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Test 3
(version A)

Maths & Logic (360-124)

The Answers

1.

1	$\forall x \forall y[G(x,y) \to \neg G(y,x)]$	
2	$\forall y G(\mathsf{d},y)$	
3	$\forall x \forall y[E(x) \wedge \neg E(y) \to G(x,y)]$	
4	$\exists x E(x)$	
5	$\neg E(\mathsf{d})$	
6	u $E(u)$	
7	$E(u) \wedge \neg E(\mathsf{d})$	($\wedge$I), 5, 6
8	$E(u) \wedge \neg E(\mathsf{d}) \to G(u,\mathsf{d})$	($\forall$E), 3
9	$G(u,\mathsf{d})$	($\to$E), 7, 8
10	$G(u,\mathsf{d}) \to \neg G(\mathsf{d},u)$	($\forall$E), 1
11	$\neg G(\mathsf{d},u)$	($\to$E), 9, 10
12	$G(\mathsf{d},u)$	($\forall$E), 2
13	$\bot$	($\neg$E), 11, 12
14	$\bot$	($\exists$E), 4, 6–13
15	$\neg\neg E(\mathsf{d})$	($\neg$I), 5–14
16	$E(\mathsf{d})$	($\neg\neg$E), 15

1	$\forall y(\exists x P(x) \to Q(y))$	
2	$\exists x P(x)$	
3	u $\forall y(\exists x P(x) \to Q(y))$	(R), 1
4	$\exists x P(x) \to Q(u)$	($\forall$E), 3
5	$\exists x P(x)$	(R), 2
6	$Q(u)$	($\to$E), 4, 5
7	$\forall y Q(y)$	($\forall$I), 3–6
8	$\exists x P(x) \to \forall y Q(y)$	($\to$I), 2–7

2.

1	$P(\mathsf{a}) \wedge Q(\mathsf{b})$	
2	$\forall x(R(x) \to \neg P(x))$	
3	$P(\mathsf{a})$	$(\wedge\text{E})$, 1
4	$Q(\mathsf{b})$	$(\wedge\text{E})$, 1
5	$\exists x Q(x)$	$(\exists\text{I})$, 4
6	$\quad$ | $\forall x R(x)$	
7	$\quad$ | $R(\mathsf{a})$	$(\forall\text{E})$, 6
8	$\quad$ | $R(\mathsf{a}) \to \neg P(\mathsf{a})$	$(\forall\text{E})$, 2
9	$\quad$ | $\neg P(\mathsf{a})$	$(\to\text{E})$, 8
10	$\quad$ | $\bot$	$(\neg\text{E})$, 3, 9
11	$\neg\forall x R(x)$	$(\neg\text{I})$, 6–10
12	$\exists x Q(x) \wedge \neg\forall x R(x)$	$(\wedge\text{I})$, 5, 11

1	$\exists x \forall y A(x,y)$	
2	u | v | $\forall y A(v,y)$	
3	| | $A(v,u)$	$(\forall\text{E})$, 2
4	| | $\exists x A(x,u)$	$(\exists\text{I})$, 3
5	| $\exists x A(x,u)$	$(\exists\text{E})$, 1, 2–4
6	$\forall y \exists x A(x,y)$	$(\forall\text{I})$, 2–5

The second is not reversible: for example, let $A(x,y)$ be "x is y's parent": then while it is certainly true that everyone has a parent, it is not necessarily true that there is someone who is everyone's parent.

3.

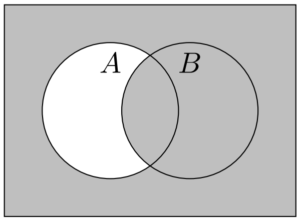
1	$\exists x(S(x) \wedge L(x,\mathsf{j}))$	
2	$\forall x \forall y\,(T(x) \wedge S(y) \to L(x,y))$	
3	$T(\mathsf{j})$	
4	u | $S(u) \wedge L(u,\mathsf{j})$	
5	| $S(u)$	$(\wedge\text{E})$, 4
6	| $L(u,\mathsf{j})$	$(\wedge\text{E})$, 4
7	| $\forall y\,(T(\mathsf{j}) \wedge S(y) \to L(\mathsf{j},y))$	$(\forall\text{E})$, 2
8	| $T(\mathsf{j}) \wedge S(u) \to L(\mathsf{j},u)$	$(\forall\text{E})$, 7
9	| $T(\mathsf{j}) \wedge S(u)$	$(\wedge\text{I})$, 3, 5
10	| $L(\mathsf{j},u)$	$(\to\text{E})$, 8, 9
11	| $L(u,\mathsf{j}) \wedge L(\mathsf{j},u)$	$(\wedge\text{I})$, 6, 10
12	| $\exists x(L(x,\mathsf{j}) \wedge L(\mathsf{j},x))$	$(\exists\text{I})$, 11
13	$\exists x(L(x,\mathsf{j}) \wedge L(\mathsf{j},x))$	$(\exists\text{E})$, 1, 4–12

OK variants for line 2: $\forall x \forall y\,(T(x) \to (S(y) \to L(x,y)))$ or $\forall x(T(x) \to \forall y\,(S(y) \to L(x,y)))$, with corresponding minor variatiations to the derivation in lines 7–10.

