REMARKABLE BIOGRAPHY

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Logical dilemmas: the life and work of Kurt Gödel
by John W. Dawson, Jr.
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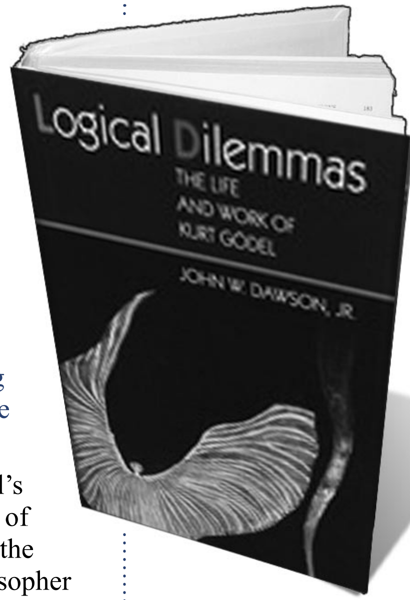
I HAVE HAD contact with Gödel only twice. In 1950, at the International Conference of Mathematicians, I attended his lecture, in which he showed that Einstein's field equations allowed a periodic solution in which the future repeats the past. The second time was in Princeton in 1960, when he phoned my wife to give his excuses for not appearing at our party for logicians, fearing that there might be too many germs.



Over the years, I have given much thought to Gödel's revolutionary contributions to the foundations of mathematics, some thoughts in discussion with the mathematician Phil Scott and some with the philosopher Jocelyne Couture. However, I never studied Gödel's ideas in their historical context and am grateful to find an integrated account of his life and work in this remarkable biography. The author manages to present Gödel's pioneering work in logic and philosophy in a technically accurate way, yet understandable by mathematicians untrained in logic and even by general readers untrained in mathematics. I don't wish to deprive the reader of this review of the pleasure of perusing the book under review, where she will learn much about Gödel's life, embellished by fascinating anecdotes. I will therefore concentrate here on Gödel's main contributions, as described by Dawson, with some elaborations of my own, which do not always agree with conventional opinion.

Gödel proved his *completeness theorem* in his doctoral dissertation of 1929. Originally, it dealt with first order classical logic and was later extended to higher order logic by Leon Henkin. It implies, in particular, that a statement in formal arithmetic is provable if and only if it is true in all models, which he originally called "realizations".

Gödel's famous *incompleteness theorem* does not contradict this. What its proof shows is that it is not enough to look only at models with the so-called ω-property (where S denotes the successor function): if $\phi(S^n0)$ holds for each natural number n, then so does $\forall_{x\in N}\,\phi(x)$.

The crucial role of this property was first pointed out by Hilbert, when trying to react positively to Gödel's challenge. Most people who cite the incompleteness theorem put it more strongly: there are true statements of arithmetic which are not provable. But this formulation presumes truth in a Platonic universe, which is here seen as a distinguished model with the ω-property.

There is a classically, though not intuitionistically, equivalent ω*-property:
if $\exists_{x\in N}\,\phi(x)$ holds, then so does $\phi(S^n0)$ for some natural number n.

Although Gödel first announced his incompleteness theorem for classical arithmetic, it also holds for intuitionistic arithmetic (as I first learned from Dirk van Dalen). We now know that this does have a distinguished model with the ω*-property, in which all true statements are provable. We will return to this in the Postscript below.

The incompleteness theorem was to be submitted for Gödel's so-called habilitation (a prerequisite for permission to lecture at a university); but he first announced it at a conference in Königsberg (now Kaliningrad). There Rudolf Carnap, Arend Heyting and John von Neumann were to defend the three prevailing mathematical philosophies: logicism, intuitionism and formalism respectively, when Gödel threw his bombshell.

Of course, logicism may be attacked on the grounds that arithmetic requires one extra-logical axiom, the so-called axiom of infinity; but intuitionistic arithmetic was shown to be formalizable by Heyting. Gödel's Platonism, at first sight, seems to contradict Hilbert's formalist program, at least as long as one confines attention to classical arithmetic and as long as one does not possess a model of the latter with the ω-property. While intuitionists have no problem with the ω*-property, they may have difficulty with the ω-property. They might argue that we can accept the truth of $\forall_{x\in N}\,\phi(x)$ only if there is a uniform way of getting to know the truth of each $\phi(S^n0)$. After all, the proofs of the formulas $\varphi(S^n0)$ might get more complicated as n increases. Perhaps this helps to explain Gödel's lifelong interest in intuitionism.

