GAPS BETWEEN ZEROS OF $\zeta(s)$ AND THE DISTRIBUTION OF ZEROS OF $\zeta'(s)$

MAKSYM RADZIWIŁŁ

Abstract. We settle a conjecture of Farmer and Ki in a stronger form. Roughly speaking we show that there is a positive proportion of small gaps between consecutive zeros of the zeta-function $\zeta(s)$ if and only if there is a positive proportion of zeros of $\zeta'(s)$ lying very closely to the half-line. Our work has applications to the Siegel zero problem. We provide a criterion for the non-existence of the Siegel zero, solely in terms of the distribution of the zeros of $\zeta'(s)$. Finally on the Riemann Hypothesis and the Pair Correlation Conjecture we obtain near optimal bounds for the number of zeros of $\zeta'(s)$ lying very closely to the half-line. Such bounds are relevant to a deeper understanding of Levinson's method, allowing us to place one-third of the zeros of the Riemann zeta-function on the half-line.

1. Introduction.

The inter-relation between the *horizontal* distribution of zeros of $\zeta(s)$ (denoted $\rho = \beta + i\gamma$) and the *horizontal* distribution of the zeros of $\zeta'(s)$ (denoted $\rho' = \beta' + i\gamma'$) is the basis of Levinson's method [12] allowing us to place one third of the zeros of $\zeta(s)$ on the critical line.

Recently it has been understood that the *horizontal* distribution of the zeros of $\zeta'(s)$ is also related to the *vertical* distribution of zeros of $\zeta(s)$. As an first attempt at capturing such a relationship we have the following conjecture of Soundararajan [16].

Note: Throughout we assume the Riemann Hypothesis. We recall that $\beta' \geq \frac{1}{2}$ for all non-trivial zeros of $\zeta'(s)$ (see [18]) and that this is equivalent to the Riemann Hypothesis.

Conjecture 1 (Soundararajan [16]). *We have*

$$\liminf_{\gamma \to \infty} (\gamma^+ - \gamma) \log \gamma = 0 \tag{A}$$

with γ^+ the ordinate suceeding γ, if and only if

$$\liminf_{\gamma' \to \infty} (\beta' - \tfrac{1}{2}) \log \gamma' = 0 \tag{B}$$

Zhang [19] shows that $A \implies B$ (see also [8] for a partial converse). Ki [11] obtained a necessary and sufficient condition for the negation of B. Ki's result shows that zeros ρ' with $(\beta' - \frac{1}{2}) \log \gamma' = o(1)$ arise not only from small gaps between zeros of $\zeta(s)$ but also, for example, from clusters of regularly spaced zeros of $\zeta(s)$. Therefore given our current knowledge about the zeros of $\zeta(s)$ it is possible for B and the negation of A to co-exist. The assertion A is arithmetically very interesting, since, following an idea of Montgomery (made explicit by Conrey and Iwaniec in [2]) if there are many small gaps between consecutive zeros of $\zeta(s)$ then the class number of $\mathbb{Q}(\sqrt{-d})$ is large and there are no Siegel zeros.

1991 *Mathematics Subject Classification.* Primary: 11M06, Secondary: 11M26.

The author is partially supported by a NSERC PGS-D award.

A more recent attempt at capturing the relation between the distribution of zeros of $\zeta(s)$ and $\zeta'(s)$ is due to Farmer and Ki [4]. Let $w(x)$ be the indicator function of the unit interval. Following Farmer and Ki we introduce two distribution functions,

$$m'(\varepsilon) := \liminf_{T\to\infty} \frac{2\pi}{T\log T} \sum_{T\le\gamma'\le 2T} w\Big(\frac{(\beta'-\frac{1}{2})\log T}{\varepsilon}\Big)$$
$$m(\varepsilon) := \liminf_{T\to\infty} \frac{2\pi}{T\log T} \sum_{T\le\gamma\le 2T} w\Big(\frac{(\gamma^{+}-\gamma)\log T}{\varepsilon}\Big).$$

These are indeed distribution functions, since in a rectangle of length T, both $\zeta(s)$ and $\zeta'(s)$ have asymptotically $N(T) \sim (T/2\pi)\log T$ zeros (see [1]), and it is conjectured that $m'(v) \to 1$ as $v \to \infty$, whereas it is known that $m(v) \to 1$ as $v \to \infty$ (see [16], [7]).

Zhang shows in [19] that if $m(\varepsilon) > 0$ for all $\varepsilon > 0$, then $m'(\varepsilon) > 0$. An analogue of Soundararajan's conjecture would assert that $m(\varepsilon) > 0$ for all $\varepsilon > 0$ if and only if $m'(\varepsilon) > 0$ for all $\varepsilon > 0$. As explained by Farmer and Ki in [4] if for example the zeros are well-spaced with sporadic large gaps, something we cannot rule out at present, then in principle $m'(\varepsilon) > 0$ is not enough to imply $m(\varepsilon) > 0$. Farmer and Ki propose the following alternative conjecture.

Conjecture 2 (Farmer and Ki [4]). *If $m'(\varepsilon) \gg \varepsilon^{v}$ with a $v < 2$ as $\varepsilon \to 0$ then $m(\varepsilon) > 0$ for all $\varepsilon > 0$.*

This is a realistic conjecture since we expect that $m'(\varepsilon) \sim (8/9\pi)\varepsilon^{3/2}$ as $\varepsilon \to 0$ (see [3]). Farmer and Ki comment "we intend this as a general conjecture, applying to the Riemann zeta-function but also to other cases such as a sequence of polynomials with all zeros on the unit circle" and that "stronger statements should be true for the zeta function". Our main result is a proof of Conjecture 2 in a stronger and quantitative form for the Riemann zeta-function.

Main Theorem. *Let $A, \delta > 0$ be given.*
- *If $m'(\varepsilon) \gg \varepsilon^{A}$ as $\varepsilon \to 0$ then $m(\varepsilon^{1/2}) \gg \varepsilon^{A+\delta}$ for all $\varepsilon \le 1$.*
- *If $m(\varepsilon^{1/2}) \gg \varepsilon^{A}$ as $\varepsilon \to 0$ then $m'(\varepsilon) \gg \varepsilon^{A+\delta}$ for all $\varepsilon \le 1$.*

We conjecture that $m'(\varepsilon) \asymp m(\varepsilon^{1/2})$ provided that one of $m(\varepsilon)$ or $m'(\varepsilon)$ is $\gg \varepsilon^{A}$ for some $A > 0$. This is consistent with the expectation that $m(\varepsilon) \sim (\pi/6)\varepsilon^{3}$ and $m'(\varepsilon) \sim (8/9\pi)\varepsilon^{3/2}$ as $\varepsilon \to 0$ (see [3]). Our Main Theorem could be restated as saying that

$$\log m(\varepsilon) \sim \log m'(\varepsilon^{1/2})$$

as $\varepsilon \to 0$ provided that one of $m(\varepsilon)$ or $m'(\varepsilon)$ is greater than ε^{A}. As a consequence of the Main Theorem we obtain estimates for $m'(\varepsilon)$ assuming the Pair Correlation Conjecture.

Corollary 1. *Assume the Pair Correlation Conjecture. Let $\delta > 0$. Then*

$$\varepsilon^{3/2+\delta} \ll m'(\varepsilon) \ll \varepsilon^{3/2-\delta}$$

as $\varepsilon \to 0$.