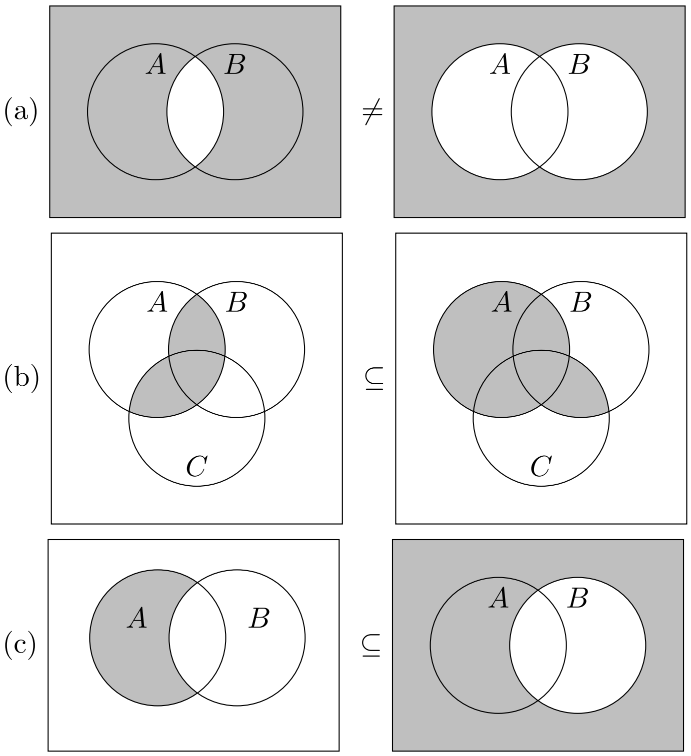
1	$\exists x(S(x) \land L(x,\mathrm{j}))$	
2	$\forall x \forall y\,(T(x) \to (S(y) \to L(x,y)))$	
3	$\underline{T(\mathrm{j})}$	
4	u | $\underline{S(u) \land L(u,\mathrm{j})}$	
5	| $S(u)$	($\land$E), 4
6	| $L(u,\mathrm{j})$	($\land$E), 4
7	| $\forall y\,(T(\mathrm{j}) \to (S(y) \to L(x,y)))$	($\forall$E), 2
8	| $T(\mathrm{j}) \to (S(u) \to L(\mathrm{j},u)$	($\forall$E), 7
9	| $S(u) \to L(\mathrm{j},u)$	($\to$E), 3, 8
10	| $L(\mathrm{j},u)$	($\to$E), 5, 9
11	| $L(u,\mathrm{j}) \land L(\mathrm{j},u)$	($\land$I), 6, 10
12	| $\exists x(L(x,\mathrm{j}) \land L(\mathrm{j},x))$	($\exists$I), 11
13	$\exists x(L(x,\mathrm{j}) \land L(\mathrm{j},x))$	($\exists$E), 1, 4–12

1	$\exists x(S(x) \land L(x,\mathrm{j}))$	
2	$\forall x(T(x) \to \forall y\,(S(y) \to L(x,y)))$	
3	$\underline{T(\mathrm{j})}$	
4	u | $\underline{S(u) \land L(u,\mathrm{j})}$	
5	| $S(u)$	($\land$E), 4
6	| $L(u,\mathrm{j})$	($\land$E), 4
7	| $T(\mathrm{j}) \to \forall y\,(S(y) \to L(\mathrm{j},y))$	($\forall$E), 2
8	| $\forall y\,(S(y) \to L(\mathrm{j},y))$	($\to$E), 3, 7
9	| $S(u) \to L(\mathrm{j},u)$	($\forall$E), 8
10	| $L(\mathrm{j},u)$	($\to$E), 5, 9
11	| $L(u,\mathrm{j}) \land L(\mathrm{j},u)$	($\land$I), 6, 10
12	| $\exists x(L(x,\mathrm{j}) \land L(\mathrm{j},x))$	($\exists$I), 11
13	$\exists x(L(x,\mathrm{j}) \land L(\mathrm{j},x))$	($\exists$E), 1, 4–12

4. $(A \setminus B)^c = \{x | \neg[x \in A \wedge x \notin B]\} = \{x | x \notin A \vee x \in B\} = A^c \cup B$. The Venn diagram:



5. (a) False (b, c) True. Venn diagrams:



6. $\mathcal{P}\{p,q,r\} = \big\{\emptyset, \{p\}, \{q\}, \{r\}, \{p,q\}, \{p,r\}, \{q,r\}, \{p,q,r\}\big\}$. $\mathcal{P}(\mathcal{P}(A))$ has $2^8 = 256$ elements, including these: $\emptyset, \{\emptyset\}, \{\{q,r\}\}, \{\{p\},\{q\}\}, \{\{p,q,r\},\{p\},\{q\}\}$, (and 251 others ...)

7. Try to cover many of the points in the various readings and discussions on these topics. I did deduct marks for not answering the question asked!