Tips on completing the square

The key thing to remember about completing the square is that the method works best if the coefficient of x^2 is 1, and then you will essentially do a substitution, where the new variable u is x plus half the coefficient of x. The rest may be done "by inspection".

In Cal II, we usually want to complete the square in order to do an integral by substitution; the comment above tells us exactly what the substitution will be, and we shall emphasise that point here. Notice that $du = dx$ with these substitutions, since $u = x$ + half the x-coefficient.

Let's illustrate the process with an example:

Example 1: Suppose the expression you are working with is $x^2 - 4x - 5$. This could occur in an integral like $\int \frac{dx}{\sqrt{x^2-4x-5}}$ for example.

We then set $u = x - 2$ (the "-2" coming from half the x-coefficient "-4"). So $u^2 = x^2 - 4x + 4$, and we compare this to $x^2 - 4x - 5$:

$$\underline{x^2 - 4x}\ \underline{\underline{+\ 4 + ??}} \quad = \quad \underline{x^2 - 4x}\ \underline{\underline{-\ 5}}$$

Looking at this (subtract $x^2 - 4x$ from both sides to see why) and it's clear that the ?? must be $-5 - 4 = -9$:

$$(x^2 - 4x + 4) - 9 \ = \ u^2 - 9 \ = \ x^2 - 4x - 5$$

So $x^2 - 4x - 5 = u^2 - 9$. And then the integral in u-terms becomes $\int \frac{du}{\sqrt{u^2-9}}$, which can be done by trig sub, for example.

No need for a formula: just **look** at the expression you started with and the u^2 and it's easy to see what you need to add or subtract.

Example 2: In many exercises you will need to "prepare" the quadratic first. For example, suppose you wished to work with $5+4x-x^2$ (for example, this might come from an integral like $\int \frac{dx}{\sqrt{5+4x-x^2}}$). Then you would start by "removing" – or factoring out – the minus sign: $5+4x-x^2 = -(x^2-4x-5)$. You would then proceed as above, to get $x^2-4x-5 = u^2-9$, and hence you'd end up with $-(u^2-9)$ which is really $9 - u^2$. Then your new integral would be $\int \frac{du}{\sqrt{9-u^2}}$.

Example 3: Similarly if you started with $3x^2 + 6x + 13$, (coming from $\int \frac{x\,dx}{3x^2+6x+13}$ for example), you'd factor out the 3, get the correct substitution, use it to get the right quadratic in u equal to the appropriate quadratic in x, and finally use that to get your original integral equal to a u-integral. Here are the steps:

$$\begin{aligned}
3x^2 + 6x + 13 &= 3(x^2 + 2x + \tfrac{13}{3}) \quad \text{(so } u = x + 1\text{, and } x = u - 1\text{)}\\
u^2 &= x^2 + 2x + 1\\
\text{so}\ \ x^2 + 2x + \tfrac{13}{3} &= u^2 + \tfrac{10}{3} \quad \text{(since } -1 + \tfrac{13}{3} = \tfrac{10}{3}\text{)}\\
\text{so}\ \ 3x^2 + 6x + 13 &= 3(u^2 + \tfrac{10}{3})\\
&= 3u^2 + 10
\end{aligned}$$

So your new integral would be $\int \frac{(u-1)\,du}{3u^2+10}$, which splits into a simple substitution and an arctan (or trig sub).