Algebraic Topology, midterm 2 solutions

Problem 1. Show that for the subspace $\mathbf{Q} \subset \mathbf{R}$, the relative homology group $H_1(\mathbf{R}, \mathbf{Q})$ is free abelian and find a basis.

Solution 1. Consider the long exact sequence of reduced homology groups for the pair $(\mathbf{R}, \mathbf{Q})$ involving maps

$$\widetilde{H}_1(\mathbf{R}) \to H_1(\mathbf{R}, \mathbf{Q}) \to \widetilde{H}_0(\mathbf{Q}) \to \widetilde{H}_0(\mathbf{R}).$$

Since $\mathbf{R}$ is contractible, we have $\widetilde{H}_1(\mathbf{R}) = \widetilde{H}_0(\mathbf{R}) = 0$, and consequently $\partial\colon H_1(\mathbf{R}, \mathbf{Q}) \to \widetilde{H}_0(\mathbf{Q})$ is an isomorphism. Recall that $\widetilde{H}_0(\mathbf{Q})$ is the kernel of the augmentation homomorphism from $H_0(\mathbf{Q}) = \bigoplus_{\mathbf{Q}} \mathbf{Z}$ to $\mathbf{Z}$ mapping $(n_q)_{q \in \mathbf{Q}}$ to $\sum_{q \in \mathbf{Q}} n_q$. Let σ_q denote the singular 0-simplex that sends the unique vertex to $q \in \mathbf{Q}$. Then $[\sigma_q - \sigma_0]$ for $q \in \mathbf{Q} - 0$ form a free basis of $\widetilde{H}_0(\mathbf{Q})$. Let τ_q denote the linear singular 1-simplex sending v_0 to 0 and v_1 to q. Then $\partial \tau_q = \sigma_q - \sigma_0$ and consequently $[\tau_q]$ for $q \in \mathbf{Q} - 0$ form a free basis of $H_1(\mathbf{R}, \mathbf{Q})$.

Problem 2. Suppose that $f\colon S^n \to S^n$ has no fixed points. Prove that $\deg f = (-1)^{n+1}$. Deduce that for n even $\mathbf{Z}_2$ is the only nontrivial group that can act freely on S^n.

Solution 2. Consider the homotopy $H\colon S^n \times I \to S^n$ from f to the antipodal map given by $H(x,t) = ((1-t)f(x) - tx)/\|(1-t)f(x) - tx\|$, which is well defined since for all $x \in S^n$ we have $f(x) \neq x$. We proved in recitation that the antipodal map has degree $(-1)^{n+1}$, which justifies the first assertion. For the second assertion, let $\varphi\colon G \to \mathrm{Homeo}(S^n)$ be a homomorphism. Then for each $g \in G$ we have $\deg \varphi(g) = \pm 1$ and the map $\psi\colon G \to \mathbf{Z}_2$ defined by $\deg \varphi(g) = (-1)^{\psi(g)}$ is a homomorphism. By the first assertion, for n even ψ is injective, which proves that G is trivial or $\mathbf{Z}_2$.

Problem 3. Show that a nonorientable closed surface Σ cannot be embedded as a subspace of $\mathbf{R}^3$ in such a way as to have a neighbourhood V homeomorphic to the mapping cylinder of some map from a closed orientable surface Σ' to Σ, and where $\partial V = \Sigma'$.

Solution 3. Decompose $\mathbf{R}^3$ as the union of open sets $A = \mathrm{Int} V$, $B = \mathbf{R}^3 - \Sigma$. Note that A deformation retracts to Σ, and $A \cap B$ deformation retracts to Σ'. Thus the Mayer Vietoris sequence

$$H_2(A \cup B) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(A \cup B)$$

shows that $H_1(\Sigma') \cong H_1(\Sigma) \oplus H_1(B)$. Since $H_1(\Sigma)$ is the abelianisation of $\pi_1(\Sigma)$, which is presented by a single relator of form $a_1^2 \cdots a_n^2$, it contains an element of order 2. On the other hand, we computed in recitation that $H_1(\Sigma')$ is free abelian, which is a contradiction.

Problem 4. Compute the cohomology with $\mathbf{Z}$ coefficients of the space obtained from the unit cube I^3 by identifying opposite faces via a one-quarter twist.

Solution 4. Identifications lead to the CW structure of the space X in question indicated in the figure. We have two 0-cells (v and w), four 1-cells (labelled a, b, c, d), three 2-cells (labelled R, S, T) and one 3-cell. Consequently, the cellular chain complex has form

$$\mathbf{Z} \to \mathbf{Z}^3 \to \mathbf{Z}^4 \to \mathbf{Z}^2.$$

The first map is trivial since the opposite squares in the attaching map of the 3-cell cancel each other. The second map has matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{pmatrix},$$

since for example the first column describes the boundary of R which is $a - b + c - d$. Finally, the last map sends each a, b, c, d to $v - w$, thus has matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

The dual maps have transposed matrices. Consequently

- $H^3(X) = \mathbf{Z}$,
- $H^2(X) = \mathbf{Z}^3/\mathrm{span}\{(1,1,1),(0,0,2),(2,0,0)\} = \mathbf{Z}_2 \oplus \mathbf{Z}_2$,
- $H^1(X) = \mathrm{span}\{(1,1,1,1)\}/\mathrm{span}\{(1,1,1,1)\} = 0$,
- $H^0(X) = \mathbf{Z}$.