Journal of Topology Advance Access published September 1, 2013

Journal of Topology (2013) Page 1 of 17
doi:10.1112/jtopol/jtt009

Graph manifolds with boundary are virtually special

Piotr Przytycki and Daniel T. Wise

Abstract

Let M be a graph manifold. We prove that fundamental groups of embedded incompressible surfaces in M are separable in $\pi_1 M$, and that the double cosets for crossing surfaces are also separable. We deduce that if there is a 'sufficient' collection of surfaces in M, then $\pi_1 M$ is virtually the fundamental group of a special non-positively curved cube complex. We provide a sufficient collection for graph manifolds with boundary, thus proving that their fundamental groups are virtually special, and hence linear.

1. *Introduction*

A *graph manifold* is an oriented compact connected 3-manifold that is irreducible and has only Seifert-fibred pieces in its JSJ decomposition. Hempel proved that the fundamental groups of all Haken 3-manifolds, in particular all graph manifolds, are residually finite [**9**, Theorem 1.1]. Throughout the article we assume that a graph manifold is not a single Seifert-fibred space and not a Sol manifold. For background on graph manifolds we refer to the survey article by Buyalo and Svetlov [**3**].

We are interested in separability properties of surfaces properly embedded in graph manifolds. A subgroup F of a group G is *separable* if, for each $g \in G - F$, there is a finite index subgroup H of G with $g \notin HF$. Let S be an oriented incompressible surface embedded in a graph manifold M. Then $\pi_1 S$ embeds in $\pi_1 M$ by the loop theorem.

Theorem 1.1. *Let M be a graph manifold (with or without boundary). Let S be an oriented incompressible surface embedded in M. Then $\pi_1 S$ is separable in $\pi_1 M$.*

More generally, consider subgroups $F_1, F_2 \subset G$. The double coset $F_1 F_2$ is *separable* if, for each $g \in G - F_1 F_2$, there is a finite index subgroup H of G with $g \notin H F_1 F_2$.

We identify the group of covering transformations of the universal cover $\tilde{M}$ of M with $\pi_1 M$.

Theorem 1.2. *Let M be a graph manifold. Let $S, P \subset M$ be oriented incompressible surfaces whose intersections with each block are horizontal or vertical (see Section 2). Let $\tilde{S}, \tilde{P} \subset \tilde{M}$ be intersecting components of the preimages of S, P. Then $\mathrm{Stab}(\tilde{S})\,\mathrm{Stab}(\tilde{P})$ is separable in $\pi_1 M$.*

We apply Theorems 1.1 and 1.2 to obtain the following corollary.

Received 19 November 2011; revised 28 November 2012.

2010 *Mathematics Subject Classification* 20F65 (primary).

Piotr Przytycki was partially supported by MNiSW grant N201 012 32/0718 and the Foundation for Polish Science. Daniel T. Wise was supported by NSERC.

Corollary 1.3. *Let M be a graph manifold with non-empty boundary. Then $\pi_1 M$ is virtually the fundamental group of a special cube complex.*

A *special* cube complex is a non-positively curved cube complex that admits a local isometry into the Salvetti complex of a right-angled Artin group (see [**6**, **7**]). As a consequence, the fundamental groups of special cube complexes (which are also called *special*) are subgroups of right-angled Artin groups. The latter have various outstanding properties. To mention just a few, they are linear [**10**] and residually torsion-free nilpotent [**5**]. Moreover, they virtually satisfy Agol's RFRS condition [**1**].

It was proved that fundamental groups of closed hyperbolic 3-manifolds are virtually special [**2**, **18**]. A relative version of this theorem says that the fundamental groups of hyperbolic 3-manifolds with boundary are virtually special as well [**18**]. Our theorem treats the complementary case, with an eye towards eventually analysing the case of manifolds with both hyperbolic and Seifert-fibred pieces [**15**].

The class of graph manifolds with boundary has been studied by Wang and Yu who prove [**17**, Theorem 0.1] that they all virtually fibre over the circle. (Note that we do not exploit that result in our article.) A closed graph manifold might not virtually fibre [**13**]. Hence, by Agol's virtual fibring criterion [**1**] such a manifold cannot have virtually special fundamental group. Thus some restriction is needed in Corollary 1.3.

In fact, we have recently learned that independently Yi Liu has proved [**12**, Theorem 1.1] that the graph manifolds with virtually special fundamental groups are exactly the ones that admit a non-positively curved Riemannian metric. It was proved by Leeb [**11**, Theorem 3.2] that graph manifolds with boundary admit a non-positively curved Riemannian metric (with geodesic boundary). Hence our Corollary 1.3 is a special case of the theorem of Liu.

In order to obtain Corollary 1.3 we prove, using Theorems 1.1 and 1.2, the following criterion involving a 'sufficient' collection of surfaces. (For definitions see Section 2.)

Definition 1.4. Let $\mathcal{S}$ be a collection of incompressible oriented surfaces embedded in a graph manifold M that are not ∂-parallel annuli and satisfy the property that the intersection of each surface from $\mathcal{S}$ with each block of M is vertical or horizontal. We say that $\mathcal{S}$ is *sufficient* if:

(1) for each block $B \subset M$ and each torus $T \subset \partial B$, there is a surface $S \in \mathcal{S}$ such that $S \cap T$ is non-empty and vertical with respect to B;
(2) for each block $B \subset M$ there is a surface $S \in \mathcal{S}$ such that $S \cap B$ is horizontal.

Note that property (1) automatically implies property (2). Indeed, let B_0 be a block and let B_1 be any adjacent block. Let $T \subset B_0 \cap B_1$. By (1) there is a surface $S \in \mathcal{S}$ such that $S \cap T$ is vertical in B_1. Then $S \cap B_0$ is horizontal.

We have the following criterion.

Theorem 1.5. *Assume that a graph manifold M admits a sufficient collection $\mathcal{S}$. Then $\pi_1 M$ is virtually special.*

Once we prove Theorem 1.5, in order to derive Corollary 1.3, it remains to construct a sufficient collection for graph manifolds with boundary.

In the course of his proof [**12**, Lemma 4.7], Liu constructs a set of cohomology classes giving rise to a sufficient collection. Hence, combining this with Theorem 1.5, one can get an alternate argument for Liu's theorem that all graph manifolds admitting a non-positively curved Riemannian metric have virtually special fundamental groups. Liu suggests to us that cut-and-paste operations on the surfaces obtained in [3, Section 5.5.3] also yield a sufficient collection.