An assumption on the zero distribution in Corollary 1 is inevitable, since $m'(\varepsilon) \to 0$ implies that almost all the zeros of $\zeta(s)$ are simple. Corollary 1 allows one to quantify the loss in Levinson's method coming from the zeros of $\zeta'(s)$ lying closely to the half-line. Unfortunately

Corollary 1 is a conditional result, and as such it cannot be used to put a greater proportion of the zeros of $\zeta(s)$ on the half-line (see [5] for related work).

A final consequence of our work is a criterion for the non-existence of the Siegel zero in terms of the zeros of $\zeta'(s)$. We state it only for completeness since a stronger result has been obtained by Farmer and Ki [4].

Corollary 2. *Let $A > 0$. If $m'(\varepsilon) \gg \varepsilon^A$, for all $\varepsilon > 0$, then for primitive characters χ modulo q,*

$$L(1;\chi) > (\log q)^{-18}.$$

for all q sufficiently large.

Proof. If $m'(\varepsilon) \gg \varepsilon^A$ for every $\varepsilon > 0$ then $m(1/4) > 0$ by our Main Theorem, hence $L(1;\chi) > (\log q)^{-18}$ for all q sufficiently large by Theorem 1.1 of Conrey-Iwaniec, [2]. $\square$

With some care it is possible to turn the above Corollary into an effective result. By Dirichlet's formula Corollary 2 also implies that the class number of $\mathbb{Q}(\sqrt{-d})$ is at least as large as $c\sqrt{d}(\log d)^{-18}$ with c constant.

Farmer and Ki show that if $m'(\varepsilon) \gg \exp(-\varepsilon^{-1/2+\delta})$ as $\varepsilon \to 0$, for some $\delta > 0$, then there are $N(T)/\log\log T$ ordinates of zeros of $\zeta(s)$ lying in $[T;2T]$ and such that $(\gamma^+ - \gamma)\log\gamma = o(1)$ as $T \to \infty$. Using the result of Conrey and Iwaniec [2] this is enough to rule out the existence of Siegel zeros. It is an interesting question to determine whether, given the current technology, one can increase the exponent $\frac{1}{2}$ in Farmer and Ki's assumption $m'(\varepsilon) \gg \exp(-\varepsilon^{-1/2+\delta})$ and still guarantee the non-existence of Siegel zeros.

2. Main ideas

The first part of our Main Theorem follows from the stronger Theorem 1 below.

Theorem 1. *Let $A, \delta > 0$. There is a constant $C = C(\delta, A)$ such that if $0 < \varepsilon < C$ and $m'(\varepsilon) \geq c\varepsilon^A$ then $m(\varepsilon^{1/2-\delta}) \geq (c/8)\varepsilon^A$.*

The approximate value of $C(\delta, A)$ is $(B\delta/A)^{32A/\delta}$ with B an absolute constant. Theorem 1 follows from two technical Propositions which we now describe. Given a zero $\rho' = \beta' + i\gamma'$ of $\zeta'(s)$ we denote by $\rho_c = \frac{1}{2} + i\gamma_c$ the zero of $\zeta(s)$ lying closest to ρ'. If there are two choices of ρ_c then we pick the one lying closer to the origin. For any ordinate γ of a zero of $\zeta(s)$ we denote by γ^+ the ordinate suceeding γ and by γ^- the ordinate preceeding γ. We denote by $\gamma^\pm$ the ordinate closest to γ. Theorem 1 follows quickly from the following Proposition.

Proposition 1. *Let $0 < \delta, \varepsilon < 1$. Let $S_{\varepsilon,\delta}(T)$ be a set of zeros $\rho' = \beta' + i\gamma'$ of $\zeta'(s)$ such that $T \leq \gamma' \leq 2T$, $\beta' - \frac{1}{2} \leq \varepsilon/\log T$ and*

$$|\gamma_c - \gamma_c^\pm| > \varepsilon^{1/2-\delta}/\log T$$

There is a $C = C(\delta, A)$ such that if $0 < \varepsilon < C$ then $|S_{\varepsilon,\delta}(T)| \leq \varepsilon^A \cdot T\log T$.

The proof of Proposition 1 rests on a Proposition describing the structure the roots of $\zeta'(s)$ lying close to the half-line. The Proposition which we are about to state complements with a corresponding upper bound the classical lower bound,

$$|\rho' - \rho_c| \geq \sqrt{\frac{2\left(\beta' - \frac{1}{2}\right)}{\log T}}$$

valid for all $\rho' = \beta' + i\gamma'$ (see [16]). It might be of independent interest.

Proposition 2. *Let $0 < \delta < 1$, $0 < \varepsilon < c$ with $c > 0$ an absolute constant. Let T be large and $\mathcal{Z} := \mathcal{Z}_{\varepsilon,\delta}(T)$ be a set of $\delta/\log T$ well-spaced ordinates of zeros ρ' such that $\rho' \neq \rho$, $\beta' - \frac{1}{2} \leq \varepsilon/\log T$ and $T \leq \gamma' \leq 2T$. If $|\mathcal{Z}_{\varepsilon,\delta}(T)| \gg \varepsilon^A \cdot T\log T$ then, for any given $\kappa > 0$, all but $\kappa|\mathcal{Z}|$ elements $\rho' \in \mathcal{Z}$ satisfy the inequality,*

$$\sqrt{\frac{\beta' - \frac{1}{2}}{\log T}} \ll |\rho' - \rho_c| \ll \sqrt{A\log(\varepsilon\kappa\delta)^{-1}} \cdot \sqrt{\frac{\beta' - \frac{1}{2}}{\log T}}.$$

The proof of the converse part of our Main Theorem builds on ideas of Zhang, and follows from the following more precise statement valid for any fixed $\varepsilon > 0$.

Theorem 2. *Let $A, \delta > 0$. There is a $C = C(\delta, A)$ such that if $0 < \varepsilon < C$ and $m(\varepsilon^{1/2}) \geq c\varepsilon^A$ then $m'(\varepsilon) \geq (c/4)\varepsilon^{A+\delta}$.*

The paper is organized as follows. Most of the paper, all the way until section 7, is devoted to the proof of the propositions above and the deduction of Theorem 1 from them. Following section 7 we prove Theorem 2 and Corollary 1.

3. Lemma on Dirichlet polynomials

Define,

$$A_N(s) := \sum_{n \leq N} \frac{\Lambda(n)W_N(n)}{n^s}$$

with

$$W_N(n) = \begin{cases} 1 & \text{for } 1 \leq n \leq N^{1/2} \\ \log(N/n)/\log N & \text{for } N^{1/2} < n \leq N \end{cases}$$

The lemma below is due to Selberg.

Lemma 1. *Let $\sigma = \frac{1}{2} + 2/\log N$, with $N \leqslant T$. Then for $T \leqslant t \leqslant 2T$,*

$$\sum_{\rho} \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \ll |A_N(s)| + \log T$$

Proof. This is equation (2.2) in [15]. □

Using the explicit formula we obtain an upper bound for the number of zeros in a small window $[t - 2\pi K/\log t; t + 2\pi K/\log t]$, in terms of the Dirichlet polynomial

$$B_N(s) := \sum_{n \leq N} \frac{\Lambda(n)}{n^s} \cdot \left(1 - \frac{\log n}{\log N}\right).$$

We have the following lemma.

