

Algebraic Topology, problem list 9

Problem 1. Compute the cellular cohomology groups $H^i(X; \mathbf{Z})$ and $H^i(X; \mathbf{Z}_2)$ for

- (i) $X = S^n$,
- (ii) $X = \mathbf{R}P^n$,
- (iii) $X = \mathbf{C}P^n$,

where $n \geq 1$.

Problem 2. Prove that $\text{Ext}(H_1 \oplus H_2; G) = \text{Ext}(H_1; G) \oplus \text{Ext}(H_2; G)$.

Problem 3. Suppose that the homology groups H_n of a chain complex with C_n free abelian are finitely generated. Denote $T_n \subseteq H_n$ the maximal torsion subgroups. Prove that $H^n(C; \mathbf{Z}) = (H_n/T_n) \oplus T_{n-1}$.

Problem 4. Let X be path-connected CW complex. Find a bijection between $H^1(X; G)$ and the set of basepoint-preserving homotopy classes of maps from X to $K(G, 1)$.

Problem 5. Prove that $\text{Tor}(G, H) = \text{Tor}(H, G)$.

Hint: for short exact sequences $J \rightarrow I \rightarrow H$ and $F_1 \rightarrow F_0 \rightarrow G$ with F_0, F_1 free, use the snake lemma to get an exact sequence

$$0 \rightarrow \text{Tor}(G, J) \rightarrow \text{Tor}(G, I) \rightarrow \text{Tor}(G, H) \rightarrow J \otimes G \rightarrow I \otimes G \rightarrow H \otimes G \rightarrow 0.$$