Three derivations in predicate logic

Some students like Roger; all teachers like any student; Roger is a teacher. Therefore there is someone who both likes and is liked by Roger.

Take $T(x) = \text{“}x\text{ is a teacher}\text{”}$, $A(x, y) = \text{“}x\text{ likes } y\text{”}$, $S(y) = \text{“}y\text{ is a student}\text{”}$, and $R = \text{“}Roger\text{”}$. Then

\begin{align*}
1 & \quad \exists x (S(x) \land A(x, R)) \\
2 & \quad \forall x \forall y (T(x) \land S(y) \rightarrow A(x, y)) \\
3 & \quad T(R) \\
4 & \quad u \quad S(u) \land A(u, R) \\
5 & \quad \forall y (T(R) \land S(y) \rightarrow A(R, y)) \quad (\forall E), 2 \\
6 & \quad T(R) \land S(u) \rightarrow A(R, u) \quad (\forall E), 5 \\
7 & \quad S(u) \quad (\land E), 4 \\
8 & \quad T(R) \land S(u) \quad (\land I), 3, 7 \\
9 & \quad A(R, u) \quad (\rightarrow E), 6, 8 \\
10 & \quad A(u, R) \quad (\land E), 4 \\
11 & \quad A(u, R) \land A(R, u) \quad (\land I), 9, 10 \\
12 & \quad \exists x (A(x, R) \land A(R, x)) \quad (\exists I), 11 \\
13 & \quad \exists x (A(x, R) \land A(R, x)) \quad (\exists E), 1 \\
\end{align*}
All horses are animals; therefore all heads of horses are heads of animals

Take \( H(y) = "y \text{ is a horse}" \), \( A(y) = "y \text{ is an animal}" \), and \( T(x, y) = "x \text{ is the head of } y" \). Then

\[
\begin{array}{c|c|c}
1 & \forall y (H(y) \rightarrow A(y)) & \\
2 & u & \exists y (T(u, y) \land H(y)) \\
3 & v & T(u, v) \land H(v) \\
4 & & T(u, v) \\
5 & & H(v) \\
6 & & H(v) \rightarrow A(v) \\
7 & & A(v) \\
8 & & T(u, v) \land A(v) \\
9 & & \exists y (T(u, y) \land A(y)) \\
10 & & \exists y (T(u, y) \land A(y)) \\
11 & & \exists y (T(u, y) \land H(y)) \rightarrow \exists y (T(u, y) \land A(y)) \\
12 & \forall x (\exists y (T(x, y) \land H(y)) \rightarrow \exists y (T(x, y) \land A(y))) & \end{array}
\]

(\( \land E \)), 3  
(\( \land E \)), 3  
(\( \forall E \)), 1  
(\( \rightarrow E \)), 5, 6  
(\( \land I \)), 4, 7  
(\( \exists I \)), 8  
(\( \exists E \)), 2  
(\( \rightarrow I \)), 2–10  
(\( \forall I \)), 2–11
Some teachers like all students; no teacher likes any jerk. Therefore no students are jerks.

Take $T(x) =$ “$x$ is a teacher”, $A(x, y) =$ “$x$ likes $y$”, $S(y) =$ “$y$ is a student”, and $J(y) =$ “$y$ is a jerk”. Then

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<tr>
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<tbody>
<tr>
<td>1</td>
<td>$\exists x (T(x) \land \forall y (S(y) \rightarrow A(x, y)))$</td>
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<tr>
<td>2</td>
<td>$\forall x \forall y (T(x) \land J(y) \rightarrow \neg A(x, y))$</td>
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</tr>
<tr>
<td>3</td>
<td>$\exists y (S(y) \land J(y))$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$S(u) \land J(u)$</td>
<td></td>
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<tr>
<td>5</td>
<td>$T(v) \land \forall y (S(y) \rightarrow A(v, y))$</td>
<td></td>
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<tr>
<td>6</td>
<td>$\forall y (T(v) \land J(y) \rightarrow \neg A(v, y))$ (\forall E), 2</td>
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<tr>
<td>7</td>
<td>$T(v) \land J(u) \rightarrow \neg A(v, u)$ (\forall E), 6</td>
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<tr>
<td>8</td>
<td>$T(v)$ (\wedge E), 5</td>
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<td>9</td>
<td>$J(u)$ (\wedge E), 4</td>
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<td>10</td>
<td>$T(v) \land J(u)$ (\wedge I), 8, 9</td>
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<td>11</td>
<td>$\neg A(v, u)$ (\rightarrow E), 7, 10</td>
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<td>12</td>
<td>$\forall y (S(y) \rightarrow A(v, y))$ (\forall E), 5</td>
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<td>13</td>
<td>$S(u) \rightarrow A(v, u)$ (\forall E), 12</td>
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<tr>
<td>14</td>
<td>$S(u)$ (\wedge E), 4</td>
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<tr>
<td>15</td>
<td>$A(v, u)$ (\rightarrow E), 13, 14</td>
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<tr>
<td>16</td>
<td>$\bot$ (\neg E), 11, 15</td>
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<td>17</td>
<td>$\bot$ (\exists E), 1, 5–16</td>
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<td>18</td>
<td>$\bot$ (\exists E), 3, 4–17</td>
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<tr>
<td>19</td>
<td>$\neg \exists y (S(y) \land J(y))$ (\neg I), 3–18</td>
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Practice derivations in predicate logic

The first three are from the text, Exercise 5.5.6. Use the notation of the text if you want to compare your answer to mine.

The last three are new — use the suggested notation (and you can find the solutions on the course webpage, under “Assignments and Answers”).

1. Generous people are happy; Albie is intelligent, but not happy. Everyone is either generous or they’re not very free with their money. Hence, someone is intelligent but not very free with their money.

2. Bruce is charismatic. Bruce will retire to Australia only if everyone is satisfied. Everyone is happy if they are satisfied. Everyone will retire to Australia if someone is charismatic. Hence everyone is happy.

3. France is a country bigger than Luxembourg; some country is bigger than France. If something is bigger than a second thing, and the second is bigger than a third, then the first is bigger than the third. So, some country is bigger than either France or Luxembourg (i.e. bigger than France and bigger than Luxembourg).

4. Some students like Roger; all teachers like any student; Roger is a teacher. Therefore there is someone who both likes and is liked by Roger.

Take $T(x) = \text{“}x\text{ is a teacher”}$, $A(x,y) = \text{“}x\text{ likes }y\text{”}$, $S(y) = \text{“}y\text{ is a student”}$, and $R = \text{“Roger”}$.

5. All horses are animals; therefore all heads of horses are heads of animals

Take $H(y) = \text{“}y\text{ is a horse”}$, $A(y) = \text{“}y\text{ is an animal”}$, and $T(x,y) = \text{“}x\text{ is the head of }y\text{”}$.

6. Some teachers like all students; no teacher likes any jerk. Therefore no students are jerks.

Take $T(x) = \text{“}x\text{ is a teacher”}$, $A(x,y) = \text{“}x\text{ likes }y\text{”}$, $S(y) = \text{“}y\text{ is a student”}$, and $J(y) = \text{“}y\text{ is a jerk”}$. 