The article is organized as follows: In Section 2, we discuss notation. In Section 3, we derive Corollary 1.3 from Theorems 1.1 and 1.2. More precisely, we first prove Theorem 1.5 and then prove that graph manifolds with boundary virtually have a sufficient collection (Proposition 3.1). In Section 4, we prepare the background for the proofs of Theorem 1.1 in Section 5 and Theorem 1.2 in Section 6.

2. *Notation*

A graph manifold will be denoted by M. The JSJ tori decompose M into pieces called *blocks* (denoted usually by M_v or B). By passing to a finite degree cover [13, Proposition 4.4], we can assume that all the blocks are products $M_v = S^1 \times F_v$, where F_v is an oriented surface with at least two boundary components and non-zero genus. Then M is called *simple*. The induced quotient map $\pi_1 M_v \to \pi_1 F_v$ does not depend on the choice of the product structure.

Let S be an oriented incompressible surface embedded in M. An *elevation* $S' \to M'$ of the embedding $S \to M$ is an embedding of a cover S' of S into a cover M' of M such that the diagram below commutes. (A *lift* is an elevation with $S' = S$.)

$$\begin{array}{ccc} S' & \longrightarrow & M' \\ \downarrow & & \downarrow \\ S & \longrightarrow & M \end{array}$$

The surface S can be homotoped so that each component of $S \cap M_v$ (called a *piece*) is either *vertical* (fibred by the circles of the Seifert fibration and essential) or *horizontal* (transverse to the fibres thus covering F_v). The only exception is when S is a ∂-parallel annulus. We discuss this case separately in Remark 6.1.

Since S is embedded, for each block M_v the components of $S \cap M_v$ are either all horizontal or all vertical, or else $S \cap M_v$ is empty. We accordingly call the block S*-horizontal*, S*-vertical* or S*-empty*. When the surface S in question is understood, we simply call the block *horizontal*, *vertical* or *empty*.

We shall consider (possibly non-compact) covers $M' \to M$ of graph manifolds. The connected components in M' of the preimage of blocks of M will also be called *blocks*. When a specified elevation of S crosses a block $M'_v \to M_v$, then this block will be called *horizontal* or *vertical* if M_v is such. Other blocks of M' will be called *empty*.

3. *Cubulation*

Proof of Theorem 1.5. Complete the collection $\mathcal{S}$ to $\mathcal{S}'$ by adding all JSJ tori, and adding a collection of vertical tori in each block M_v whose base curves on the surface F_v *fill* F_v. With respect to some hyperbolic metric on F_v this means that the complementary regions of the union of the geodesic representatives of the base curves are discs or annular neighbourhoods of the boundary. Note that if we add to that family of curves the base arcs of the annuli guaranteed by property (1) of a sufficient collection, all the complementary regions become discs. We call such a family *strongly filling*.

After a homotopy we can assume that the surfaces in $\mathcal{S}'$ are pairwise transverse. Each elevation of an incompressible surface from $\mathcal{S}'$ to the universal cover $\tilde{M}$ of M splits $\tilde{M}$ into two components (up to a set of measure 0). This gives $\tilde{M}$ the structure of a 'space with walls' (see [4] or [14]). We can consider the action of $\pi_1 M$ on the associated dual CAT(0) cube complex X.

We claim that $\pi_1 M$ acts freely on X. To justify this, pick $g \in \pi_1 M$. If g does not stabilize some block $\tilde{B} \subset \tilde{M}$, then it acts freely on the tree that is the underlying graph of the graph manifold structure of $\tilde{M}$. Hence g also acts freely on X, since we have included the JSJ tori in $\mathcal{S}'$. Otherwise, suppose that g belongs to the stabilizer of $\tilde{B}$ identified with $\pi_1 B$ for some block $B \subset M$. If g is not central in $\pi_1 B$, then by the strong filling property for the vertical pieces within B, the element g acts freely on the tree that is dual to the preimage in $\tilde{B}$ of one of the pieces. Since every elevation of a surface in $\mathcal{S}'$ to $\tilde{M}$ has connected intersection with $\tilde{B}$, this implies that g acts freely on X. Otherwise, g is central in $\pi_1 B$ and the claim follows from the existence of a horizontal piece in B among the surfaces in $\mathcal{S}'$ (property (2) of a sufficient collection). Note that in most cases the action of $\pi_1 M$ on X is not cocompact.

We now invoke [7, Theorem 4.1], which is a criterion for $\pi_1 M \setminus X$ to be virtually special. In the case of a cube complex X arising from a collection $\mathcal{S}'$ of compact π_1-injective surfaces in a 3-manifold M, this criterion is satisfied when:

(1) $\mathcal{S}'$ is finite;
(2) for each surface $S \in \mathcal{S}'$, in the $\pi_1 S$ cover $M^S = \pi_1 S \setminus \tilde{M}$ of M, there are only finitely many elevations of surfaces in $\mathcal{S}'$ disjoint from S, but not separated from S by another elevation of a surface from $\mathcal{S}'$;
(3) for each $S \in \mathcal{S}'$ the subgroup $\pi_1 S$ is separable in $\pi_1 M$;
(4) for each pair of intersecting elevations $\tilde{S}, \tilde{P} \subset \tilde{M}$ of $S, P \in \mathcal{S}'$, the double coset $\mathrm{Stab}(\tilde{S})\,\mathrm{Stab}(\tilde{P})$ is separable in $\pi_1 M$.

Condition (1) is immediate; conditions (3) and (4) are supplied by Theorems 1.1 and 1.2. It remains to discuss condition (2):

Fix $S \in \mathcal{S}'$ and let P^S be an elevation of a surface in $\mathcal{S}'$ to the $\pi_1 S$ cover M^S of M. Assume that P^S is disjoint from (the lift of) S but not separated from S by another elevation of a surface from $\mathcal{S}'$. Then P^S must intersect at least one (of the finitely many) block B of M^S intersecting $S \subset M^S$ (otherwise an elevation of a JSJ torus separates S and P^S). We fix the block B. Assume first that $P^S \cap B$ is horizontal and that a component of $P^S \cap B$ projects to a specified piece of the finitely many pieces of $\mathcal{S}'$. Then there can be at most 2 such P^S, since they are all nested.

Now assume that $P^S \cap B$ is vertical. Thus $S \cap B$ is also vertical. The entire configuration can then be analysed using the base curves on the base surface F of B. For a strongly filling family of curves, it is easy to check that each pair of their elevations to the universal cover of F not separated by a third one has to be at a uniformly bounded distance. Hence there are again only finitely many possible P^S. This concludes the argument for condition (2).