Lemma 2. *For $T \leq t \leq 2T$ and $N \leq T$,*

$$N\big(t + \frac{\pi}{\log N}\big) - N\big(t - \frac{\pi}{\log N}\big) \ll \frac{\log T}{\log N} + \frac{|B_N(\frac{1}{2} + it)|}{\log N}$$

Proof. Let

$$F_\Delta(v) = \left(\frac{\sin \pi\Delta v}{\pi\Delta v}\right)^2$$

be the Fejer kernel. The Fourier transform of $F_\Delta(v)$ is for $|x| < \Delta$

$$\widehat{F}_\Delta(x) := \int_{-\infty}^{\infty} F_\Delta(t)e^{-2\pi itx}dx = \frac{1}{\Delta}\left(1 - \frac{x}{\Delta}\right)$$

and $\widehat{F}_\Delta(x) = 0$ for $|x| > \Delta$. By the explicit formula (see Lemma 1 in [9]),

$$(3) \quad \sum_\gamma F_\Delta(\gamma - t) = O(e^{\pi\Delta/2}\cdot t^{-2} + 1/\Delta) + \frac{1}{2\pi}\int_{-\infty}^{\infty} F_\Delta(u-t)\cdot \Re\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right)du$$
$$- \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left(\widehat{F}_\Delta\left(\frac{\log n}{2\pi}\right) + \widehat{F}_\Delta\left(\frac{-\log n}{2\pi}\right)\right)$$

The integral over u is bounded by $\ll (\log t)/\Delta$. On the other hand the prime sum is bounded by,

$$\left|\frac{1}{2\pi\Delta}\sum_{n\le e^{2\pi\Delta}}\frac{\Lambda(n)}{n^{1/2+it}}\cdot\left(1 - \left|\frac{\log n}{2\pi\Delta}\right|\right)\right|$$

Finally $(\pi/2)^2\sum_\gamma F_\Delta(\gamma - t)$ is an upper bound for the number of zeros in the interval going from $t - 1/(2\Delta)$ to $t + 1/(2\Delta)$. If $T \le t \le 2T$ we choose $2\pi\Delta = \log N$ and we are done. $\square$

In order to understand the average behavior of the Dirichlet polynomials $A_N(s)$ and $B_N(s)$ we use a version of the large sieve.

Lemma 3. *Let $A(s)$ be a Dirichlet polynomial with positive coefficients and of length x. Let $s_r = \sigma_r + it_r$ be points with $T \le t_r \le 2T$ and $0 \le \sigma_r - \frac{1}{2} \le \varepsilon/\log T$ for some small $\varepsilon > 0$. Suppose that $|t_i - t_j| \ge \delta/\log T$ for $i \ne j$, with $100\varepsilon < \delta < 1$. Then, for $x^k \le T$,*

$$\sum_{s_r}|A(s_r)|^{2k} \le \frac{20\log T}{\delta}\int_{-2T}^{2T}|A(\tfrac{1}{2} + it)|^{2k}dt$$

Proof. Let $D(s) = A(s)^k$. For any s we have, with $\mathcal{C}$ a circle of radius $\delta/(2\log T)$ around s,

$$|D(s)|^2 \le \frac{4(\log T)^2}{\pi\delta^2}\iint_{\mathcal{C}}|D(x+iy)|^2dxdy$$

Summing over all $s = s_r$, since the circles are disjoint we obtain,

$$\sum_{s_r}|D(s_r)|^2 \le \frac{4(\log T)^2}{\pi\delta^2}\int_{\frac{1}{2}-\delta/\log T}^{\frac{1}{2}+\delta/\log T}\int_{T-1}^{2T+1}|D(\sigma+it)|^2dtd\sigma$$

Since the coefficients of $D(s)$ are positive, and D is of length at most T, by a majorant principle (see Chapter 3, Theorem 3 in [13]), the inner integral is bounded by

$$\le 3e^{2\delta}\int_{-2T}^{2T}|D(\tfrac{1}{2}+it)|^2dt$$

Since in addition $\delta < 1$, the claim follows (we obtain a constant of $8e^2/\pi < 20$). $\square$

Combining the above lemma with Chebyschev's inequality allows us to understand the average size of the Dirichlet polynomials $A_N(s)$ and $B_N(s)$.

Lemma 4. *Let $s_r = \sigma_r + it_r$ be a set of well-spaced points as appearing in the statement of Lemma 3. Suppose that $N^k \le T/\log T$. The number of points s_r for which we have*

$$|A_N(s_r)| > (k/e)\log N \ \text{ or } |B_N(s_r)| > (k/e)\log N$$

is bounded above by $\ll (e^{-k}/\delta) T \log T$.

Proof. Let $L_N(s)$ be either $A_N(s)$ or $B_N(s)$. Let

$$D_N(s) = \sum_{n\le N} \frac{\Lambda(n)}{n^s}.$$

By a majorant principle (see Chapter 3, Theorem 3 in [13]) we have,

$$\int_{-2T}^{2T} |L_N(\tfrac{1}{2}+it)|^{2k} dt \le 3 \int_{-2T}^{2T} |D_N(\tfrac{1}{2}+it)|^{2k} dt$$

By Soundararajan's lemma 3 in [17], for $N^k \le T/\log T$ we have,

$$\int_{-2T}^{2T} |D_N(\tfrac{1}{2}+it)|^{2k} dt \ll k! T (\log N)^{2k}$$

Therefore, for $N^k \le T/\log T$, by the previous lemma,

$$\sum |L_N(s_r)|^{2k} \ll \frac{k!}{\delta} \cdot T \log T (\log N)^{2k}$$

It follows that for $N^k \le T/\log T$, the number of points s_r for which $|L_N(s_r)| > B\log N$ is less than,

$$\ll \left(\frac{k}{B}\right)^k \cdot (T/\delta)\log T$$

Choosing $B = k/e$ we conclude that the number of points for which $|L_N(s_r)| > k/e$ is bounded by $(e^{-k}/\delta) T\log T$ as desired. $\square$

Lemma 5. *Let $0 < c < 1$. Uniformly in $T \leqslant t \leqslant 2T$ and $N \leqslant T$,*

$$\frac{\zeta'}{\zeta}(s) = \sum_{|s-\rho|<c/\log T} \frac{1}{s-\rho} + O\left(\frac{\log T}{c} \cdot \mathcal{E}_{T,N}(s)\right)$$

where

$$\mathcal{E}_{T,N}(s) := \frac{1}{\log N} \cdot \left(|A_N(s)| + |B_N(\tfrac{1}{2}+it)|\right) + \frac{\log T}{\log N}.$$

Furthermore, if s_r is a set of well-spaced points as in Lemma 3, and $N^k \le T/\log T$, then

$$\sum_{s_r} |\mathcal{E}_{T,N}(s_r)|^{2k} \ll (k^{2k}/\delta) T \log T.$$

Proof. Selberg shows in [14] (see equation (14) on page 4) that

$$\frac{\zeta'}{\zeta}(s) = \sum_{|s-\rho|<(\log T)^{-1}} \frac{1}{s-\rho} + O\bigg(\frac{\log T}{\log N}\cdot |A_N(s)| + \frac{\log^2 T}{\log N}\bigg)$$

It suffices to notice that the contribution of the zeros ρ with $c(\log T)^{-1} < |s-\rho| < (\log T)^{-1}$ is bounded above by

$$\ll \frac{\log T}{c}\cdot\bigg(N\big(t+\frac{\pi}{\log N}\big) - N\big(t-\frac{\pi}{\log N}\big)\bigg) \ll \frac{\log T}{c}\cdot\bigg(\frac{\log T}{\log N} + \frac{1}{\log N}\cdot|B_N(\tfrac{1}{2}+it)|\bigg)$$

Combining the above two equations we obtain the first part of the lemma. Now it remains to estimate the moments of $\mathcal{E}_{T,N}$. We have,

$$\sum_{s_r} |\mathcal{E}_{T,N}(s_r)|^{2k} \ll \bigg(\frac{C}{\log N}\bigg)^{2k}\cdot\bigg(\sum_{s_r}|A_N(s_r)|^{2k} + \sum_{s_r}|B_N(s_r)|^{2k}\bigg) + ((Ck)^{2k}/\delta)T\log T$$

with $C > 0$ an absolute constant. Using Lemma 3 and proceeding as in Lemma 4 we find that the $2k$-th moments of the Dirichlet polynomials A_N and B_N is bounded above by $(k!/\delta)T\log T(\log N)^{2k}$. Hence we conclude that the $2k$-th moment of $\mathcal{E}_{N,T}$ is bounded above by $((Ck)^{2k}/\delta)T\log T$

□

4. Proof of Proposition 2

The proof of Proposition 2 rests on the following classical lemma.

