Algebraic Topology Math 582

Final exam material

(1) Definitions of homotopy groups and relative homotopy groups. Propositions on the behaviour of homotopy groups with respect to covering spaces and products. Complete proof of the exactness of the long homotopy sequence of a pair.

(2) Definition of a fibre bundle, example of the Hopf bundle. Definition of the homotopy lifting property for discs and proof of that property for fibre bundles. Statement and proof of the theorem on $\pi_n(E, F) = \pi(B)$. Consequences for π_2 of spheres.

(3) Definition of a CW complex and a CW pair. Proof of the lemmas characterising open sets and showing that compact subsets lie in finite subcomplexes. Definition of the homotopy extension property and its proof for CW pairs. Definition of a cellular map, statement and full proof of the cellular approximation theorem.

(4) Proof of the compression lemma. Statement and proof of Whitehead's theorem. Proof of the theorem on existence and uniqueness of $K(G, 1)$.

(5) Definition of n-connected spaces and pairs. Statement and proof of Freudenthal's theorem for S^n. Consequences for homotopy groups of spheres.

(6) Definition of singular n-chains $C_n(X)$ and the boundary map ∂. Proof of $\partial^2 = 0$. Definition of $H_n(X)$. Proof of $H_0(X) = \mathbf{Z}$ for X path-connected and of $\widetilde{H}_n(X) = 0$ for X a point. Statement and proof of the theorem relating $\pi_1(X)$ and $H_1(X)$.

(7) Definitions of chain maps and chain homotopies. Proof of the theorem that homotopic maps induce chain homotopic maps on C_n, hence equal maps on H_n (without the computation that the prism operator is a chain homotopy). Definition of relative homology groups. Statement of the theorem on the exactness of the long homology sequence of a pair. Statement and proof of the excision theorem and its corollary.

(8) Statement and proof of the lemma characterising $H_n(S^n)$ and $H_n(D^n, \partial D^n)$. Definitions of a Δ-complex and its simplicial homology. Proof of the theorem that simplicial and singular homologies coincide.

(9) Definition of cellular homology. Complete proof that cellular and singular homologies coincide. Statement and proof of Hurewicz theorem.

(10) Proof of the exactness of the Mayer–Vietoris sequence. Definition of homology with coefficients. Proofs of Borsuk's theorem and Borsuk–Ulam's theorem.

(11) Definition of cohomology with coefficients. Statement and proof of the universal coefficient theorem for cohomology, including the lemma that $\mathrm{Ext}(H, G)$ is well-defined.

(12) Definition of the cup product. Statement and proof of the lemma describing $\partial(\varphi \cup \psi)$. Proof that $\cup$ induces maps on homology. Statement and proof of the theorem on the commutativity of $\cup$ (without the computation that P' is a chain homotopy).

(13) Definition of a manifold, a local orientation, and an orientation. Full proof of the theorem on the existence of the fundamental class. Definition of the cap product.

(14) Statement of the theorem on Poincaré duality. Definition of $H_c^i(X)$. Definition of directed sets and direct limits. Description of $H_c^i(X)$ using direct limits. Example of $X = \mathbf{R}^n$. Definition of $D_M \colon H_c^i(M) \to H_{n-i}(M)$.

(15) Lemma on the relation between cap and cup products with corollaries. Definition of the cross product for cohomology, and the statement of Künneth formula in the special case we discussed. Definition of a division algebra structure on $\mathbf{R}^n$ and the proof of the theorem that it cannot exist for n not a power of 2 (assuming Poincaré duality and Künneth formula).