Hence all the conditions of the virtual specialness criterion above are satisfied and the cube complex $\pi_1 M \setminus X$ is virtually special. □

As an application, we will consider graph manifolds with boundary.

PROPOSITION 3.1. *A graph manifold M with non-empty boundary has a finite cover with a sufficient collection.*

Note that combining Proposition 3.1 with Theorem 1.5 yields Corollary 1.3.

In the proof of Proposition 3.1 we will need the following lemma.

LEMMA 3.2 (version of [17, Lemma 1.1]). *Let $T_1, \ldots, T_n$ be the boundary components of a block M_v. Assume that we are given families of disjoint, identically oriented circles $C_1 \subset T_1, \ldots, C_{n-1} \subset T_{n-1}$ such that the oriented intersection number between C_i and the vertical*

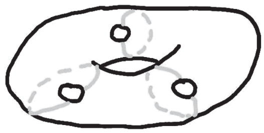

FIGURE 1. *Bases of A_e in common M_w from different i.*

fibre is non-zero and independent of i. Then there is a family of disjoint, identically oriented circles $C_n \subset T_n$ such that $\bigcup_{i=1}^n C_i$ is the boundary of an oriented horizontal surface embedded in M_v.

Proof of Proposition 3.1. Let Γ be the underlying graph of M, i.e. the graph dual to the JSJ decomposition. A vertex w of Γ is called a *boundary vertex* if its block M_w has a torus boundary component contained in ∂M. Note that a boundary vertex exists since ∂M is non-empty.

We first pass to a finite cover of M that is simple (see Section 2). Moreover, we shall pass to a finite cover whose underlying graph Γ has the following property:

(antennas): For each pair of adjacent vertices $v_0, v_1 \in \Gamma$ there is an embedded edge path $(v_0, v_1, v_2, \ldots, v_n)$ such that:

(i) the subpath $(v_1, v_2, \ldots, v_n)$ is a full subgraph (that is, induced subgraph) of Γ;
(ii) v_n is a boundary vertex.

We will first construct a sufficient collection under the assumption that (antennas) property holds. We later explain how to pass to a cover satisfying (antennas).

As discussed in the introduction, it suffices to obtain property (1) of a sufficient collection. Let $B = M_{v_0}$ be a block and T be a torus in its boundary. Let C_0 be the circle on T that is vertical with respect to B. If T is a boundary torus of the whole M, then we put $n = 0$, otherwise we define v_1 so that M_{v_1} is the block distinct from M_{v_0} containing T. Applying (antennas), we obtain an edge path satisfying (i) and (ii). We will find a properly embedded surface S_n intersecting T along circles in the direction of C_0.

For $i = 0$ to n, we inductively define surfaces $S_0 \subset \cdots \subset S_i \subset \cdots \subset S_n$ embedded in M, but not necessarily properly: S_i might have boundary components in $M_{v_i} \cap M_{v_{i+1}}$.

We define the surface S_0 to be a vertical annulus in $B = M_{v_0}$ joining T to itself, not separating M_{v_0} (this uses that M is simple). If $n = 0$, then we are done.

Otherwise, let $i \geqslant 1$ and assume that S_{i-1} has already been constructed but is not proper. Let C_{i-1} denote one of the boundary circles of S_{i-1} in $M_{v_{i-1}} \cap M_{v_i}$. If C_{i-1} is vertical in M_{v_i}, then we can complete S_{i-1} immediately to S_n by adding several vertical annuli in M_{v_i}.

Otherwise, let $\mathcal{E}$ be the family of all edges adjacent to v_i, distinct from the edges joining it to v_{i-1} and v_{i+1} (if it is defined). Since by (antennas) property the path $(v_j)_{j=1}^n$ is full, all the edges in $\mathcal{E}$ join v_i to a vertex outside the path $(v_j)_{j=1}^n$. Similarly as we have done for the edge (v_1, v_0), for each edge $e = (v_i, w) \in \mathcal{E}$ we take a vertical annulus A_e in M_w joining the boundary torus of M_w corresponding to e to itself. We require again that A_e does not separate M_w and because of that we can take it disjoint from all the annuli in M_w constructed for smaller values of i, assigned to other boundary components (see Figure 1).

The annuli A_e specify circles C_e on the tori $M_w \cap M_{v_i}$. Thus far, for all but one (or all if $i = n$) boundary tori of M_{v_i} that are not in the boundary of M, we have constructed non-vertical circles C_{i-1} or C_e. For all the boundary tori K of M in M_{v_i}, except for one (call it Q) when $i = n$, we pick arbitrary horizontal circles C_K.

By Lemma 3.2, if we take appropriate orientations on the circles C_{i-1}, C_e, C_K and we take appropriately many copies, we can find an oriented circle C_i on the remaining boundary torus

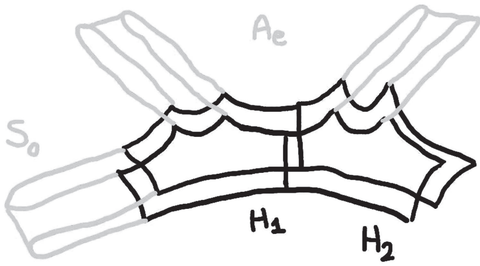

FIGURE 2. *The surface S_2.*

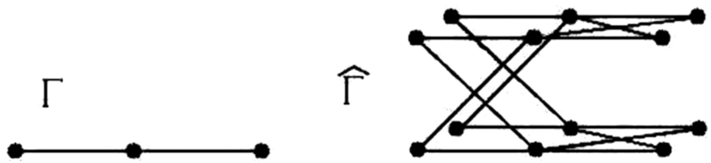

FIGURE 3. *The graphs Γ and $\hat{\Gamma}$.*

of M_{v_i} (connecting to $M_{v_{i+1}}$, or being Q), such that appropriately many copies of C_i together with the copies of C_{i-1}, C_e and C_K bound an embedded horizontal surface H_i.

Taking appropriately many copies of A_e, S_{i-1} and H_i, we form the surface S_i. If it is non-orientable, we replace it by the boundary of its regular neighbourhood.

Inductively, we arrive at the required surface S_n needed for property (1) of a sufficient collection; see Figure 2.

