Algebraic Topology, problem list 6

Problem 1. Prove the *five lemma*, saying that if the following diagram is commutative, rows are exact, β, δ are isomorphisms, α is surjective, and ε is injective, then γ is an isomorphism.

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \varepsilon} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Problem 2. For all $n \geq 0$ compute $H_n^{\Delta}(X)$ for X

(i) the torus T^2,

(ii) the projective plane $\mathbf{R}P^2$,

(iii) the Klein bottle.

Use a convenient Δ-complex structure.

Problem 3. Let $n \geq 1$. Show that a reflection of $\mathbf{R}^{n+1}$ preserving S^n induces an automorphism of $H_n(S^n)$ sending a generator to its negative. Using that, describe the automorphism induced on $H_n(S^n)$ by the antipodal map.

Problem 4. Let $n \geq 1$. Prove that S^n has a continuous field of nonzero tangent vectors if and only if n is odd.

Problem 5. Let $n \geq 1$. Prove that a map $S^n \to S^n$ induces the same maps on $\pi_n(S^n)$ and $H_n(S^n)$ (under any identification with $\mathbf{Z}$).