Lemma 6. *If $\rho' \neq \rho$ then,*

$$\frac{1}{2}\cdot\log\gamma' = \sum_{\rho} \frac{\beta'-\frac{1}{2}}{(\beta'-\frac{1}{2})^2+(\gamma'-\gamma)^2} + O(1).$$

Proof. See Zhang [19], Lemma 3. □

We will show that on average the zero $\rho = \rho_c$ dominates, the claim then follows shortly. In order to simplify the notation we define, as in the previous section,

$$A_N(s) := \sum_{n\le N} \frac{\Lambda(n)W_N(n)}{n^s}$$

with $W_N(n)$ the same smoothing as defined in the previous section. We also define

$$B_N(s) := \sum_{n\le N}\frac{\Lambda(n)}{n^s}\cdot\bigg(1-\frac{\log n}{\log N}\bigg).$$

On average both Dirichlet polynomials are of size $\log N$.

Proof of Proposition 2. Let $N \le T$ to be fixed later. In the formula

$$\tfrac{1}{2}\cdot\log\gamma' = \sum_{\rho}\frac{\beta'-\frac{1}{2}}{(\beta'-\frac{1}{2})^2+(\gamma'-\gamma)^2} + O(1) \tag{4}$$

The contribution of the ρ's for which $|\gamma - \gamma'| < \pi(\log N)^{-1}$ is bounded above by

$$(5) \quad \ll \left(N\big(\gamma' + \frac{\pi}{\log N}\big) - N\big(\gamma' - \frac{\pi}{\log N}\big) \right) \cdot \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2}$$
$$\ll \left(\frac{\log T}{\log N} + \frac{1}{\log N} \cdot |B_N(\tfrac{1}{2} + i\gamma')| \right) \cdot \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2}.$$

by Lemma 2. On the other hand, to bound the contribution of the ρ's for which $|\gamma - \gamma'| > \pi(\log N)^{-1}$ we notice that if $|\gamma' - \gamma| > \pi(\log N)^{-1}$ then

$$(\beta' - \tfrac{1}{2})^2 + (\gamma - \gamma')^2 \gg (2/\log N)^2 + (\gamma - \gamma')^2.$$

Therefore the contribution of the ρ's with $|\gamma - \gamma'| > \pi(\log N)^{-1}$ to (4) is bounded above by

$$(6) \quad \ll (\beta' - \tfrac{1}{2}) \log N \cdot \left(\sum_{\rho} \frac{2/\log N}{(2/\log N)^2 + (\gamma - \gamma')^2} \right)$$
$$\ll (\beta' - \tfrac{1}{2}) \log N \cdot \left(\log T + \left| A_N \left(\frac{1}{2} + \frac{1}{\log N} + i\gamma' \right) \right| \right)$$

by Lemma 1. Combining (4), (5) and (6) we conclude that

$$(7) \quad \log T \ll \left(\frac{\log T}{\log N} + \frac{1}{\log N} \cdot |B_N(\tfrac{1}{2} + i\gamma')| \right) \cdot \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2} +$$
$$+ (\beta' - \tfrac{1}{2}) \log N \cdot \left(\log T + \left| A_N \left(\frac{1}{2} + \frac{1}{\log N} + i\gamma' \right) \right| \right)$$

Suppose that $N^k \leq T/\log T$ with a k to be fixed later and N the largest integer such that $N^k < T/\log T$. By Lemma 4 the number of $\rho' \in \mathcal{Z}_{\varepsilon,\delta}$ for which $|B_N(\frac{1}{2} + i\gamma')| > (k/e)\log N$ is bounded above by $c(e^{-k}/\delta)T\log T$ with c a constant. Similarly the number of $\rho' \in \mathcal{Z}_{\varepsilon,\delta}$ for which $|A_N(\frac{1}{2} + 1/\log N + i\gamma')| > (k/e)\log N$ is also bounded by above by $c(e^{-k}/\delta)T\log T$. Choose k so that $ce^{-k}/\delta T \log T \leq (\kappa/2)|\mathcal{Z}_{\varepsilon,\delta}|$. Since $|\mathcal{Z}_{\varepsilon,\delta}| \geq c_1\varepsilon^A T\log T$ we can take k to be the closest integer to $c_2 A \log(\kappa\varepsilon\delta)^{-1}$ with c_2 an absolute constant. Choose N to be the largest integer such that $N^k \leq T/\log T$. With this choice of k and N it follows that for at most $\kappa|\mathcal{Z}_{\varepsilon,\delta}|$ elements $\rho' \in \mathcal{Z}_{\varepsilon,\delta}$ we have $|B_N(\frac{1}{2}+i\gamma')| \geq (k/e)\log N$ or $|A_N(\frac{1}{2}+i\gamma')| \geq (k/e)\log N$. It follows that for all but at most $\kappa|\mathcal{Z}_{\varepsilon,\delta}|$ of the $\rho' \in \mathcal{Z}_{\varepsilon,\delta}$ we have,

$$c\log T \leq k \cdot \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2} + \frac{1}{k} \cdot (\beta' - \tfrac{1}{2}) \cdot (\log T)^2$$

with $c > 0$ an absolute constant. If ε is choosen so that $\varepsilon < (c/2)$ then (since $\beta' - \frac{1}{2} < \varepsilon/\log T$) we obtain

$$(c/2)\log T \leq k \cdot \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2}$$

hence $|\rho_c - \rho'|^2 \leq (k(\beta' - \frac{1}{2})/\log T)^{1/2}$ which gives the desired bound for all but at most $\kappa|\mathcal{Z}_{\varepsilon,\delta}|$ elements $\rho' \in \mathcal{Z}_{\varepsilon,\delta}$. (Recall that $k \ll A\log(\varepsilon\delta\kappa)^{-1}$)). $\square$

5. Proof of Proposition 1

The lemma below is critical, in that it allows us to produce a sufficiently dense well-spaced sequence of zeros of $\zeta'(s)$.

Lemma 7 (Soundararajan [16]). *Suppose that $\rho_1 = \frac{1}{2} + \mathrm{i}\gamma_1$ and $\rho_2 = \frac{1}{2} + \mathrm{i}\gamma_2$ are two consecutive zeros of $\zeta(s)$ with $T \leqslant \gamma_1 < \gamma_2 \leqslant 2T$ for large T. Then the box,*

$$\{s = \sigma + \mathrm{i}t : \tfrac{1}{2} \leqslant \sigma < \tfrac{1}{2} + 1/\log T, \gamma_1 < t < \gamma_2\}$$

contains at most one zero (counted with multiplicity) of $\zeta'(s)$.

