Calculus III (Maths 201–DDB)

(Marks)

Note: Justify all your answers — don't make me guess your thoughts!

(7) 1. Construct a power series for $\displaystyle\int_0^x \frac{t^2-\sin t^2}{t^6}\,dt$; use this series to approximate $\displaystyle\int_0^{1/2} \frac{x^2-\sin x^2}{x^6}\,dx$ to within $\pm 10^{-6}$. Justify your error estimate.

(6) 2. (a) Use the Binomial Theorem to find the Maclaurin series for the function $f(x) = \dfrac{1}{\sqrt{1-x^2}}$. What is the interval of convergence for this series?

(b) Use this series to find the Maclaurin series for the function $\arcsin(x)$. What is the radius of convergence for this series?

(c) Use the first 4 terms of the series in part (b) to approximate $\pi/6$.

(7) 3. What is the third degree Taylor polynomial $T_3(x)$ for the function $f(x) = \sqrt[5]{x}$ centered at $x = 32$? Use $T_3(x)$ to approximate $\sqrt[5]{33}$. Use Taylor's inequality to estimate the possible error of this approximation.

4. Two "quickies" *(Don't spend a lot of time on these!)*:

(2) (a) Use a known power series (one of the basic ones you have learned about) to evaluate the sum of the following series:

$$\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!} \quad = \quad 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} \pm \cdots$$

Give your answer in simplified exact form (not a decimal, please).

(2) (b) Suppose $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(x+3)^n}{n(n!)}$; find $f^{(10)}(-3)$ (without calculation).

5. Consider the curve given by the following parametric equations: $\begin{cases} x = t^2 - 9 \\ y = t^2 + 3t \end{cases}$

(10) (a) Find the x and y intercepts.
Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and all points with horizontal and vertical tangents.
Sketch the graph, showing all these points. Indicate the direction of increasing t (the "orientation").

(6) (b) This curve forms a closed region in Quadrant III, the region between the x-axis and the curve.

i. Find the area bounded by this region (in Quadrant III).

ii. Set up (but **do not** evaluate) the integral necessary to find the arc length of the part of the curve in Quadrant III.

(10) 6. Draw a rough sketch of the graph of the polar curve $r = 1 + 2\cos\theta$.
(Hint: first find where $r = 0$ and so where the graph "flips" across the origin.)
Calculate the area inside the inner loop.
Set up (but **do not** evaluate) the integral necessary to find the arc length of the outer loop (*i.e.* the outside circumference).
(*Hint: use symmetry to simplify the range of the integrals.)*

(Total: 50)