



Calculus III (Maths 201–DDB)

(Marks)

Note: Justify all your answers — don't make me guess your thoughts!

- (7) 1. Construct a power series for $\int_0^x \frac{1 - \cos t^2}{t^4} dt$; use this series to approximate $\int_0^{1/2} \frac{1 - \cos x^2}{x^4} dx$ to within $\pm 10^{-6}$. Justify your error estimate.
- (6) 2. (a) Use the Binomial Theorem to find the Maclaurin series for the function $f(x) = \frac{1}{1+x^2}$. What is the interval of convergence for this series?
 (b) Use this series to find the Maclaurin series for the function $\arctan(x)$. What is the interval of convergence for this series?
 (c) Use the first 5 terms of the series in part (b) to approximate $\pi/4$.
- (7) 3. What is the third degree Taylor polynomial $T_3(x)$ for the function $f(x) = \sqrt[5]{x}$ centered at $x = 32$? Use $T_3(x)$ to approximate $\sqrt[5]{33}$. Use Taylor's inequality to estimate the possible error of this approximation.
4. Two “quickies” (*Don't spend a lot of time on these!*):
- (2) (a) Use a known power series (one of the basic ones you have learned about) to evaluate the sum of the following series:
- $$\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} \pm \dots$$
- Give your answer in simplified exact form (not a decimal, please).
- (2) (b) Suppose $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x+3)^n}{n(n!)}$; find $f^{(10)}(-3)$ (without calculation).
5. Consider the curve given by the following parametric equations: $\begin{cases} x = t^2 - 4 \\ y = t^2 + 2t \end{cases}$
- (10) (a) Find the x and y intercepts.
 Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and all points with horizontal and vertical tangents.
 Sketch the graph, showing all these points. Indicate the direction of increasing t (the “orientation”).
- (6) (b) This curve forms a closed region in Quadrant III, the region between the x -axis and the curve.
 i. Find the area bounded by this region (in Quadrant III).
 ii. Set up (but **do not** evaluate) the integral necessary to find the arc length of the part of the curve in Quadrant III.
- (10) 6. Draw a rough sketch of the graph of the polar curve $r = 1 - 2 \sin \theta$.
 (*Hint: first find where $r = 0$ and so where the graph “flips” across the origin.*)
 Calculate the area inside the inner loop.
 Set up (but **do not** evaluate) the integral necessary to find the arc length of the outer loop (*i.e.* the outside circumference).
 (*Hint: use symmetry to simplify the range of the integrals.*)

(Total: 50)