To carry out his proof, Gödel had to re-invent the theory of *primitive* recursive functions, which may already have been known to Dedekind and Peano. Prompted by Herbrand, he later lectured on *general* recursive functions in Princeton, to fit in with the Church-Turing thesis. Today, these may be more easily described as recursively enumerable relations which happen to be one-to-one and universally defined, although the latter property need not always be provable.

Gödel barely beat von Neumann to the second incompleteness theorem, which asserts that the consistency of arithmetic, suitably codified, cannot be proved within arithmetic. Many people, including Gödel himself, saw this as destroying Hilbert's program. I am not so impressed with this result; for, if a formal language is inconsistent, then anything can be proved, including its consistency.

In his 1938 Princeton lectures, published in 1940, Gödel proved the consistency of the continuum hypothesis, in what came to be known as Gödel-Bernays set theory. He used the constructible hierarchy of sets as his model. This model depends on the axiom of choice and is not really constructible, in spite of its name. Gödel tried very hard to prove the independence of the continuum hypothesis, but did not succeed. This was proved about twenty years later by Paul Cohen, whom Gödel generously encouraged to publish his proof immediately. Unlike some famous mathematicians in similar circumstances, he did not claim priority.

In a letter to von Neumann in 1955, we find the first known statement of what is now called the $P = NP$ problem, although Gödel professed no interest in the emerging Computer Science.

His last important contribution was the 1958 Dialectica Interpretation. In this paper he outlined the notion of a computable function of finite type and stressed how it can be applied to provide a constructive proof of the consistency of classical arithmetic. The paper also enunciated a number of constructivist principles, most of which are now known to be provable in higher order intuitionistic arithmetic.

Although raised as a freethinker, Gödel later declared: "in religion there is much more that is rational than is generally believed". He thought he had found a proof of the existence of God, an updated version of the famous argument by Anselm of Canterbury, which essentially asserted that God is defined to be a perfect being and that perfection implies existence. Gödel hesitated to publish his proof for fear that a belief in God might be ascribed to him, whereas he only wanted to show that such a proof could be carried out on the basis of accepted principles of formal logic. Still, it turned up in his Nachlass and has given rise to some recent discussion.

The Italian algebraist Magari had found a flaw in the argument, but the Czech logician Hajek claimed to have fixed this. One wonders whether the existence of elephants can be proved by the same method.

Gödel's life ended sadly by self-induced starvation, like that of Eratosthenes, famous for his sieve and for being the first to measure the circumference of the earth. Yet their reasons were different: Eratosthenes was losing his eyesight and did not wish to live as a blind man; Gödel believed that his food had been poisoned.

An appendix to Dawson's book contains some interesting biographical vignettes of other logicians, from Paul Bernays to Ernst Zermelo. One is struck by how many of them also suffered from nervous breakdowns, depression or even paranoia: Cantor, Post and Zermelo. Dawson speculates that there is "a deep connection between rationalism and permanent unshakable delusional system."

POSTSCRIPT: Some categorical afterthoughts may be appropriate. It is now evident that many of Gödel's ideas can be illuminated by contributions from category theory. Bill Lawvere, searching for a characterization of the category of sets, was led to the notion of an elementary topos. This first saw the light of day in the Proceedings of the 1971 Halifax conference, in a joint article with Myles Tierney, offering a categorical proof of Paul Cohen's independence theorem. Candidates for the classical category of sets are now recognized as elementary toposes (with natural numbers object) in which the terminal object is a generator. They may be viewed as Henkin models of classical higher order arithmetic. Gödel and other Platonists would wish to single out one such model with the ω-property as *the* category of sets. It is not clear whether a distinguished such model can be constructed, and classical mathematicians may have to live with a whole sheaf of such models. The situation is different when it comes to models of intuitionistic higher order arithmetic.

These are elementary toposes in which the terminal object is a non-trivial indecomposable projective, as was first pointed out by Peter Freyd. The Platonist Gödel might have been pleased that a distinguished such model exists, the so-called *free* topos, the initial object in the (large) category of all (small) toposes. Being opposed to nominalism, Gödel might have been less pleased that the free topos can be constructed linguistically as what might be called the Tarski-Lindenbaum category of higher order intuitionistic arithmetic. Phil Scott and I wrote our book "Introduction to higher order categorical logic" with the explicit aim of showing that the free topos is a constructible model with the ω*-property in which all true statements are provable. Hilbert might have been pleased with this result if he could have overcome his antagonism to Brouwer's intuitionism. Phil and I were motivated by Gödel's Dialectica Interpretation, inasmuch as we attempted to show that the principles of constructive mathematics outlined there do indeed hold in the free topos. Unfortunately, we never completed this project.