It remains to explain how to obtain property (antennas). Fullness is automatic if:

(1) Γ has no double edges or edges joining a vertex to itself (this is attained using residual finiteness of $\pi_1\Gamma$) and
(2) the path $(v_j)_{j=1}^n$ is always chosen to be geodesic.

It thus suffices to pass to Γ where, for each vertex v_1, there is a geodesic terminating at a boundary vertex v_n that does not pass through a prescribed neighbour v_0 of v_1.

To do this, we take the following degree 2^k cover $\hat{M}$ of M, where k is the number of blocks of M. The cover $\hat{M}$ is defined by the mapping of $H_1(M, \mathbf{Z})$ into $\mathbf{Z}_2^k$ determined by the cohomology classes of closed non-separating vertical tori, one in each of the k blocks. Let $\hat{\Gamma}$ be the underlying graph of the graph manifold $\hat{M}$; see Figure 3.

Fix vertices $v_0, v_1 \in \hat{\Gamma}$ and let γ be a geodesic path in $\hat{\Gamma}$ from v_1 to a boundary vertex v_n. If γ passes through v_0, then we alter it as follows. Let g denote the non-trivial element of the group of covering transformations of $\hat{M}$ fixing v_1. Then g maps γ to a geodesic path disjoint from v_0 terminating at a boundary vertex. This shows that $\hat{M}$ satisfies (antennas) and completes the proof of Proposition 3.1. □

We record the following consequence of the proof of Proposition 3.1, which will be used in [**15**].

COROLLARY 3.3. *Let M be a graph manifold with non-empty boundary. There exists a finite cover $\hat{M}$ of M such that, for each circle C_0 in a boundary torus $T \subset \hat{M}$, there is an incompressible surface S embedded in $\hat{M}$ that is not a ∂-parallel annulus with $S \cap T$ consisting of a non-empty set of circles parallel to C_0.*

4. *Separability: preliminaries*

This section prepares the background for the proofs of Theorems 1.1 and 1.2.

Hempel's theorem. We begin by discussing consequences of Hempel's theorem. The results of this subsection follow also immediately from [8, Theorem 4.1].

THEOREM 4.1 (special case of [9, Theorem 1.1]). *Fundamental groups of graph manifolds are residually finite.*

COROLLARY 4.2. *If T is an incompressible vertical torus in a block of a simple graph manifold M, then $\pi_1 T$ is separable in $\pi_1 M$. If T is a JSJ or boundary torus, then any finite index subgroup of $\pi_1 T$ is separable in $\pi_1 M$.*

The proof of Corollary 4.2 uses characteristic covers. The *n-characteristic* cover of a manifold B is the finite cover corresponding to the intersection of all subgroups of $\pi_1 B$ of index n. For example, the n-characteristic cover of a torus T corresponds to $n\mathbf{Z} \times n\mathbf{Z} \subset \mathbf{Z} \times \mathbf{Z} = \pi_1 T$. If B is a simple block, then since it retracts onto its boundary tori, its n-characteristic cover restricts to n-characteristic covers over its boundary tori.

Proof of Corollary 4.2. The group $\pi_1 T$ is separable in $\pi_1 M$ since it is a maximal abelian subgroup and $\pi_1 M$ is residually finite. A finite index subgroup $H \subset \pi_1 T = \mathbf{Z} \times \mathbf{Z}$ contains some $n\mathbf{Z} \times n\mathbf{Z}$. Hence it suffices to consider a finite cover M' of M formed by gluing n-characteristic covers of the blocks. Since $n\mathbf{Z} \times n\mathbf{Z}$ is separable in $\pi_1 M'$, we have that H is separable in $\pi_1 M$. □

We now prove the analogous result for annuli.

COROLLARY 4.3. *Let T be a JSJ or boundary torus in a graph manifold M. Then every cyclic subgroup of $\pi_1 T$ is separable in $\pi_1 M$.*

Proof. Let $\mathbf{Z}$ be a cyclic subgroup of $\pi_1 T$ and let $g \in \pi_1 M - \mathbf{Z}$. There is a finite index subgroup $H \subset \pi_1 T$ containing $\mathbf{Z}$, but not g. We apply Corollary 4.2 to H. □

Corollary 4.2 has two further consequences.

COROLLARY 4.4. *Let $S \subset M$ be an incompressible surface in a graph manifold. Then there is a finite cover of M where each elevation of S is* straight, *in the sense that its vertical annular pieces always join two distinct boundary components of the block.*

COROLLARY 4.5. *Let γ be a path in a graph manifold M such that its lift $\tilde{\gamma}$ to the universal cover $\widetilde{M}$ passes through as few blocks of $\widetilde{M}$ as possible in its path-homotopy class. Then there is a finite cover M' of M, where γ' (the quotient of $\tilde{\gamma}$) does not pass through the same JSJ torus more than once.*

Omnipotency. In the proof of Theorem 1.2, we will need the following 'omnipotency' lemma.

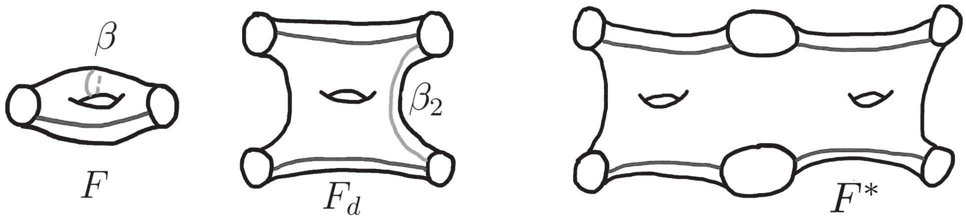

FIGURE 4. *β, β_2 and the lifts of α to F_d and F^* with $d_1 = 1, d_2 = 2$.*

LEMMA 4.6. *Let F be a surface of non-zero genus. Assign a number $n_i > 0$ to each boundary component C_i of F. Then there is a finite cover F^* of F having degree n_i on each component of the preimage of C_i.*

Proof. Since F has non-zero genus, there is a non-separating simple closed curve $\beta \subset F$. Take the double cover F_d determined by the cohomology class $[\beta] \in H^1(F, \mathbf{Z}_2)$. Each boundary component C_i of F lifts to a pair of boundary components C_i^1, C_i^2 of F_d. Choose a family of disjoint embedded arcs β_i joining C_i^1 to C_i^2. Take the cover F^* determined by the mapping of $H_1(F_d, \mathbf{Z})$ to $\prod \mathbf{Z}_{n_i}$ determined by the cohomology classes $[\beta_i] \in H^1(F_d, \mathbf{Z}_{n_i})$. □

REMARK 4.7. There is an extra feature to the construction in the proof of Lemma 4.6. If $\alpha \subset F$ is an arc joining two distinct boundary components of F, then no two lifts of α to F^* join the same pair of boundary components; see Figure 4.