Proof. The only way that ρ' can lie on the critical line is if $\rho' = \rho$. Since $\gamma_1 < t < \gamma_2$ this possibility is excluded. As for the box $\frac{1}{2} < \sigma < \frac{1}{2} + 1/\log T$ we know by Soundararajan's work [16] (see Proposition 6) that the box $\frac{1}{2} < \sigma < \frac{1}{2} + 1/\log T$ with t in $[\gamma_1, \gamma_2]$ can contain at most on zero of $\zeta'(s)$, counted with multiplicity. □

We are now ready to prove Proposition 1.

Proof of Proposition 1. Suppose that $S = S_{\varepsilon,\delta}(T) > \varepsilon^A \cdot T \log T$. We will show that this leads to a contradiction when $0 < \varepsilon < C(\delta, A)$ with $C(\delta, A)$ some explicit constant depending only on δ and A (for example we could take $C(\delta, A) = (c\delta/A)^{32A/\delta}$ with $c > 0$ an absolute constant). Since each $\rho' \in S$ satisfies $\gamma_c^- \leq \gamma' \leq \gamma_c^+$ and $|\gamma_c - \gamma_c^{\pm}| > \varepsilon^{1/2-\delta}/\log T$ by the above lemma for each $\rho' \in S$ there is at most one zero of $\zeta'(s)$ in $[\gamma_c^-, \gamma_c]$ and at most one zero of $\zeta'(s)$ in $[\gamma_c, \gamma_c^+]$.

We construct a subset S' of S by skipping every second element in S. This produces a subset of at least $(1/2)|S|$ elements, with the property that the ordinates of elements of S' are $\varepsilon^{1/2-\delta}/\log T$ well-spaced, because $|\gamma_c - \gamma_c^{\pm}| \geq \varepsilon^{1/2-\delta}/\log T$ for each $\rho' \in S$.

By Proposition 2, we have for at least half of the $\rho' \in S'$,

$$|\gamma' - \gamma_c| \leq |\rho' - \rho_c| \leq \frac{C\sqrt{A\varepsilon \log(\varepsilon)^{-1}}}{\log T}. \tag{8}$$

with $C > 0$ an absolute constant. We call S'' the subset of S' satisfying the above inequality. Since $|\gamma_c^{\pm} - \gamma_c| > \varepsilon^{1/2-\delta}/\log T$ for each $\rho' \in S''$ the interval $|\gamma' - t| \leq \varepsilon^{1/2-\delta}/\log T$ contains exactly one ordinate of a zero of $\zeta(s)$ (namely γ_c) once ε is choosen so small so as to make the right-hand side of (8) less than $\varepsilon^{1/2-\delta}/\log T$ (for example $\varepsilon < (\delta/CA)^{2/\delta}$ would suffice).

Using Lemma 5, we have at $s = \rho' \in S$,

$$\sum_{|s-\rho|<c/\log T} \frac{1}{s-\rho} \ll \frac{\log T}{c} \cdot |\mathcal{E}_{T,N}(s)|$$

Choose $s = \rho' \in S''$, $c = \varepsilon^{1/2-\delta}$ and N the largest integer such that $N^k \leq T/\log T$ with a k to be fixed later (ultimately $k = \lceil (A+1)/\delta \rceil$). By our previous remark the left-hand side of the above expression consists of only one term $(\rho' - \rho_c)^{-1}$. Raising the above expression to the $2k$-th power and then summing over all $\rho' \in S''$ we obtain

$$\sum_{\rho' \in S'} \frac{1}{|\rho' - \rho_c|^{2k}} \ll \varepsilon^{-k+2k\delta} \cdot (C \log T)^{2k} \sum_{\rho' \in S'} |\mathcal{E}_{T,N}(\rho')|^{2k} \tag{9}$$

$$\ll \varepsilon^{-k+2k\delta} \cdot ((Ck)^{2k}/\varepsilon^{1/2-\delta}) \cdot T(\log T)^{2k+1} \tag{10}$$

by Lemma 5, with $C > 0$ an absolute constant (not necessarily the same in each occurence). Since for each $\rho' \in S''$ we have,

$$|\rho' - \rho_c| \ll \frac{\sqrt{A\varepsilon \log(\varepsilon)^{-1}}}{\log T}$$

the left-hand side of (9) is at least

$$\sum_{\rho' \in S'} \frac{1}{|\rho' - \rho_c|^{2k}} \gg |S''| \cdot (C/A)^k \cdot \varepsilon^{-k} (\log(\varepsilon)^{-1})^{-k} (\log T)^{2k} \tag{11}$$

$$\gg (C/A)^k \cdot \varepsilon^{A-k} \cdot (\log(\varepsilon)^{-1})^{-k} \cdot T(\log T)^{2k+1} \tag{12}$$

since $|S''| \gg \varepsilon^A T \log T$. Combining the upper bound (9) and the lower bound (11) we get

$$\varepsilon^{A-k}(\log(\varepsilon)^{-1})^{-k} \le \varepsilon^{-k-1/2+(2k+1)\delta} \cdot (CAk)^{2k}$$

with $C > 0$ an absolute constant. The above inequality simplifies to

$$\varepsilon^{A+1/2} \le (CAk)^{2k} \cdot \varepsilon^{(2k+1)\delta} \cdot (\log(\varepsilon)^{-1})^k.$$

Using the inequality $(\log x) \le x^\delta/\delta$ we obtain

$$\varepsilon^{A+1/2} \le (CAk/\delta)^{2k} \cdot \varepsilon^{k\delta}$$

Choosing k to be the smallest integers with $k\delta > A + 1$ we obtain a contradiction once $\varepsilon < (2C\delta^2/A^2)^{16A/\delta}$ with C an absolute constant. (Note: We have certainly not tried to optimize the constant $C(\delta, A)$). $\square$

6. Proof of Theorem 1.

Let T be large. By assumption each interval $[T; 2T]$ contains at least $c\varepsilon^A N(T)$ ordinates $T \le \gamma' \le 2T$ with $\beta' - \frac{1}{2} < \varepsilon/\log T$. If $\rho' = \rho$ for more than half of these zeros of $\zeta'(s)$, then we have $\ge (c/2)\varepsilon^A N(T)$ zeros ρ with $\gamma^+ = \gamma$ and so we are done.

Thus we can assume that there are $\ge (c/2)\varepsilon^A N(T)$ zeros ρ' with $T \le \gamma' \le 2T$, $\rho' \ne \rho$ and $\beta' - \frac{1}{2} < \varepsilon/\log T$. We call the set of such ρ' by S. By Lemma 7 between any two consecutive zeros of $\zeta(s)$ there is at most one $\rho' \in S$. For each $\rho' \in S$ consider two possibilities

(1) $|\gamma_c^{\pm} - \gamma_c| \le \varepsilon^{1/2-\delta}/\log T$
(2) $|\gamma_c^{\pm} - \gamma_c| > \varepsilon^{1/2-\delta}/\log T$

Call S_2 the subset of S for which the second possibility holds. If the second possibility holds for at least one half of the elements in S then $|S_2| \ge (c/2)\varepsilon^A T \log T$. But this is impossible by Proposition 1 once ε is less than $(c/4)C(\delta, A+1)$, with $C(\delta, A)$ as in the statement of Proposition 1. Therefore the second possibility can hold for *at most* one half of the elements in S. Hence the first possibility holds for *at least* a half of the elements in S. Call S_1 the subset of S for which the first possibility holds.

