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Test 3
(version A)

Calculus III (Maths 201–DDB)

Justify all your answers — just having the correct answer is not sufficient.
Pace yourself — a rough guide is to spend not more than 2m minutes or so on a question worth m marks.

(8) 1. Suppose $F(x, y, z) = x^2 y + y^2 z + \cos(xz)$.

(a) Find the directional derivative of F at the point $P_0(0, 2, 1)$ in the direction from P_0 to the origin.

(b) For the level surface (contour surface) $F(x, y, z) = 5$ find the equation of the tangent plane at P_0.

(c) On the level surface $F(x, y, z) = 5$ find $\dfrac{\partial z}{\partial y}$.

(6) 2. Suppose $w = w(x, y)$ is differentiable, $x = \mathrm{e}^u \cos v$, $y = \mathrm{e}^u \sin v$. Show that

$$\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 = \mathrm{e}^{2u}\left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right)$$

(6) 3. Find and classify the critical points of $f(x, y) = 4xy - 2x^4 - y^2$.

(6) 4. Use Lagrange Multipliers to find the extreme values of $f(x, y, z) = x^2 + y^2 + z^2$ where (x, y, z) lie on the plane $x + 2y + 3z = 7$.

(6) 5. Evaluate the following (change coordinates as appropriate):

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1 - x^3}\, dx\, dy$$

(6) 6. Evaluate the double integral $\iint_{\mathcal{R}} \mathrm{e}^{-(x^2+y^2)}\, dx\, dy$, where $\mathcal{R}$ is the entire xy plane.

(For a bonus mark, use this to derive the value of $\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\, dx$.*)*

(6) 7. Sketch the solid region of integration for the following:

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \int_{r/\sqrt{3}}^{\sqrt{4-r^2}} r\sqrt{r^2 + z^2}\, dz\, dr\, d\theta$$

Convert the integral to spherical coordinates. Evaluate the triple integral by whatever method you prefer.

(6) 8. Sketch the solid region $\mathcal{S}$ that lies inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $2x^2 + 2y^2 + z^2 = 18$. Find the volume of $\mathcal{S}$.

(Total: 50)