Surface-injective covers. In the proof of Theorems 1.1 and 1.2, we will need 'surface-injective' covers. As preparation, we discuss the structure of the following infinite cover. As usual, we assume that $S \subset M$ is an incompressible oriented surface embedded in a graph manifold M, and that each non-empty intersection of S with a block of M is either vertical or horizontal.

DEFINITION 4.8. Let M^S denote the infinite cover $\pi_1 S \setminus \tilde{M}$ of S corresponding to $\pi_1 S \subset \pi_1 M$. Let us describe the topology of non-empty blocks of M^S. For each horizontal component S_0 of $S \cap M_v$, there is in M^S an associated horizontal block $M_v^{S_0} \cong S_0 \times \mathbf{R}$. Similarly, for each vertical component S_0 of $S \cap M_v$, there is in M^S an associated vertical block $M_v^{S_0} \cong S^1 \times \tilde{F}_v$. The annulus S_0 embeds inside $M_v^{S_0}$ as a product of the factor S^1 and a proper arc on $\tilde{F}_v$.

DEFINITION 4.9. Let S be a surface in M. A finite cover M' of M is called *S-injective* with respect to the JSJ tori if S lifts to M' and $S \cap B'$ is connected for each block B' of M'. Moreover, we require that each horizontal component of $S \cap B'$ maps with degree 1 onto the base surface of B'. We allow $M' = M$.

In particular, M' arises from M^S by quotienting each non-empty block separately (though empty blocks are identified). Observe that the intersection of the lift of S with each JSJ or boundary torus of M' is connected. The property of being S-injective is not preserved under passage to covers. Nevertheless, in Construction 4.12 we provide (high-degree) S-injective covers. We need the following terminology and lemma.

DEFINITION 4.10. A *semicover* of a graph manifold M with respect to the JSJ tori is a graph manifold $\bar{M}$ together with a local embedding $\bar{M} \to M$ restricting to a covering map over each JSJ torus and over each open block. We say that the semicover is *finite* if $\bar{M}$ is compact. Then $\bar{M} \to M$ can only fail to be a covering map at a torus $\bar{T}$ of $\partial \bar{M}$ that covers a JSJ torus of M. We refer to such a $\bar{T}$ as a *halt torus*.

LEMMA 4.11. *Let $p : \bar{M} \to M$ be a finite semicover. Suppose that all halt tori of $\bar{M}$ map homeomorphically onto JSJ tori of M. Then we can embed $\bar{M}$ in a graph manifold M' such that the semicover p extends to a finite cover $M' \to M$.*

Proof. For each M_v let d_v be the degree of the (possibly disconnected) cover $p^{-1}(M_v) \to M_v$. Similarly, let $d_{v,w}$ be the degree of $p^{-1}(T) \to T$ for the torus $T = M_v \cap M_w$. Let $D = \max_{v,w}\{d_{v,w}\}$. For each v, take $D - d_v$ copies of M_v and glue these copies to $\bar{M}$ to form M'. □

CONSTRUCTION 4.12. Let S be a straight incompressible surface in M (see Corollary 4.4) and let $N > 0$ be divisible by all the degrees of (possibly disconnected) covering maps $S \cap M_v \to F_v$. Then there is a finite cover M_N^S of M which is

(1) S-injective with respect to the JSJ tori and
(2) such that each JSJ or boundary torus of M_N^S intersected by the lift of S maps to a torus T of M with degree $N/|S \cap T|$.

Here $|S \cap T|$ denotes the number of connected components of $S \cap T$. The construction also works if S is disconnected and we will need this in [**16**].

Proof. Consider a horizontal block M_v. Let $n = |S \cap M_v|$. Let S_0 be a component of $S \cap M_v$ (they are all parallel). We take the unique degree N/n cover of M_v to which S_0 lifts. It is the quotient of the $M_v^{S_0}$ block of M^S (see Definition 4.8) by the N/nth power of the generator of covering transformations. By the divisibility hypothesis, the result of this over the boundary of M_v is that: if there are k components of intersection of S with a boundary torus, then, in the cover, we get k tori components projecting with degree N/k. Hence, for two adjacent horizontal blocks we have a matching between the elevations of the JSJ tori crossed by (the lifts of) the various S_0.

Now consider a vertical block M_v. Fix a component S_0 of $S \cap M_v$. Let F_v' be the double cover of F_v determined by the $\mathbf{Z}_2$-cohomology class of a non-separating simple closed curve on F_v. Each boundary component C of F_v is covered in F_v' by a pair C_1, C_2. Suppose that S_0 connects tori T, Q with base boundary circles of F_v denoted by C^T, C^Q, respectively. Let $t = |S \cap T|$ and $q = |S \cap Q|$. Pick disjoint embedded arcs θ and ω on F_v' joining C_1^T with C_2^T and C_1^Q with C_2^Q, respectively. Take $N/t - 1$ extra copies of F_v' containing copies of θ and $N/q - 1$ extra copies of F_v' containing copies of ω. Cutting and regluing along these arcs in cyclic order gives a cover of F_v whose boundary components project homeomorphically, except two degree N/t covers of C^T and two degree N/q covers of C^Q. To get a cover of M_v, we form the product with S^1.

We now take one such covering block for each component S_0 of $S \cap M_v$ for vertical M_v and two blocks described above for horizontal M_v. All boundary components match except that there are some hanging boundary components giving rise to halt tori.

This concludes the construction of a semicover. Note that the lift of S crosses all blocks of this semicover that cover vertical blocks of M and half of the blocks covering a horizontal

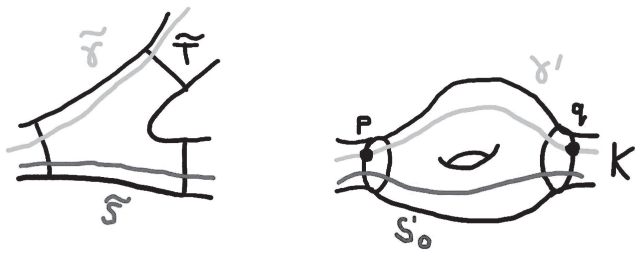

FIGURE 5. *The intersection q needs to be removed from K.*

one. This semicover satisfies the hypothesis of Lemma 4.11. We use it to obtain a (non-unique) S-injective cover M_N^S. □

5. *Separability of a surface*

Proof of Theorem 1.1. We can assume that M is simple (see Section 2) and S is straight (see Corollary 4.4).