By Lemma 7, there are no two $\rho' \in S_1$ lying between the same tuple of consecutive zeros of $\zeta(s)$. Every $\rho' \in S_1$ lies either between $[\gamma_c^-, \gamma_c]$ or $[\gamma_c; \gamma_c^+]$ and moreover one of these intervals is of length $\le \varepsilon^{1/2-\delta}/\log \gamma_c$. Skipping every second $\rho' \in S_1$ we make sure that no two $\rho_1 \in S_1$ and $\rho_2 \in S_1$ lie between the same set of consecutive zeros. Therefore every second $\rho' \in S_1$ gives rise to one (new) zero γ (namely γ_c or γ_c^-) with $(\gamma^+ - \gamma)\log\gamma \le \varepsilon^{1/2-\delta}$. Thus we have at least $(1/2)|S_1| \ge (c/8)\varepsilon^A \cdot T \log T$ zeros $T \le \gamma \le 2T$ such that $(\gamma^+ - \gamma)\log\gamma \le \varepsilon^{1/2-\delta}$.

7. Lemma: Zeros of the Riemann zeta-function

In this section we collect a few facts concerning the zeros of the Riemann zeta-function. They will be used in the proof of Theorem 2 and Corollary 1. We first need Gonek's lemma.

Lemma 8 (Gonek [10]). *If $x = a/b \neq 1$ and $a, b \leq N$, then,*

$$\sum_{T \leq \gamma \leq 2T} x^{i\gamma} \ll N \log^2 T$$

Proof. As noted by Ford and Zaharescu (Lemma 1, [6]), it follows from Gonek's work that,

$$\sum_{T \leq \gamma \leq 2T} x^{1/2+i\gamma} = -\frac{\Lambda(n_x)}{2\pi} \frac{e^{iT\log(x/n_x)} - 1}{i\log(x/n_x)} + O\left(x \log^2(2xT) + \frac{\log 2T}{\log x}\right).$$

Since x is not an integer we have $x \neq n_x$. Therefore the closest that $|x/n_x| = |a/(bn_x)|$ can be to 1 is when bn_x is equal to $a \pm 1$. This shows that $|\log(x/n_x)| \gg a^{-1} \gg N^{-1}$. Therefore the main term in the above equation is bounded by $N \log T$, This gives a bound of $\sum_{T \leq \gamma \leq 2T} x^{i\gamma} \ll N/\sqrt{x} \log T + \sqrt{x} \log^2 T$ for $x > 1$. For $x < 1$ this bound is reversed to $\sqrt{x} N \log T + \log^2 T/\sqrt{x}$. In either case the final bound is $\ll N \log^2 T$ because $N^{-1} \leq |x| \leq N$. ∎

An quick consequence of the above lemma is a bound for Dirichlet polynomials.

Lemma 9. *Let $B_N(s)$ be as in Lemma 2. If $N^k \leq \sqrt{T}$ then,*

$$\sum_{T \leq \gamma \leq 2T} |B_N(\tfrac{1}{2} + i\gamma)|^{2k} \ll (Ck)^k \cdot T \log T \cdot (\log N)^{2k}$$

for some absolute constant $C > 0$.

Proof. First notice that for $T \leq t \leq 2T$

$$\sum_{\substack{p^k \leq N \\ k > 1}} \frac{\log p}{p^{k/2+kit}} \cdot \left(1 - \frac{\log p^k}{\log N}\right) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\zeta'}{\zeta}(s + 1 + 2it) \cdot \frac{N^{s/2} ds}{s^2 \log \sqrt{N}} + O(1)$$

$$= -\frac{N^{-it}}{2t^2 \log N} + \frac{\zeta'}{\zeta}(1 + 2it) + O\left(1 + \frac{\log T}{\log N} \cdot N^{-1/8}\right)$$

and that the above expression is less than $\ll \log\log T$ by a classical estimate for the size of ζ'/ζ on the Riemann Hypothesis. Therefore,

$$\sum_{T \leq \gamma \leq 2T} |B_N(\tfrac{1}{2} + i\gamma)|^{2k} \ll C^k \sum_{T \leq \gamma \leq 2T} \left| \sum_{p \leq N} \frac{\log p}{p^{1/2+i\gamma}} \cdot \left(1 - \frac{\log p}{\log N}\right) \right|^{2k} + T \log T \cdot (C \log\log T)^{2k}.$$

with $C > 0$ some absolute constant. We denote the coefficients of the Dirichlet polynomial over primes by $a(p)$. We have,

$$\sum_{T \leq \gamma \leq 2T} \left| \sum_{p \leq N} a(p) p^{-i\gamma} \right|^{2k} = \sum_{\substack{p_1, \ldots, p_k \leq N \\ q_1, \ldots, q_k \leq N}} a(p_1) \ldots a(p_k) a(q_1) \ldots a(q_k) \sum_{T \leq \gamma \leq 2T} \left(\frac{p_1 \ldots p_k}{q_1 \ldots q_k}\right)^{i\gamma}$$

The diagonal terms $p_1 \ldots p_k = q_1 \ldots q_k$ contribute at most

$$\ll T \log T \cdot k! \cdot \Big(2 \sum_{p \le N} |a(p)|^2 \Big)^k \ll (Ck)^k T \log T \cdot (\log N)^{2k}$$

because given $q_1, \ldots, q_k$ all the solutions to the equation $p_1 \ldots p_k = q_1 \ldots q_k$ are obtained by pairing together each prime p_i with some other prime q_j, and there is at most $k!$ such pairings. To bound the off-diagonal terms $p_1 \ldots p_k \neq q_1 \ldots q_k$ we notice that $p_1 \ldots p_k \le N^k \le \sqrt{T}$ and similarly that $q_1 \ldots q_k \le N^k \le \sqrt{T}$. Therefore by Gonek's lemma

$$\sum_{T \le \gamma \le 2T} \Big(\frac{p_1 \ldots p_k}{q_1 \ldots q_k} \Big)^{i\gamma} \le \sqrt{T} \log^2 T.$$

Since $\sum_{p \le N} a(p) \ll \sqrt{N}$ it follows that the off-diagonal terms contribute at most $C^k N^k \cdot \sqrt{T} \log^2 T \ll C^k T \log^2 T$, which is less than the main term as soon as $k > 0$ $\square$

An immediate consequence of the above lemma is the following.

Lemma 10. *Let $T \le t \le 2T$. Then,*

$$\sum_{T \le \gamma \le 2T} \Big| N\big(\gamma + \frac{2\pi}{\log T}\big) - N\big(\gamma - \frac{2\pi}{\log T}\big) \Big|^{2k} \ll (Ck)^{2k} \cdot T \log T.$$

with $C > 0$ an absolute constant.

