PCA Module

Maths DDB — Cal III

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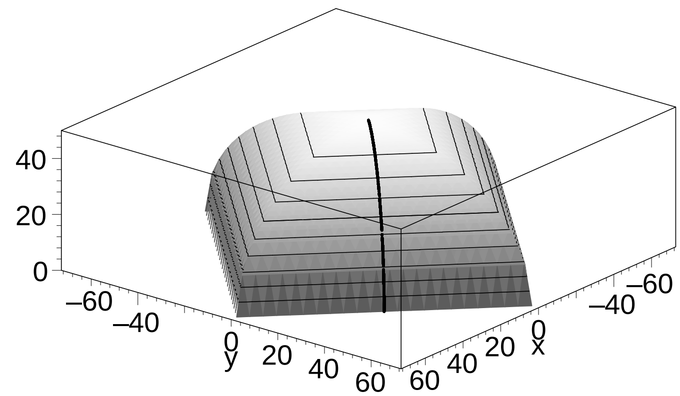
The Bent Pyramid
A module assignment[1]



The archaelogist Flinders Petrie (1853–1942) visited the Bent Pyramid in Dahshur, Egypt. The pyramid looks like the figure at the right — notice the slanted side at the "front"; its equation is

$$z = f(x, y) = \frac{1}{\sqrt{2}}\sqrt{70^2 - (x+y)^2}$$

(lengths measured in meters). Petrie decides to climb to the top of the pyramid following the curve

$$\boldsymbol{r}(t) = 35\cos\left(\frac{\pi}{40}t\right)\boldsymbol{i} + 35\cos\left(\frac{\pi}{40}t\right)\boldsymbol{j} + 35\sqrt{2}\sin\left(\frac{\pi}{40}t\right)\boldsymbol{k}$$

which lies on the front side of the pyramid, as shown by the heavy black curve on that face. He starts at time $t = 0$ (time measured in minutes). At time $t = 10$, Petrie is at a point A, described by the position vector

$$\overrightarrow{OA} = \frac{35\sqrt{2}}{2}\boldsymbol{i} + \frac{35\sqrt{2}}{2}\boldsymbol{j} + 35\boldsymbol{k}$$

Questions:

1. How long does it take Petrie to climb to the peak of the pyramid?
2. Find the velocity $\boldsymbol{v}(t)$, the unit tangent vector $\boldsymbol{T}(t)$, and the curvature $\kappa(t)$ of Petrie's trajectory.
3. What is his rate of ascent (rate of increase of the height as a function of t) at time $t = 10$ minutes?
4. Write the equation of the tangent plane to the pyramid at A.
5. What is the direction of fastest ascent at the point A?
6. What is the directional derivative at the point A in the direction of the vector $\boldsymbol{i} - \boldsymbol{j}$?
7. Use linear approximation at the point A to estimate the height of the pyramid where $x = 18\sqrt{2}$ and $y = 17.25\sqrt{2}$.
8. The front side of the pyramid may be considered as the level surface $G(x, y, z) = 70^2$ for the function $G(x, y, z) = 2z^2 + (x + y)^2$. Find the equation of the tangent plane to the pyramid at the point A using the function $G(x, y, z)$.
9. Write a double or triple integral (your choice) for the volume of the part of the pyramid in the first octant (under the "front side"). (Do *not* evaluate the integral.)
10. Change the integral above to polar (or cylindrical) coordinates, depending on whether you wrote a double or triple integral, and compute the volume in the first octant.
    Hint: for the line segment that is the edge of the base of the pyramid you need to use the identity $\cos\theta + \sin\theta = \sqrt{2}\cos(\theta - \frac{\pi}{4})$, and the formula $\sec u = 1/\cos u$.

[1] Taken from Math 189-260A Final Exam, Dec. 2000, McGill University.