If S is a vertical torus or annulus contained in a single block, then $\pi_1 S$ is separable by Corollaries 4.2 and 4.3. Otherwise, S contains a horizontal piece. Choose the basepoint $\tilde{m}$ of the universal cover $\tilde{M}$ of M in the interior of a horizontal piece of the elevation $\tilde{S}$ of S to $\tilde{M}$ stabilized by $\pi_1 S \subset \pi_1 M$. Let $g \in \pi_1 M - \pi_1 S$ and let $\tilde{\gamma}$ be a path in $\tilde{M}$ representing g, that is, joining $\tilde{m}$ to $g\tilde{m}$. Then $\tilde{\gamma}$ does not end on $\tilde{S}$. Our goal is to find a finite cover with the same property.

We can assume that $\tilde{\gamma}$ crosses as few elevations of JSJ tori as possible in its homotopy class. Let $\tilde{B}$ denote the last non-empty block of $\tilde{M}$ entered by $\tilde{\gamma}$ (when all blocks crossed by $\tilde{\gamma}$ are non-empty, we take $\tilde{B}$ to be the last one).

We first consider the case where $\tilde{B}$ is the last block of $\tilde{M}$ entered by $\tilde{\gamma}$. Then $\tilde{B}$ is horizontal (by the choice of $\tilde{m}$). In the quotient $B^S \subset M^S$ of $\tilde{B}$, the projection of the endpoint of $\tilde{\gamma}$ is still disjoint from S and the same is true in a sufficiently large cyclic quotient of the block B^S. This quotient coincides with an appropriate block of the cover M_N^S from Construction 4.12. Hence, for N sufficiently large, the cover M_N^S is as desired.

We now consider the case where $\tilde{B}$ is not the last block entered by $\tilde{\gamma}$, in which case $\tilde{B}$ is vertical. By Corollary 4.5, we can pass to a finite cover M' where the projection γ' of $\tilde{\gamma}$ does not backtrack, that is, γ' does not cross the same JSJ torus twice. This property will be preserved under taking further covers.

Let S' denote the elevation of S to M'. For the separability of $\pi_1 S$ we will prove that γ' does not end in S' within M' or after passing to a further finite cover.

Let $\tilde{T}$ denote the universal cover of the JSJ torus through which $\tilde{\gamma}$ leaves $\tilde{B}$. Let T' be its quotient in M'. Our first step is to guarantee that, in M' (or its finite cover), the surface S' does not cross T'.

The quotient block $B' \subset M'$ of $\tilde{B}$ is vertical. Let p and q denote the projections to B' of the first and last point of the intersection of $\tilde{\gamma}$ with $\tilde{B}$. Let S'_0 be the quotient in B' of $\tilde{S} \cap \tilde{B}$ (there might be some other components of $S' \cap B'$). Let K be the JSJ torus crossed by S'_0 other than the one containing p. Any further S'-injective cover (for example, $M_N'^{S'}$ from Construction 4.12) satisfies our condition $S' \cap T' = \emptyset$ unless $q \in K$ (that is, $T' = K$); see Figure 5. In that case, we first use Corollary 4.2 to pass to a cover where (keeping the same notation) the point q does not lie in K, and so S'_0 does not cross T'. There might still be an accidental component of $S' \cap B'$ intersecting T'. We can remove it by passing to an S'-injective cover.

Summarizing, we have constructed a cover M' where T' is disjoint from S'. It suffices now to pass to a degree 2 cover determined by the cohomology class $[T'] \in H^1(M', \mathbf{Z}_2)$. In that cover the portion of γ' after q is contained entirely in the union of empty blocks. In particular, its end lies outside the appropriate lift of S', as desired. □

6. *Separability of intersecting surfaces*

Outline of the argument. Let $\tilde{S}$ and $\tilde{P}^0$ be intersecting elevations of S and P to the universal cover $\tilde{M}$ of M. We reserve the notation $\tilde{P}$ for a different elevation of P. By our hypothesis, $\tilde{S} \cap \tilde{P}^0$ is non-empty, and so we can choose the basepoint $\tilde{m} \in \tilde{S} \cap \tilde{P}^0$.

We fix $g \in \pi_1 M - \mathrm{Stab}(\tilde{S})\,\mathrm{Stab}(\tilde{P}^0)$ and take a path $\tilde{\gamma}$ starting at $\tilde{m}$ representing g in $\tilde{M}$. Let $\tilde{P}$ denote the elevation of P through the terminal point $g\tilde{m}$ of $\tilde{\gamma}$. We aim to find a finite quotient of $\tilde{M}$, where the projections of $\tilde{S}$ and $\tilde{P}$ are 'disjoint' in the sense that they do not intersect at a basepoint-translate.

The main object we work with is the 'core' $\bar{M} \subset \tilde{M}$ consisting of blocks simultaneously intersecting $\tilde{S}$ and $\tilde{P}$. In Step 1, we prove that, for each $\pi_1 S$ orbit in $\bar{M}$ of a core block, we can quotient it to a finite block with 'disjoint' quotients of $\tilde{S}$ and $\tilde{P}$.

In Step 2, we use Step 1 to show how to simultaneously quotient the whole core (or rather its extension) to a finite quotient $\hat{M}'$ where the images of $\tilde{S}$ and $\tilde{P}$ are 'disjoint' (Step 2(i)). Moreover, we arrange that the images of $\tilde{S}$ and $\tilde{P}$ never simultaneously intersect the same halt torus of the semicover (see Definition 4.10) $\hat{M}' \to M$ (Step 2(iii)). The semicover $\hat{M}'$ extends to a finite cover M' by Step 2(ii).

Finally, in Step 3, we use Step 2(iii) to pass to a further cover, where the quotients of $\tilde{S}$ and $\tilde{P}$ can meet only inside the image of the core. But this is excluded by Step 2(i).

Proof of Theorem 1.2. We choose $\tilde{m} \in \tilde{S} \cap \tilde{P}^0$ as in the outline of the argument. Without loss of generality, we can assume that if $\tilde{m}$ lies in a vertical piece of $\tilde{P}^0$, then it also lies in a vertical piece of $\tilde{S}$. We identify $\mathrm{Stab}(\tilde{S}) \subset \pi_1 M$ with $\pi_1 S$. Note that, by Corollary 4.4, by passing to a finite cover we can assume that S and P are straight. As usual, M can be also assumed to be simple and S-injective.

