PCA Module
Maths DDB — Cal III

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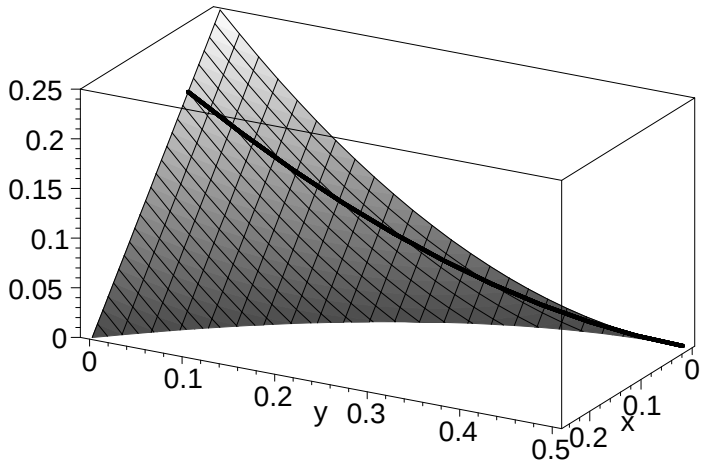
The Sweet Eiffel Tower
A module assignment[1]



I bought a bottle of maple syrup shaped like the Eiffel tower. When I was pouring syrup on my pancakes I noticed, as shown in the figure on the right, that one of the four sides was given by the equation

$$z = \left(\frac{1}{2} - y\right)^2 - x$$

for (x, y) in the first quadrant of the xy plane. The base of the bottle is the xz plane.

A bee called Maya was walking on the curved side of the bottle following the curve

$$\boldsymbol{r}(t) = \left(\frac{1}{4} - t\right)^2 \boldsymbol{i} + 2t\boldsymbol{j} + 3\left(\frac{1}{4} - t\right)^2 \boldsymbol{k}$$

starting at $t = 0$ (t measured in minutes); the bee's trajectory is shown by the thick black curve on the surface. At time $t = \frac{1}{20}$ Maya is at a point A described by the position vector

$$\overrightarrow{OA} = \frac{1}{25}\boldsymbol{i} + \frac{1}{10}\boldsymbol{j} + \frac{3}{25}\boldsymbol{k}$$

(all measurements x, y, z are in meters).

Questions:

1. Write the equation of the tangent plane to the bottle at A.
2. What is the direction of fastest ascent at the point A?
3. What is the directional derivative at the point A in the direction of the vector $2\boldsymbol{i} + 3\boldsymbol{j}$?
4. Write a double integral that computes the volume of the part of the bottle which lies in the first octant; what is the volume of the entire bottle?
5. Write the equation of the tangent line to Maya's trajectory at A.
6. What is Maya's rate of ascent (rate of increase in height as a function of time), at time $t = \frac{1}{20}$?
7. Write an integral that computes the arclength Maya travels on the bottle from time $t = 0$ to $\frac{1}{4}$. Do *not* evaluate the integral.
8. Find the curvature of Maya's trajectory (as a function of t).
9. The curved side of the bottle may be considered as the level surface $G(x, y, z) = \frac{1}{2}$ for the function $G(x, y, z) = y + \sqrt{x + z}$. Find the equation of the tangent plane at the point A using the function $G(x, y, z)$.

[1] Taken from Math 189-222B Final Exam, Apr. 2002, McGill University.