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Assignment 1
Answers

Calculus III (Maths 201–DDB)

1. (a) $R = 1; I = [0, 2]$ (b) $R = \infty; I = (-\infty, \infty)$ (c) $R = \infty; I = (-\infty, \infty)$

2. (a) $R = 5; I = (0, 10]$ (b) $R = 5; I = (0, 10)$ (c) $R = 5; I = [0, 10]$

3. I have given formulas where I expect you to have found them; in 3(d) no formula is given: I expect you just to calculate several terms (not necessarily as many as I have — I used MAPLE!).

   (a) $x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \frac{1}{315}x^8 + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n-1}x^{2n}}{(2n)!}$ $I = (-\infty, \infty)$

   (b) $x^2 - \frac{1}{3}x^6 + \frac{1}{5}x^{10} - \frac{1}{7}x^{14} + \frac{1}{9}x^{18} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}$ $I = [-1, 1]$

   (c) $x^3 - x^4 + \frac{1}{2}x^5 - \frac{1}{6}x^6 + \frac{1}{24}x^7 - \frac{1}{120}x^8 + \frac{1}{720}x^9 + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+3}}{n!}$ $I = (-\infty, \infty)$

   (d) $1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{2}x^4 + \frac{3}{10}x^5 + \frac{19}{90}x^6 + \frac{13}{105}x^7 + \frac{31}{360}x^8 + \frac{163}{3240}x^9 + \cdots$ $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

   (e) $x^3 - 3x^4 + 6x^5 - 10x^6 + 15x^7 - 21x^8 + 28x^9 + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{(n+2)!}{2\cdot n!}\,x^{n+3}$ $I = (-1, 1)$

   (f) $x - \frac{1}{9}x^3 + \frac{1}{25}x^5 - \frac{1}{49}x^7 + \frac{1}{81}x^9 + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)^2}$ $I = [-1, 1]$

   (g) $1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \frac{35}{128}x^8 + \cdots = 1 + \sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{n!\,2^n}\,x^{2n}$ $I = [-1, 1]$

4. (a) $x - \frac{1}{4}x^2 + \frac{1}{18}x^3 - \frac{1}{96}x^4 + \frac{1}{600}x^5 - \frac{1}{4320}x^6 + \frac{1}{35280}x^7 + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n\cdot n!}$

   (b) $(-\infty, \infty)$ (c) 0.3633 (only need 4 terms; the 5th is 0.000017)

5. Hint: Break the fraction into two, and then subtract them as appropriate.

   (a) $-\frac{3}{2} - \frac{3}{4}(x-1) - \frac{9}{8}(x-1)^2 - \frac{15}{16}(x-1)^3 - \frac{33}{32}(x-1)^4 - \frac{63}{64}(x-1)^5 - \frac{129}{128}(x-1)^6$

   $$-\frac{255}{256}(x-1)^7 + \cdots = -\sum_{n=0}^{\infty}\frac{2^{n+1} - (-1)^{n+1}}{2^{n+1}}\,(x-1)^n$$

   (b) $(0, 2)$ (c) $-\frac{129}{128}\,6! = -\frac{5805}{8}$

6. (a) $1+\dfrac{1}{2}\,x-\dfrac{1}{8}\,x^2+\dfrac{1}{16}\,x^3-\dfrac{5}{128}\,x^4+\dfrac{7}{256}\,x^5-\dfrac{21}{1024}\,x^6+\dfrac{33}{2048}\,x^7+\cdots$

$$=1+\frac{1}{2}x+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{(2n-3)!!}{n!\,2^n}\,x^n \qquad I=[-1,1]$$

(b) Use the series for $4\sqrt{1+\frac{x}{16}}:=4+\dfrac{1}{8}\,x-\dfrac{1}{512}\,x^2+\dfrac{1}{16384}\,x^3+\cdots$ with $x=\frac{1}{2}$, to get $4\frac{8129}{131072}=4.062019$.

7. The series is for $\sin(x)$ is

$$x-\frac{1}{6}\,x^3+\frac{1}{120}\,x^5-\frac{1}{5040}\,x^7+\frac{1}{362880}\,x^9-\frac{1}{39916800}\,x^{11}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

If $|x|\le 0.5$, then the term $\frac{1}{5040}\,x^7\le 1.5\times 10^{-6}$, so the first 3 terms will suffice: this is the 5^{th} Maclaurin polynomial, (*i.e.* $n=5$).

8. The series is

$$2+\frac{1}{4\cdot 2^3}\,(x-16)-\frac{1\cdot 3}{4^2\cdot 2^7\cdot 2!}\,(x-16)^2+\frac{1\cdot 3\cdot 7}{4^3\cdot 2^{11}\cdot 3!}\,(x-16)^3-\frac{1\cdot 3\cdot 7\cdot 11}{4^4\cdot 2^{15}\cdot 4!}\,(x-16)^4+\cdots$$

The remainder $R_4(x)$ is then

$$\sqrt[4]{x}-2-\frac{1}{32}\,(x-16)+\frac{3}{4096}\,(x-16)^2-\frac{7}{262144}\,(x-16)^3+\frac{77}{67108864}\,(x-16)^4$$

Using $T_4(15)$ as an approximation of $\sqrt[4]{15}$, we get $1\frac{64960691}{67108864}=1.9679897279$. We can estimate the error by Taylor's Inequality: $f^{(5)}(x)=\frac{1\cdot 3\cdot 7\cdot 11\cdot 15}{4^5}x^{-19/4}$ and since $x^{-19/4}$ is decreasing, this is $\le$ the value for $x=15$ for $15\le x\le 17$. So we can take $M=\frac{1\cdot 3\cdot 7\cdot 11\cdot 15}{4^5}15^{-19/4}$ giving an error estimate of $\frac{M}{5!}=7.3\times 10^{-8}$.