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(April 2005)

Test 2
(version A)

Calculus III (Maths 201–DDB)

Justify all your answers — just having the correct answer is not sufficient.
Pace yourself — a rough guide is to spend not more than $2m$ minutes or so on a question worth m marks.

(4×2) 1. Let Λ be the line passing through the point $P_0(1,2,3)$, parallel to the vector $\boldsymbol{v} = 3\boldsymbol{i}+9\boldsymbol{j}+\boldsymbol{k}$, and let Π be the plane given by $2x - y + 3z = 5$.

(a) What are the equations of Λ? (b) Show that Λ is parallel to the plane Π.

(c) What is the distance from Λ to Π? (Hint: Choose a point Q on Π, then project the vector $\overrightarrow{P_0Q}$ onto the normal of the plane Π. You know the length of the projection, right?)

(d) What is the angle between Π and the plane $3x + 2y + z = 1$?

(3) 2. Prove that $|\boldsymbol{u}+\boldsymbol{v}|^2 + |\boldsymbol{u}-\boldsymbol{v}|^2 = 2(|\boldsymbol{u}|^2 + |\boldsymbol{v}|^2)$, for any vectors $\boldsymbol{u}, \boldsymbol{v}$.

(9) 3. Name and sketch the following surfaces in 3-space. Show all your work, including traces, intercepts (and contour curves if you use them).

(a) $z = y^2 - 2x^2$ (b) $z^2 = r^2 + 1$ (c) $\rho = 4\csc\phi$

(5) 4. Find the parametrization of $\boldsymbol{r}(t) = \langle 3\cos t\,,\ 3\sin t\,,\ 4\rangle$ in terms of the arclength s.
Use $t = 0$ for the "starting point".
This curve lies on a quadric surface: identify such a surface (give its equation) and draw a sketch of the graph of the curve and the surface on which it lies.

(11) 5. A particle P moves along a curve $\boldsymbol{r}(t) = t^3\,\boldsymbol{i} + 3t^2\,\boldsymbol{j} + 6t\,\boldsymbol{k}$. Find the unit tangent vector $\boldsymbol{T}(t)$, the unit normal vector $\boldsymbol{N}(t)$, the curvature $\kappa(t)$, the tangential and the normal components a_T, a_N of acceleration.

Verify your answers by calculating $a_T\boldsymbol{T} + a_N\boldsymbol{N}$ and checking that it equals $\boldsymbol{a}(t)$.

(Hint: do the algebra carefully, and you will find there is a lot of simplification—all the square roots work out easily if you factor where appropriate.)

(6) 6. Find the (radius, center, and) equation of the osculating circle for the curve $y = \sec x$ at the point $(0,1)$. Draw a graph of the curve and the osculating circle (on the same axes) near $(0,1)$.

(4) 7. Calculate the following limits (if possible; if the limit does not exist, say so, and justify your answer).

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$

(Hints: For (a), consider a parabolic path, among other possibilities. For (b) consider the fractions $\frac{x^3}{x^2+y^2}$ and $\frac{y^3}{x^2+y^2}$ separately. You may use symmetry to reduce this to just calculating one limit and then deducing the other.)

(4) 8. Show that $z = \sin x \cos ay$ is a solution of the equation $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{1}{a^2}\dfrac{\partial^2 z}{\partial y^2}$.

(Total: 50)

(3) **Bonus:** For a space curve given by a vector function $\boldsymbol{r} = \boldsymbol{r}(t)$, show that if $\boldsymbol{a}$ is parallel to $\boldsymbol{r}$, then the path of the space curve is planar, (where as usual $\boldsymbol{a}$ is the acceleration vector).