Algebraic Topology, problem list 1

In all the problems you are allowed to use the fact that $\pi_2(S^3)$ is trivial (we will prove next week that $\pi_i(S^n) = 0$ for all $i < n$).

Problem 1. Let X be a space with a group structure such that the multiplication map $\mu\colon X \times X \to X$ is continuous, with basepoint x_0 the identity element. Prove that the group operation in $\pi_n(X, x_0)$ can be also defined by the rule $(f + g)(s) = \mu(f(s), g(s))$.

Problem 2. Let X be a space that is path-connected and let

$$CX = (X \times I)/X \times \{0\}$$

be the cone on X with the embedding of X as $X \times \{1\}$. Prove that $\pi_2(CX, X) = \pi_1(X)$.

Problem 3. Prove that the identity map $S^2 \to S^2$ generates $\pi_2(S^2)$ by analyzing its image in $\pi_1(S^1)$ in the long exact sequence of the Hopf bundle.

Problem 4. Let S^∞ be the direct limit of $S^1 \subset S^2 \subset \cdots$. This means that S^∞ is the space of sequences $(x_1, x_2, \ldots)$ of real numbers with only finite number of nonzero x_i and $\sum x_i^2 = 1$, and that a set is open in S^∞ if its intersection with each S^n is open. Prove that S^∞ is contractible by exhibiting a homotopy of the identity map with a trivial map.

Problem 5. The *complex projective space* $\mathbf{C}P^n$ is the quotient of $S^{2n+1} \subset \mathbf{C}^{n+1}$ by the relation $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$ for $\lambda \in S^1 \subset \mathbf{C}$. The *infinite complex projective space* $\mathbf{C}P^\infty$ is the quotient of S^∞ by the union of these relations.

(i) Justify that $S^\infty \to \mathbf{C}P^\infty$ is a fibre bundle.

(ii) Compute all homotopy groups of $\mathbf{C}P^\infty$.

(iii) Prove that S^2 and $S^3 \times \mathbf{C}P^\infty$ have the same homotopy groups.