Algebraic Topology, problem list 8

Problem 1. For a finite CW complex X, its *Euler characteristic* $\chi(X)$ is $\sum_n (-1)^n c_n$, where c_n is the number of n-cells of X. The *rank* of a finitely generated abelian group is the number of $\mathbf{Z}$ summands when the group is expressed as a direct sum of cyclic groups.

Prove that $\chi(X) = \sum_n (-1)^n \mathrm{rank}\, H_n(X)$.

Hints: prove that if $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated abelian groups, then rank B = rank A + rank C. Apply this to $0 \to B_n \to Z_n \to H_n \to 0$ and $0 \to Z_n \to C_n \to B_{n-1} \to 0$.

Problem 2. Let $K \subset S^3$ be an embedded solid torus. Compute $H_1(S^3 - \mathrm{int} K)$ and find a representative for its generator.

Problem 3. Suppose the space X is a union of open sets $A_1, \ldots, A_n$ such that each intersection $A_{i_1} \cap \cdots \cap A_{i_k}$ is either empty or *acyclic* (i.e. with all reduced homology groups trivial). Show that $\widetilde{H}_i(X) = 0$ for $i \geq n-1$. For each $n \geq 2$ give an example where $\widetilde{H}_{n-2}(X) \neq 0$.

Problem 4. Let X be the 2-dimensional CW complex obtained in the following way. X^1 is the complete graph on the vertex set $\{1, 2, 3, 4, 5\}$. X^2 is obtained by adding 2-cells attached along the cycle $\gamma = 12345$ and all other cycles obtained from γ by the obvious action of A_5 (there are 6 such cycles including γ).

(i) Prove that X is acyclic.

(ii) Let Y (Floyd–Richardson example) be obtained from X by coning it off simultaneously $60 = |A_5|$ times. Prove that Y is contractible.

(iii) Prove that the obvious actions of A_5 on X and Y do not have global fixed-points.

Problem 5. Suppose $f\colon S^n \to S^n$ satisfies $f(x) = f(-x)$ for each $x \in S^n$. Prove that f has even degree.