Let $g \notin \mathrm{Stab}(\tilde{S})\,\mathrm{Stab}(\tilde{P}^0)$ and let $\tilde{\gamma}$ be a path representing g in $\tilde{M}$ as in the outline. We can assume that $\tilde{\gamma}$ traverses as few blocks of $\tilde{M}$ as possible.

Recall that $\tilde{S}$ is the elevation of S to $\tilde{M}$ passing through the initial point $\tilde{m}$ of $\tilde{\gamma}$, and $\tilde{P}$ is the elevation of P to $\tilde{M}$ passing through the terminal point $g\tilde{m}$ of $\tilde{\gamma}$. Our hypothesis on g says that $\tilde{S}$ and $\tilde{P}$ do not cross at any translate of the basepoint $\tilde{m}$ (we will refer to such a point or its quotient in an intermediate cover as a *basepoint-translate*). Our separability goal is to find a finite cover of M with the same property.

The *core* $\bar{M}$ of $\tilde{M}$ is the union of blocks intersecting both $\tilde{S}$ and $\tilde{P}$ (see Figure 6, but note that the core might consist of an infinite number of blocks). Assume that the core is non-empty, we will consider the other case at the very end of the proof.

Let $\bar{S} = \tilde{S} \cap \bar{M}$ and $\bar{P} = \tilde{P} \cap \bar{M}$. Let $\bar{M}^S$ be the manifold obtained from the core by identifying points in the same orbit of $\pi_1 S$ (this is not a genuine action on $\bar{M}$, only a partial one). In this identification, we treat the core as an open manifold, so we do not identify boundary components whose adjacent core blocks are not identified.

Let P^S be the quotient of $\bar{P}$ in $\bar{M}^S$ and let $\hat{S}$ be the quotient of $\bar{S}$ in $\bar{M}^S$. Note that P^S and $\hat{S}$ do not go through the same basepoint-translate. The notation $\hat{S}$ instead of S^S is justified by the fact that $\hat{S}$ is in fact a lift of a 'core' subsurface of S. Note that though the map $\bar{M} \to \bar{M}^S$ is not *proper* in the sense that a boundary component of $\bar{M}$ might be mapped into the interior $\bar{M}^S$, its restriction to $\bar{P} \to P^S$ is proper (see Figure 7). Equivalently, boundary components of $\bar{P}$ are mapped onto boundary components of P^S.

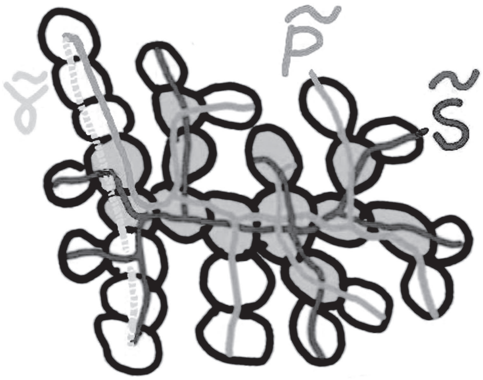

FIGURE 6. *The core of $\tilde{M}$ is shaded.*

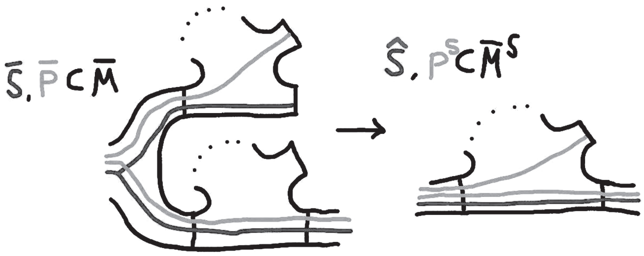

FIGURE 7. *$\bar{P} \to P^S$ is proper.*

STEP 1: *Let B^S be a block in $\bar{M}^S$ covering a block B of M. Then B^S factors through a finite cover B^* of B where quotients of P^S and $\hat{S}$ still do not intersect at a basepoint-translate. Moreover, in the case where B^S is $\hat{S}$-vertical let $K_1, K_2 \subset \partial B^S$ denote the cylinders intersected by $\hat{S}$, and let K_1^*, K_2^* be their quotients in B^*, respectively. We require that any quotient piece of P^S in B^* intersecting both K_1^* and K_2^* is a quotient of a piece in B^S intersecting K_1 and K_2.*

Loosely speaking, in Step 1 we shall achieve separability at a single core block. Note that the first property of the finite cover B^* in Step 1 is preserved by passing to a further cover that is a quotient of B^S. In the construction that we will give, in the case where B^S is P^S-horizontal, also the property in the second assertion of Step 1 is invariant under passing to a further cover.

Proof of Step 1. First, assume that the block B^S is P^S-horizontal. Let $S_0 = \hat{S} \cap B^S$. First consider the case where S_0 is vertical. Then there are only finitely many elevations of $P \cap B \subset M$ to B^S: their number is bounded by the degree of $P \cap B \to F$, where F is the base surface of B. The action of covering transformations of B on the universal cover of the block B^S factors to an action on B^S. As there are only finitely many elevations of $P \cap B$ to B^S, a finite index subgroup of the group of covering transformations preserves all of them. We quotient by this subgroup to obtain a desired finite cover B^* of B.

Now consider the case where S_0 is horizontal. Then the action of covering transformations on the universal cover of the block B factors to $\mathbf{Z} = \langle c \rangle$ action on B^S. The easy subcase is where one (hence any) elevation of $P \cap B$ to B^S is non-compact (see Figure 8). Then, as in the previous case, there are only finitely many elevations of $P \cap B$ and we can choose a finite cover B^* obtained by quotienting by a subgroup $\langle c^k \rangle$ that maps $P^S \cap B^S$ onto itself.

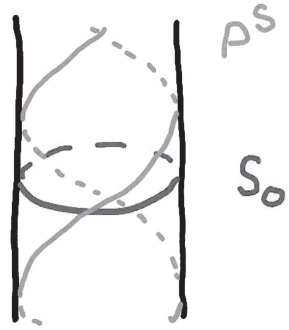

FIGURE 8. *Non-compact components of $P^S \cap B^S$: cross-section by a cylinder.*

The interesting subcase is where the elevations of $P \cap B$ to B^S are compact. Let P_0 be any component of $P^S \cap B^S$. Let $\mathcal{P}$ be the maximal connected subsurface of P^S containing P_0 consisting uniquely of horizontal pieces. First consider the situation where P_0 is properly contained in $\mathcal{P} \cap B^S$. We will show that $P^S \cap B^S$ is invariant under some c^k (as in the case of non-compact P_0).