Proof. Let N be the largest integer such that $N^k \le \sqrt{T}$. We have

$$\begin{aligned} N\big(\gamma + \frac{2\pi}{\log T}\big) - N\big(\gamma - \frac{2\pi}{\log T}\big) &\le N\big(\gamma + \frac{\pi}{\log N}\big) - N\big(\gamma - \frac{\pi}{\log N}\big) \\ &\ll \frac{\log T}{\log N} + \frac{|B_N(\frac{1}{2} + i\gamma)|}{\log N} \end{aligned}$$

by Lemma 2. Raising the above expression to the $2k$-th power and then summing over all $T \le \gamma \le 2T$ we obtain

$$\sum_{T \le \gamma \le 2T} \Big| N\big(\gamma + \frac{2\pi}{\log T}\big) - N\big(\gamma - \frac{2\pi}{\log T}\big) \Big|^{2k} \ll (Ck)^{2k} \cdot T \log T + \frac{C^{2k}}{(\log N)^{2k}} \sum_{T \le \gamma \le 2T} |B_N(\tfrac{1}{2} + i\gamma)|^{2k}$$

with $C > 0$ an absolute constant. By the previous lemma the sum over $T \le \gamma \le 2T$ is $\ll (Ck)^k \cdot T \log T \cdot (\log N)^{2k}$ and so the claim follows. $\square$

Corollary 3. *Let $A > 0$ and $\delta > 0$ be given. If $0 < \varepsilon < C(\delta, A)$, with $C(\delta, A)$ depending only on δ and A, then,*

$$\#\Big\{ T \le \gamma \le 2T : N\big(\gamma + \frac{2\pi}{\log T}\big) - N\big(\gamma - \frac{2\pi}{\log T}\big) > \varepsilon^{-\delta} \Big\} \le \varepsilon^{A+1} \cdot T \log T.$$

Proof. By the previous lemma we have for $k > 1$,

$$\sum_{T \le \gamma \le 2T} \Big| N\big(\gamma + \frac{2\pi}{\log T}\big) - N\big(\gamma - \frac{2\pi}{\log T}\big) \Big|^{2k} \ll (Ck)^{2k} \cdot T \log T$$

with $C > 0$ a positive absolute constant. Therefore the number of $T \le \gamma \le 2T$ for which the interval $[\gamma - 2\pi/\log T; \gamma + 2\pi/\log T]$ contains more than $\varepsilon^{-\delta}$ zeros is bounded above by $\varepsilon^{2k\delta}(Ck)^{2k} \cdot T \log T$. Choose $k = \lceil A/\delta \rceil$. Then $\varepsilon^{2k\delta}(Ck)^{2k} \le \varepsilon^{A}$ provided that $\varepsilon \le (cA/\delta)^{-4/\delta})$ with $c > 0$ an absolute constant. □

8. Proof of Theorem 2

We will require the following two lemma.

Lemma 11 (Zhang [19])**.** *Let $\varepsilon < 1$. If $\rho = \frac{1}{2} + i\gamma$ is a zero of $\zeta(s)$ such that γ is sufficiently large and $(\gamma^{+} - \gamma)\log\gamma < \varepsilon$ then there exists a zero ρ' of $\zeta'(s)$ such that*

$$|\rho' - \rho| \le \frac{2\varepsilon}{\log \gamma}.$$

Lemma 12 (Soundararajan [16])**.** *We have,*

$$|\rho' - \rho_c|^2 \ge \frac{2\left(\beta' - \frac{1}{2}\right)}{\log \gamma'}.$$

We are now ready to prove Theorem 2.

Proof of Theorem 2. Suppose that there are at least $c\varepsilon^{A} \cdot T\log T$ zeros $T \le \gamma \le 2T$ such that $(\gamma^{+} - \gamma)\log\gamma \le \varepsilon^{1/2}$. Call this set S. If $\gamma^{+} = \gamma$ for at least a half of the elements in S then $\rho' = \rho$ and hence $\beta' = \frac{1}{2}$ for at least $(c/2)\varepsilon^{A} \cdot T\log T$ zeros.

Hence suppose that $\gamma^{+} > \gamma$ for at least half of the elements in S and call the subset of such elements S_1. By Corollary 3, the number of $T \le \gamma \le 2T$ such that the interval $[\gamma - 2\pi/\log T; \gamma + 2\pi/\log T]$ contains more than $\varepsilon^{-\delta}$ zeros is $\le (c/4)\varepsilon^{A} \cdot T\log T$, provided that ε is small enough with respect to δ and A. Therefore there is a subset S_2 of S_1 of cardinality $\ge (c/4)\varepsilon^{A} \cdot T\log T$ with the properties that $0 < (\gamma^{+} - \gamma)\log\gamma < \varepsilon^{1/2}$ and the number of zeros in the interval $[\gamma - 2\pi/\log T, \gamma + 2\pi/\log T]$ is less than $\varepsilon^{-\delta}$.

By Lemma 10 each $\rho \in S_2$ gives rise to a zero ρ' such that $|\rho' - \rho| \le 2\sqrt{\varepsilon}/\log T$. By Lemma 11 the zero ρ' satisfies $(\beta' - \frac{1}{2})\log\gamma \le \varepsilon$. Furthermore the interval $|t - \gamma| < 2\sqrt{\varepsilon}/\log T$ contains at most $\varepsilon^{-\delta}$ zero. Therefore striking out at most $\varepsilon^{-\delta}$ zeros from S_2 we obtain each time a new and distinct zero ρ' of $\zeta'(s)$. It follows that $\varepsilon^{\delta}|S_2|$ is a lower bound for the number of zeros ρ' with $(\beta' - \frac{1}{2})\log\gamma \le \varepsilon$. Hence $m'(\varepsilon) \ge (c/4)\varepsilon^{A+\delta}$, as desired. □

9. Proof of Corollary 1

The Pair Correlation Conjecture asserts that the number of zeros $T \le \gamma_1, \gamma_2 \le 2T$ for which $2\pi\alpha/\log T < \gamma_1 - \gamma_2 \le 2\pi\beta/\log T$ is asymptotically

$$N(T) \cdot \int_{\alpha}^{\beta} \left(1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2 + \delta(u)\right) du$$

with δ denoting Dirac's delta function. Here we derive a simple consequence of the Pair Correlation Conjecture for small gaps between *consecutive* zeros of the Riemann zeta-function. The lower bound is not optimal but sufficient for our needs.

Lemma 13. *Assume the Pair Correlation Conjecture. Let $\delta > 0$ be given. Then $\varepsilon^{3+\delta} \ll m(\varepsilon) \ll \varepsilon^{3}$ provided that $0 < \varepsilon < C(\delta)$ with $C(\delta)$ a constant depending only on δ.*

Proof. The Pair Correlation Conjecture asserts that the number of distinct zeros $T \le \gamma_1, \gamma_2 \le 2T$ for which $0 \le \gamma_1 - \gamma_2 \le 2\pi\alpha/\log T$ is asymptotically $N(T) \cdot f(\alpha)$ with $f(\alpha)$ such that $f(\alpha) \sim c \cdot \alpha^3$ as $\alpha \to 0$. The number of $T \le \gamma \le 2T$ such that $(\gamma^+ - \gamma)\log\gamma \le \varepsilon$ is less than the number of distinct $T \le \gamma_1, \gamma_2 \le 2T$ for which $0 \le \gamma_1 - \gamma_2 \le 2\pi\varepsilon/\log T$ therefore $m(\varepsilon) \le f(\varepsilon) \ll \varepsilon^3$.

