Algebraic Topology

Midterm 1 preparation problems

Definition. Let γ be a closed path in a space X, based at x_0. We explained in class how γ induces an automorphism of each $\pi_n(X, x_0)$, which depends only on $[\gamma] \in \pi_1(X, x_0)$. This defines an action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

Problem 1. Describe the action of $\pi_1(X)$ on $\pi_i(X)$ for $i \geqslant 2$ in terms of the action of the deck transformation group of the universal cover $\widetilde{X}$ of X on the maps $S^i \to \widetilde{X}$. Keep track of the basepoints.

Problem 2. For which n is the action of $\pi_1(\mathbf{R}P^n)$ on $\pi_n(\mathbf{R}P^n)$ trivial?

Problem 3. Let $x_0 \in Q \subset A \subset X$, where A is path-connected. Prove that the following sequence induced by inclusions is exact (note that the last three elements are only sets):

$$\cdots \to \pi_n(A, Q, x_0) \to \pi_n(X, Q, x_0) \to \pi_n(X, A, x_0) \to \pi_{n-1}(A, Q, x_0) \to \\ \cdots \to \pi_1(X, A, x_0) \to 0.$$

Problem 4. Let E be a subspace of $\mathbf{R}^2$ obtained by deleting a subspace of $\{0\} \times \mathbf{R}$. For which such spaces E is the projection $E \to \mathbf{R}$, $(x, y) \to x$ a fibre bundle?

Problem 5. (i) Find a fibre bundle $S^7 \to S^4$ with fibre S^3. Hint: quaternions.

(ii) Prove that $\pi_7(S^4)$ contains a $\mathbf{Z}$ summand.

Problem 6. Find a fibre bundle $\mathbf{C}P^3 \to S^4$ with fibre S^2.

Problem 7. Show that if $S^m \to S^n$ is a fibre bundle with fibre S^k, then $k = n - 1$ and $m = 2n - 1$.

Problem 8. Compute $\pi_n(S^1 \vee S^n)$.

Problem 9. Let X, Y be homotopy equivalent CW complexes with no cells of dimension $n + 1$ for some $n \geqslant 0$. Prove that their n-skeleta X^n, Y^n are also homotopy equivalent.

Problem 10. Let X be the *Whitehead manifold* obtained as a direct limit $X_1 \subset X_2 \subset \cdots$, where each X_i is a solid torus $S^1 \times D^2$, and each embedding $X_i \subset X_{i+1}$ is equivalent to the embedding of one component of the Whitehead link in S^3 (see below) into the complement of the other. Prove that X is contractible.

Problem 11. Prove that $\pi_i(\mathbf{R}P^2) = \pi_i(S^2 \times \mathbf{R}P^\infty)$ for each i.

Problem 12. Let X be a connected CW complex with $\pi_i(X) = 0$ for all $i \leqslant n$. Construct a CW complex Z homotopy equivalent to X and with the n-skeleton Z^n consisting of only one vertex.

Problem 13. Compute $\pi_5(S^3 \vee S^3)$ in terms of $\pi_5(S^3)$ and describe its generators.

Problem 14. Prove that there is no retraction of the unit ball D^n onto its boundary S^{n-1}. Deduce *Brouwer fixed-point theorem* saying that each map $D^n \longrightarrow D^n$ has a fixed point.