Some Logic Problems

Albert, Brenda and Chuck are suspected of theft. They make the following assertions:

> Albert: "Brenda is guilty and Chuck is innocent."
>
> Brenda: "If Albert is guilty, then so is Chuck."
>
> Chuck: "I am innocent, but at least one of the others is guilty."

Use A (respectively B, C) to represent "Albert" (respectively "Brenda", "Chuck") "is innocent".

1. Express the testimony of each suspect in propositional logic using the atomic propositions above.
2. Assuming everybody is telling the truth, then who is innocent and who is guilty. Prove your answer is correct, by showing the corresponding entailment is valid.
3. (a) Is it possible that Chuck is lying and the others are telling the truth?
   (b) What if Brenda is lying and the others are telling the truth?
   (c) And what if Albert is lying and the others are telling the truth?

   In each case, if there is a valid conclusion, say what it is (and prove the inference is valid), and otherwise prove the situation is inconsistent.
4. Consider the three ways in which two could be liars and one a truth-sayer; finally, what if all three are lying?

Answers

1. The three statements (we'll call them α, β, γ after the initials of the speakers) are:
   $\alpha = \neg B \wedge C$; $\beta = \neg A \rightarrow \neg C$; $\gamma = C \wedge (\neg A \vee \neg B)$. Note that $\beta \equiv C \rightarrow A$, and that the negations of these statements are $\neg\alpha \equiv C \rightarrow B$; $\neg\beta \equiv \neg A \wedge C$; $\neg\gamma \equiv C \rightarrow A \wedge B$.
2. $\alpha, \beta, \gamma \vdash A \wedge \neg B \wedge C$ (you can prove this easily with a formal derivation or an analytic tableau — I'll leave that to you!)
3. (a) $\alpha, \beta, \neg\gamma \vdash \perp$ (*i.e.* this is an impossible situation)
   (b) $\alpha, \neg\beta, \gamma \vdash \neg A \wedge \neg B \wedge C$
   (c) $\neg\alpha, \beta, \gamma \vdash \perp$
4. $\alpha, \neg\beta, \neg\gamma \vdash \perp$
   $\neg\alpha, \beta, \neg\gamma \vdash$ several consistent alternatives: *e.g.* $A \wedge B \wedge C$ or $\neg C$.
   $\neg\alpha, \neg\beta, \gamma \vdash \neg A \wedge B \wedge C$
   $\neg\alpha, \neg\beta, \neg\gamma \vdash \perp$

I'll leave proving these to you (*e.g.* by analytic tableaux).