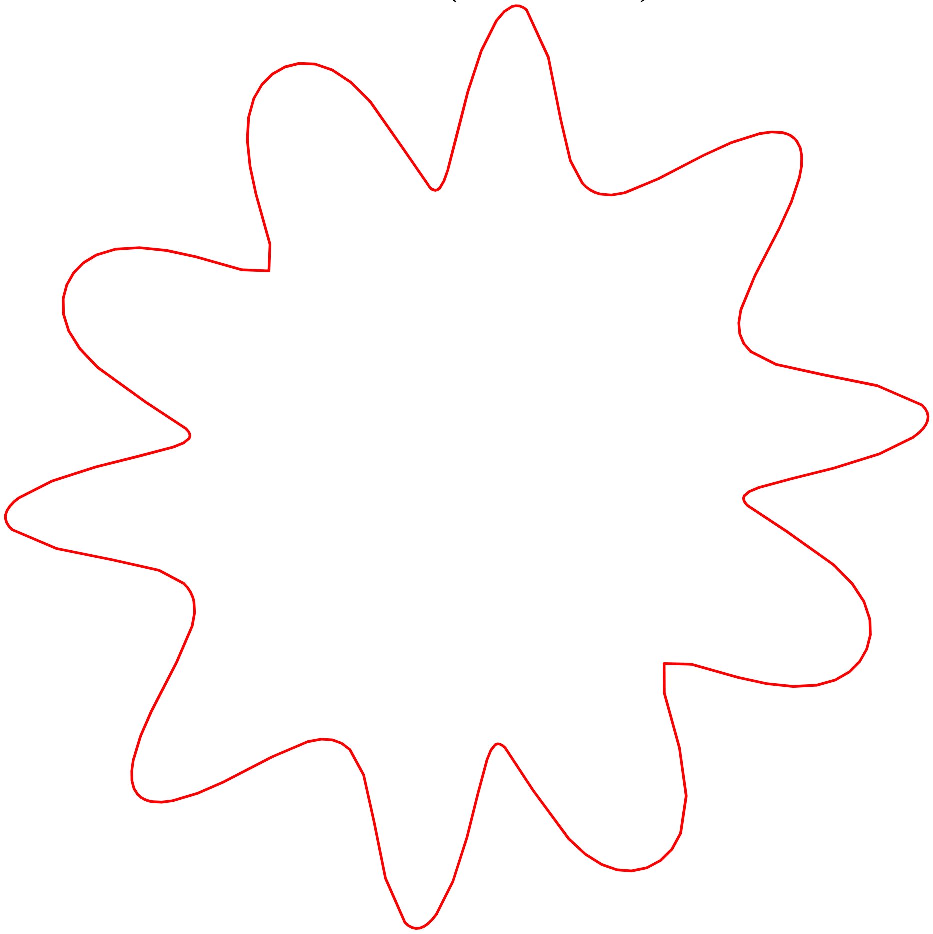
```
> polarplot(1+sin(2*Pi*sin(2*t-Pi/2)),scaling=CONSTRAINED,axes=
  NONE,title=`r = 1 + sin(2Pi sin(2t - Pi/2))`);
```

$r = 1 + sin(2Pi\ sin(2t - Pi/2))$

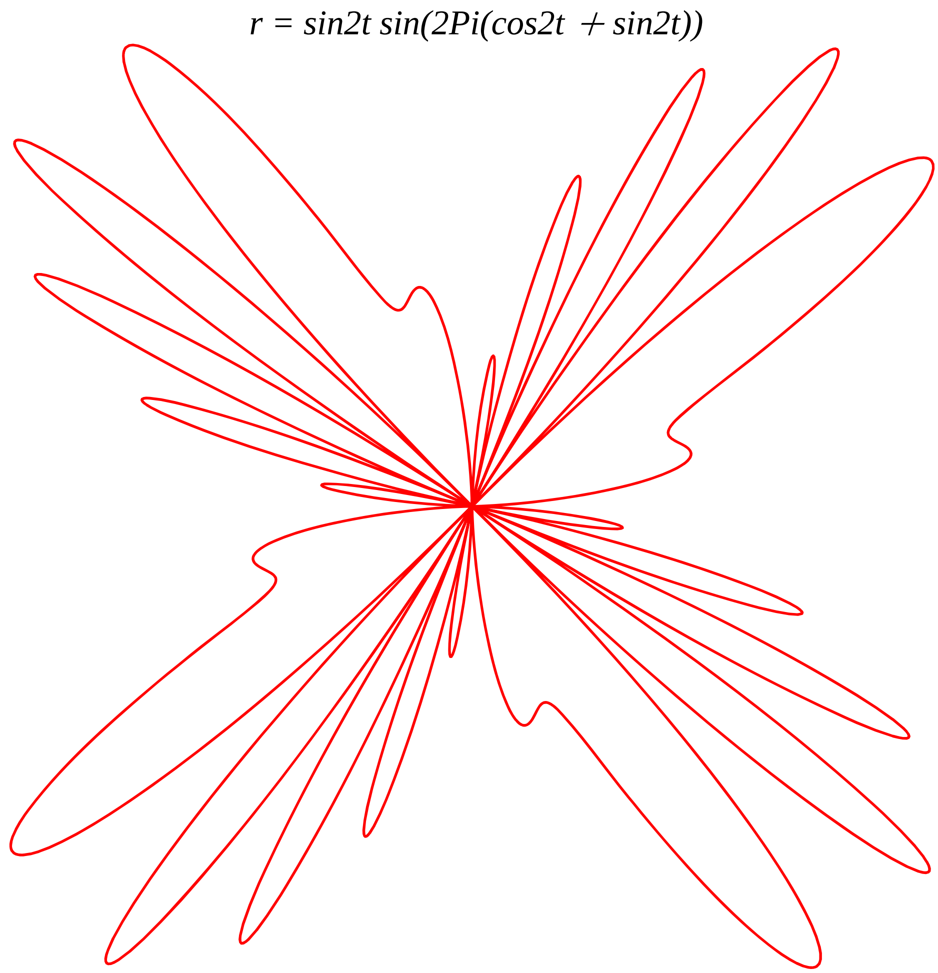


```
> polarplot(4+sin(2*t+Pi*sin(4*t)),scaling=CONSTRAINED,axes=NONE,
  title=`r = 4 + sin(2t + Pi sin4t)`);
```

$r = 4 + sin(2t + Pi\ sin4t)$



```
> polarplot(sin(2*t)*sin(2*Pi*(cos(2*t)+sin(2*t))),scaling=
  CONSTRAINED,axes=NONE,numpoints=1000,title=`r = sin2t sin(2Pi
  (cos2t + sin2t))`);
```



The beauty of the next pattern is partially due to roundoff errors for large angles.

> $plot\left(\left[0.2+\cos(20\,t),\ t,\ t=0\,..40\,\pi\right],\ coords=\mathrm{polar},\ scaling=constrained,\ axes=none\right)$