Algebraic Topology

Midterm 2 preparation problems

Problem 1. Prove that the prism operator P defined in class satisfies $\partial P + P\partial = g_\sharp - f_\sharp$, i.e. P is a chain homotopy between $f_\sharp$ and $g_\sharp$.

Problem 2. Compute the homology groups of the subspace of $I \times I$ consisting of the four boundary edges plus all points in the interior whose first coordinate is rational.

Problem 3. Show that for the subspace $\mathbf{Q} \subset \mathbf{R}$, the relative homology group $H_1(\mathbf{R}, \mathbf{Q})$ is free abelian and find a basis.

Problem 4. Show that $H_1(X, A)$ is not isomorphic to $\widetilde{H}_1(X/A)$ if $X = [0, 1]$ and A is the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots$ together with its limit 0.

Problem 5. Let X be the n-dimensional simplex with its natural Δ-complex structure. Compute the homology groups of the k-skeleta X^k.

Problem 6. Suppose that $f\colon S^n \to S^n$ has no fixed points. Prove that $\deg f = (-1)^{n+1}$. Deduce that for n even $\mathbf{Z}_2$ is the only nontrivial group that can act freely on S^n.

Problem 7. Let X be a CW complex homeomorphic to S^2. Prove that its 1-skeleton X^1 cannot be the complete graph K_5 or the complete bipartite graph $K_{3,3}$.

Problem 8. Let X be the space obtained from the cube I^3 by identifying its bottom and top faces by translation, left and right faces by translation, and front and back faces by a rotation around the axis through the centres of left and right faces. Compute the homology groups of X.

Problem 9. Let $m, n > 1$ and let $l_1, \dots, l_n$ be integers relatively prime to m. The *lens space* $L = L_m(l_1, \dots, l_n)$ is the quotient of $S^{2n-1} \subset \mathbf{C}^n$ by the action of $\mathbf{Z}_m$ generated by $(z_1, \dots, z_n) \to (e^{2\pi i l_1} z_1, \dots, e^{2\pi i l_n} z_n)$. Find a CW structure on L and compute its homology groups.

Problem 10. Let $f\colon X \to Y$ be a map between n-dimensional connected CW complexes, for some $n \geq 1$. Suppose that f induces isomorphisms on π_i for all $i \leq n$. Prove that f is a homotopy equivalence.

Problem 11. Let $n > 1$ and let G be an abelian group. X is a *Moore space* if it is simply-connected, $H_n(X) = G$ and $\widetilde{H}_i(X) = 0$ for $i \neq n$. Prove that two Moore spaces X, Y with common n, G that are CW complexes are homotopy equivalent.
Hints: proceed as for $K(G, n)$ by constructing a template Moore space X with only n-cells and $(n+1)$-cells. Construct a map $f \colon X \to Y$ inducing an isomorphism on π_n. Prove that f also induces an isomorphism on H_n by showing that $\pi_i(M_f, X) = 0$ for $i \leq n+1$.

Problem 12. (i) Use the Mayer–Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ of the torus.

(ii) Do the same for the space obtained by attaching a Möbius band to $\mathbf{R}P^2$ via a homeomorphism of its boundary circle to the standard $\mathbf{R}P^1 \subset \mathbf{R}P^2$.

Problem 13. The oriented surface S_g of genus g, embedded in $\mathbf{R}^3$ in the standard way, bounds a compact region V_g, called the *genus g handlebody*. Two copies of V_g, glued together by the identity map between their boundary surfaces S_g, form a closed 3-manifold M_g. Compute the homology groups $H_i(M_g)$ and the relative homology groups $H_i(V_g, S_g)$.

Problem 14. Consider a pair of spaces $(X, Y) = (A \cup B, C \cup D)$ with $C \subset A, B \subset D$, such that X is the union of the interiors of A and B, and Y is the union of the interiors of C and D. Prove that there is a *relative Mayer–Vietoris* exact sequence

$$\cdots \to H_{n+1}(X, Y) \to H_n(A \cap B, C \cap D) \to H_n(A, C) \oplus H_n(B, D) \to H_n(X, Y) \to \cdots$$

In particular, for $A = B = X$ we have an exact sequence

$$\cdots \to H_{n+1}(X, C \cup D) \to H_n(X, C \cap D) \to H_n(X, C) \oplus H_n(X, D) \to H_n(X, C \cup D) \to \cdots$$

Problem 15. Compute the cohomology groups with $\mathbf{Z}$ coefficients of

(i) the space obtained from S^2 under the identifications $x \sim -x$ for x in the equator S^1,

(ii) the space obtained from S^3 under the identifications $x \sim -x$ for x in the equatorial $S^2 \subset S^3$.

Problem 16. Let X be the Moore space obtained from S^n by attaching a cell e^{n+1} by a map of degree m. Show that the quotient map $p\colon X \to X/S^n = S^{n+1}$ induces the trivial map on all $\widetilde{H}_i(-;\mathbf{Z})$ but not on $H^{n+1}(-;\mathbf{Z})$. Use that to show that the splittings in the universal coefficient theorem for cohomology do not commute with the maps induced by p.

Problem 17. Let H, G be finitely generated abelian groups. Prove that $\mathrm{Tor}(G, H)$ is the largest finite group that embeds in G and in H.

Problem 18. Prove that the groups $\mathrm{Tor}(G, H)$ are well defined. Prove the universal coefficient theorem for homology.

Problem 19. Let $F = \mathbf{Z}_p$ for p prime, or $F = \mathbf{Q}$. Prove that $H^n(X; F) = \mathrm{Hom}(H_n(X; F), F)$.