Now consider the set of $T \le \gamma_1, \gamma_2 \le 2T$ for which $\frac{\varepsilon}{2} \le (\gamma_1 - \gamma_2)\log\gamma_1 \le \varepsilon$. Call S the set of $T \le \gamma_1 \le 2T$ for which the interval $[\gamma_1 - 2\pi/\log T; \gamma_1 + 2\pi/\log T]$ contains at most $\varepsilon^{-\delta}$ zeros. By Corollary 3, the zero $T \le \gamma_1 \le 2T$ with $\gamma_1 \notin S$ have cardinality $\le \varepsilon^A \cdot T\log T$ provided that $0 < \varepsilon < C(\delta, A)$ (we choose $A = 100$ for example). We have

$$\sum_{\substack{T \le \gamma_1, \gamma_2 \le 2T \\ \frac{\varepsilon}{2} \le (\gamma_1 - \gamma_2)\log\gamma_1 \le \varepsilon}} 1 = \sum_{\gamma_1 \in S} \sum_{\substack{T \le \gamma_2 \le 2T \\ \frac{\varepsilon}{2} \le (\gamma_1 - \gamma_2)\log\gamma_1 \le \varepsilon}} 1 + \sum_{\gamma_1 \notin S} \sum_{\substack{T \le \gamma_2 \le 2T \\ \frac{\varepsilon}{2} \le (\gamma_1 - \gamma_2)\log\gamma_1 \le \varepsilon}} 1 \tag{13}$$

Since $\gamma_1 \in S$ there can be at most $\varepsilon^{-\delta}$ zeros γ_2 satisfying $\varepsilon/2 \le |\gamma_1 - \gamma_2|\log\gamma_1 \le \varepsilon$. Therefore the first sum is bounded by

$$\sum_{\substack{\gamma_1 \in S \\ (\gamma_1^+ - \gamma_1)\log\gamma_1 \le \varepsilon}} \varepsilon^{-\delta} \ll \varepsilon^{-\delta} \cdot m(\varepsilon) \cdot T\log T$$

because for each $\gamma_1 \in S$ the inner sum over γ_2 is $\le \varepsilon^{-\delta}$ if $(\gamma_1^+ - \gamma_1)\log\gamma_1 \le \varepsilon$ and is 0 otherwise. On the other hand the second sum is by Cauchy-Schwarz less than,

$$|S|^{1/2} \cdot \Bigg(\sum_{T \le \gamma_1 \le 2T} \Bigg(\sum_{\substack{T \le \gamma_2 \le 2T \\ \frac{\varepsilon}{2} \le (\gamma_1 - \gamma_2)\log\gamma_1 \le \varepsilon}} 1 \Bigg)^2 \Bigg)^{1/2} \le$$

$$\le |S|^{1/2} \cdot \Bigg(\sum_{T \le \gamma_1 \le 2T} \Bigg(N\big(\gamma_1 + \frac{2\pi}{\log T}\big) - N\big(\gamma_1 - \frac{2\pi}{\log T}\big) \Bigg)^2 \Bigg)^{1/2} \ll \varepsilon^{A/2} \cdot T\log T$$

by Lemma 9. By the Pair Correlation Conjecture the left-hand side of (13) is asymptotically $C \cdot N(T) \cdot \varepsilon^3$ for some absolute constant $C > 0$. Combining the above three equations we get $C\varepsilon^3 \le m(\varepsilon)\varepsilon^{-\delta} + C_1\varepsilon^{A/2}$ for some absolute constant $C, C_1 > 0$. Therefore if ε is small enough then $\varepsilon^{3+\delta} \ll m(\varepsilon)$. □

We are now ready to prove Corollary 1.

Proof of Corollary 1. By the previous lemma, on the Pair Correlation, we have $m(\varepsilon^{1/2}) \gg \varepsilon^{3/2+\delta}$ for all $C(\delta) > \varepsilon > 0$. Therefore by the second part of our Main Theorem we get $m'(\varepsilon) \gg \varepsilon^{3/2+\delta}$ for all $C(\delta) > \varepsilon > 0$. Now suppose to the contrary that there is a $\eta > 0$ and a sequence of $\varepsilon \to 0$ such that $m'(\varepsilon) \gg \varepsilon^{3/2-\eta}$. Then, by Theorem 1 on the same subsequence of $\varepsilon \to 0$ we have $m(\varepsilon^{1/2-\delta}) \gg \varepsilon^{3/2-\eta}$. However by the Pair Correlation Conjecture we have $\varepsilon^{3/2-3\delta} \gg m(\varepsilon^{1/2-\delta}) \gg \varepsilon^{3/2-\eta}$. Choosing $0 < \delta < (1/3)\eta$ and letting $\varepsilon \to 0$ along the subsequence, we obtain a contradiction. □

10. Acknowledgments

I would like to thank Prof. Farmer, Prof. Ki and Prof. Soundararajan for comments on an early draft of this paper.

References

[1] Bruce C. Berndt. The number of zeros for $\zeta^{(k)}(s)$. *J. London Math. Soc. (2)*, 2:577–580, 1970.

[2] B. Conrey and H. Iwaniec. Spacing of zeros of Hecke L-functions and the class number problem. *Acta Arith.*, 103(3):259–312, 2002.

[3] Eduardo Dueñez, David W. Farmer, Sara Froehlich, C. P. Hughes, Francesco Mezzadri, and Toan Phan. Roots of the derivative of the Riemann-zeta function and of characteristic polynomials. *Nonlinearity*, 23(10):2599–2621, 2010.

[4] David W. Farmer and Haseo Ki. Landau–Siegel zeros and zeros of the derivative of the Riemann zeta function. *Adv. Math.*, 230(4-6):2048–2064, 2012.

[5] Shaoji Feng. A note on the zeros of the derivative of the Riemann zeta function near the critical line. *Acta Arith.*, 120(1):59–68, 2005.

[6] Kevin Ford and Alexandru Zaharescu. On the distribution of imaginary parts of zeros of the Riemann zeta function. *J. Reine Angew. Math.*, 579:145–158, 2005.

[7] Akio Fujii. On the zeros of Dirichlet L-functions. II. *Trans. Amer. Math. Soc.*, 267(1):33–40, 1981. With corrections to: "On the zeros of Dirichlet L-functions. I" [Trans. Amer. Math. Soc. **1**96 (1974), 225–235; MR **5**0 #2096] and subsequent papers.

[8] M. Z. Garaev and C. Y. Yıldırım. On small distances between ordinates of zeros of $\zeta(s)$ and $\zeta'(s)$. *Int. Math. Res. Not. IMRN*, (21):Art. ID rnm091, 14, 2007.

[9] D. A. Goldston and S. M. Gonek. A note on $S(t)$ and the zeros of the Riemann zeta-function. *Bull. Lond. Math. Soc.*, 39(3):482–486, 2007.

[10] S. M. Gonek. An explicit formula of Landau and its applications to the theory of the zeta-function. In *A tribute to Emil Grosswald: number theory and related analysis*, volume 143 of *Contemp. Math.*, pages 395–413. Amer. Math. Soc., Providence, RI, 1993.

[11] Haseo Ki. The zeros of the derivative of the Riemann zeta function near the critical line. *Int. Math. Res. Not. IMRN*, (16):Art. ID rnn064, 23, 2008.

[12] Norman Levinson. More than one third of zeros of Riemann's zeta-function are on $\sigma = 1/2$. *Advances in Math.*, 13:383–436, 1974.

[13] Hugh L. Montgomery. *Ten lectures on the interface between analytic number theory and harmonic analysis*, volume 84 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1994.

[14] A. Selberg. On the value distribution of the derivative of the Riemann zeta-function. *Unpublished manuscript available at http://publications.ias.edu/selberg/section/2483.*

[15] Atle Selberg. On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip $0 < t < T$. *Avh. Norske Vid. Akad. Oslo. I.*, 1944(1):27, 1944.

[16] K. Soundararajan. The horizontal distribution of zeros of $\zeta'(s)$. *Duke Math. J.*, 91(1):33–59, 1998.

[17] Kannan Soundararajan. Moments of the Riemann zeta function. *Ann. of Math. (2)*, 170(2):981–993, 2009.

[18] Andreas Speiser. Geometrisches zur Riemannschen Zetafunktion. *Math. Ann.*, 110(1):514–521, 1935.

[19] Yitang Zhang. On the zeros of $\zeta'(s)$ near the critical line. *Duke Math. J.*, 110(3):555–572, 2001.

Department of Mathematics, Stanford University, 450 Serra Mall, Bldg. 380, Stanford, CA 94305-2125

E-mail address: maksym@stanford.edu