Aspects of π

A module assignment

We investigate two infinite expressions which may be used to find the value of π. (We shall see a variant of the second later in the course as well.)

1. We start with the integral $\int \sin^k x\,dx$. Using integration by parts, show that the following recursion equation is true (for all values of k):

$$\int \sin^k x\,dx = -\frac{1}{k}\sin^{k-1} x\cos x + \frac{k-1}{k}\int \sin^{k-2} x\,dx$$

We shall denote the definite integral $\int_0^{\pi/2} \sin^k x\,dx$ by $\mathcal{I}(k)$.

Using this recursion formula, show that the following equations are true (for all postive integer values of n):

$$\mathcal{I}(2n) = \int_0^{\pi/2} \sin^{2n} x\,dx = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\cdots\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$\mathcal{I}(2n+1) = \int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\cdots\cdot\frac{2}{3}\cdot 1$$

Conclude that

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{\mathcal{I}(2n)}{\mathcal{I}(2n+1)} = \frac{\pi}{2}$$

Next, show that $\mathcal{I}(2n+1) \le \mathcal{I}(2n) \le \mathcal{I}(2n-1)$ (hint: just show that if $k \ge l$, then $\mathcal{I}(k) \le \mathcal{I}(l)$). Then show that

$$1 \le \frac{\mathcal{I}(2n)}{\mathcal{I}(2n+1)} \le \frac{\mathcal{I}(2n-1)}{\mathcal{I}(2n+1)} \le 1 + \frac{1}{2n}$$

and so

$$\lim_{n\to\infty} \frac{\mathcal{I}(2n)}{\mathcal{I}(2n+1)} = 1$$

This implies

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdots$$

(where this means the evident limit as $n \to \infty$).[1]

[1] It is possible to prove from this expression an even more extraordinary limit: $\lim \frac{n!}{\sqrt{n}}\left(\frac{e}{n}\right)^n = \sqrt{2\pi}$. I shall leave that to you, if you wish to pursue it!

2. Next we look again at the integral (above)

$$\mathcal{I}(2n+1) \;=\; \int_0^{\pi/2} \sin^{2n+1} x\, dx = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\cdots\cdot\frac{2}{3}\cdot 1 \tag{1}$$

together with the power series representation[2] of $\arcsin x$:

$$\arcsin x \;=\; x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1} \tag{2}$$

Using the change of variables $x = \sin\theta$ and equation (1), show that

$$\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/2} \sin^{2n+1}\theta\, d\theta = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)} \tag{3}$$

Show by direct (Cal II) integration that

$$\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \frac{\pi^2}{8} \tag{4}$$

Using the infinite series (equation (2)) for $\arcsin x$, and equation (3), show that

$$\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \tag{5}$$

Note that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty}\frac{1}{(2n)^2}$$

So conclude[3] using equations (4, 5) that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

We'll see another proof of this formula at the end of the course — it is a famous result of Euler's ("E as in e"). This was the first result obtained in summing the p series $\sum \frac{1}{n^p}$, usually[4] denoted $\zeta(p)$. It is not too difficult to extend Euler's result to obtain formulas for all the even powers $\zeta(2n)$, but to this day, no formula is known for any of the odd powers, not even for the "simplest" $\zeta(3) = \sum \frac{1}{n^3}$. About 25 years ago $\zeta(3)$ was shown to be irrational, but beyond that little is known.

[2] This is based on integrating the binomial expansion (as we did in class!): $\frac{1}{\sqrt{1-t^2}} = 1 + \sum \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} t^{2n}$

[3] Note that $\sum \frac{1}{(2n)^2} = \frac{1}{4}\sum\frac{1}{n^2}$ — why?

[4] This "zeta function" $\zeta(s)$ is one of the really famous functions of mathematics, and a conjecture concerning its behaviour (the "Riemann hypothesis") is one of several million dollar problems that challenge mathematicians. You can find out more at `http://www.claymath.org/prizeproblems/`