Calculating $\zeta(2)$—again

We start by calculating $\displaystyle\int_0^1\int_0^1 \frac{dx\,dy}{1-xy}$ —an improper integral, but we shall ignore that for now (exercise: check the appropriate limit to show this does converge).

First we use infinite series:

$$\begin{aligned}(1-xy)^{-1} &= 1+xy+\frac{(-1)(-2)}{2!}(-xy)^2+\frac{(-1)(-2)(-3)}{3!}(-xy)^3+\cdots\\ &= 1+xy+x^2y^2+x^3y^3+\cdots\end{aligned}$$

So

$$\begin{aligned}\int_0^1 \frac{dx\,dy}{1-xy} &= x+\frac{1}{2}x^2y+\frac{1}{3}x^3y^3+\frac{1}{4}x^4y^3+\cdots\Big]_0^1\\ &= 1+\frac{1}{2}y+\frac{1}{3}y^2+\frac{1}{4}y^3+\cdots\end{aligned}$$

So

$$\begin{aligned}\int_0^1\int_0^1 \frac{dx}{1-xy}\,dy &= y+\frac{1}{2^2}y^2+\frac{1}{3^2}y^3+\frac{1}{4^2}y^4+\cdots\Big]_0^1\\ &= 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\end{aligned}$$

In other words,

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-xy} = \sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2)$$

We shall now evaluate the double integral another way, ending up with an actual value therefore for $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Effectively we shall rotate the unit square (and double its area) with the transformation $x = u - v$, $y = u + v$. First, as an exercise, you should show that this transformation takes the square $\mathcal{R}$: $[0,1]\times[0,1]$ to the diamond $\mathcal{S}$ given by these four lines: $v=-u$, $v=u$, $v=u-1$, $v=1-u$. Furthermore, the Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = 2$.

So we have the following calculation (there are hints below, so you can fill in the details for yourself).

$$\begin{aligned}\iint_{\mathcal{R}} \frac{1}{1-xy}\,dx\,dy &= 2\iint_{\mathcal{S}} \frac{1}{1-(u^2-v^2)}\,du\,dv\\ &= 2\int_0^{1/2}\int_{-u}^{u}\frac{1}{1-u^2+v^2}\,dv\,du + 2\int_{1/2}^{1}\int_{u-1}^{1-u}\frac{1}{1-u^2+v^2}\,dv\,du\\ &= 2\arcsin^2(\tfrac{1}{2}) - 2\arcsin^2(0) + \pi(\arcsin(1)-\arcsin(\tfrac{1}{2}))\\ &\qquad -(\arcsin^2(1)-\arcsin^2(\tfrac{1}{2}))\\ &= \frac{\pi^2}{18} - 0 + \frac{\pi^2}{2} - \frac{\pi^2}{6} - \frac{\pi^2}{4} + \frac{\pi^2}{36}\\ &= \frac{\pi^2}{6}\end{aligned}$$

Here are the relevant hints:

- $$\int \frac{dv}{a^2+v^2} = \frac{1}{a}\arctan\left(\frac{v}{a}\right)$$
  So $\displaystyle\int \frac{1}{1-u^2+v^2}\,dv = \frac{1}{\sqrt{1-u^2}}\arctan\left(\frac{v}{\sqrt{1-u^2}}\right)$.

- $$\arctan\left(\frac{u}{\sqrt{1-u^2}}\right) = \arcsin(u)$$
  So $\displaystyle\int \frac{1}{\sqrt{1-u^2}}\arctan\left(\frac{u}{\sqrt{1-u^2}}\right)du = \int \frac{\arcsin(u)}{\sqrt{1-u^2}}\,du = \frac{1}{2}(\arcsin(u))^2$.

- $$\arctan\left(\frac{1-u}{\sqrt{1-u^2}}\right) = \frac{1}{2}\arccos(u) = \frac{\pi}{4} - \frac{1}{2}\arcsin(u)$$
  So $\displaystyle\int \frac{1}{\sqrt{1-u^2}}\arctan\left(\frac{1-u}{\sqrt{1-u^2}}\right)du = \frac{\pi}{4}\int \frac{du}{\sqrt{1-u^2}} - \frac{1}{2}\int \frac{\arcsin(u)}{\sqrt{1-u^2}}\,du = \frac{\pi}{4}\arcsin(u) - \frac{1}{4}(\arcsin(u))^2$.

- And finally $\arcsin(x) = -\arcsin(-x)$ and $\arctan(x) = -\arctan(-x)$ (so we can "double up" the integrals of the form $\int_{-\alpha}^{\alpha}$ to get $2\int_0^{\alpha}$).