Note that the pieces of $\mathcal{P} \cap B^S$ might not lie in one $\langle c \rangle$-orbit. However, we can extend this action to another cyclic action $\langle c \rangle \subset \langle \underline{c} \rangle$ on B^S by homeomorphisms for which all the pieces of $\mathcal{P} \cap B^S$ lie in one orbit.

Assume that, for some $k \neq 0$, the translate $\underline{c}^k P_0$ lies in $\mathcal{P}$. Let B_i be a sequence of blocks of $\bar{M}^S$ with $P_i \subset B_i$ a sequence of pieces in $\mathcal{P}$ connecting P_0 to $\underline{c}^k P_0$. The action of $\underline{c}$ extends to all blocks B_i crossed by $\mathcal{P}$ (some might be $\hat{S}$-vertical). Hence, for any n there is a sequence of pieces that are translates of P_i joining $\underline{c}^n P_0$ to $\underline{c}^{k+n} P_0$. Thus $\underline{c}^n P_0$ lies in P^S if and only if $\underline{c}^{k+n} P_0$ does; in other words, $P^S \cap B$ is $\underline{c}^k$-invariant. Then, for some k' we have that $P^S \cap B$ is $c^{k'}$-invariant, as desired.

It remains to consider the situation where P_0 equals $\mathcal{P} \cap B^S$. Then, by the c-action argument above, the same is true for any choice of P_0 in $P^S \cap B^S$. Moreover, since there are only finitely many vertical pieces of P^S in $\bar{M}^S$ with both boundary components in the interior of $\bar{M}^S$, only finitely many translate copies of $\mathcal{P}$ are joined together contributing to $P^S \cap B^S$. We conclude that $P^S \cap B^S$ is compact. Then, for any sufficiently large k, no element of $\langle c^k \rangle$ maps a basepoint-translate in S_0 onto a point in P^S. This completes the argument for Step 1 under the assumption that the block B^S is P^S-horizontal. If B^S is P^S-vertical and $\hat{S}$-horizontal, then we can take any cyclic quotient, since B^S does not contain a basepoint-translate.

Finally, assume that B^S is both P^S-vertical and $\hat{S}$-vertical. Let K_1, K_2 be the boundary cylinders of B^S crossed by $\hat{S}$. By the definition of the core, except for the exceptional situation where the core is a single block and hence $P^S \cap B^S$ has just one component, each piece of $P^S \cap B^S$ intersects some K_i. By the c-action argument applied to adjacent (P^S-horizontal) blocks of B^S, for each $i = 1, 2$, the intersection $P^S \cap K_i$ is either compact or periodic. Then, after quotienting B^S by finite index subgroups of one, both or none of the stabilizers of K_i we obtain $\check{B}$, in which the quotient of P^S is compact and still does not intersect $\hat{S}$ in a basepoint-translate. Let $\check{F} \to F$ be the cover induced between the base surfaces of $\check{B} \to B$. Let $C_i \subset \partial F$ be the images of K_i under $B^S \to B \to F$. By the separability of $\pi_1 \check{F}$ and $\pi_1 C_i \pi_1 \check{F}$ in $\pi_1 F$, the cover $\check{B}$ quotients further to a desired cover B^*. This completes the argument for Step 1. □

Let $\hat{M}$ denote the quotient of $\bar{M}$ (and hence $\bar{M}^S$) in M. However, if there is a JSJ torus K in M outside the image of the interior of $\bar{M}$ but with both of its adjacent blocks within the image of $\bar{M}$, then we put in $\hat{M}$ two copies of K, each compactifying one of the adjacent blocks. In other words, $\hat{M}$ is contained in M only in the sense of manifolds open at the boundary. Since

M is S-injective, each block of $\hat{M}$ is covered by exactly one block of $\bar{M}^S$. We note, however, that $\bar{M}^S \to \hat{M}$ is only an infinite semicover.

Step 2: *There is a finite semicover $\hat{M}'$ of $\hat{M}$ through which the map $\bar{M} \to \hat{M}$ factors, with the following properties. Let $\hat{S}', \hat{P}' \subset \hat{M}'$ be the extensions of the quotients of $\bar{S}, \bar{P}$ in $\hat{M}'$ to properly embedded connected surfaces mapping to $S \cap \hat{M}$ and $P \cap \hat{M}$.*

(i) *Each block B' of $\hat{M}'$ is a further cover of a cover B^* satisfying Step 1. Moreover, the images of $\hat{S}' \cap B'$, $\hat{P}' \cap B'$ in B^* are contained in the quotients of P^S and $\hat{S}$.*

(ii) *Over all boundary tori the semicover $\hat{M}' \to \hat{M}$ is $n!$-characteristic (for some uniform n), that is, it corresponds to the subgroup $n!\mathbf{Z} \times n!\mathbf{Z} \subset \mathbf{Z} \times \mathbf{Z}$.*

(iii) *Each halt torus of the semicover $\hat{M}' \to M$ intersects at most one of $\hat{S}', \hat{P}'$.*

Moreover, $\hat{M}'$ is $\hat{S}'$-injective.

Note that in view of Step 1, Step 2(i) implies immediately that $\hat{S}'$ and $\hat{P}'$ do not intersect at a basepoint-translate.

Proof of Step 2. The value n is the maximum of n needed to execute the following construction over each of the finitely many blocks of $\hat{M}$.

First suppose that $B \subset \hat{M}$ is P-horizontal and S-horizontal. Take the cyclic cover B^* guaranteed by Step 1. It may be taken with any degree $n!$ for n sufficiently large. To make the quotient to B characteristic on the boundary tori, we pass from B^* to a cover B' induced by any cover of $S \cap B$ of degree $n!$ on each boundary component (use Lemma 4.6).

Now assume that $B \subset \hat{M}$ is P-horizontal but S-vertical. Again take the cover B^* guaranteed by Step 1. By Lemma 4.6, it may be chosen to be degree $n!$ on boundary tori for n sufficiently large. In order to make the quotient to B characteristic on the boundary tori, we pass from B^* to a cyclic cover B' of degree $n!$ determined by an arbitrary degree 1 horizontal surface.

In the case where the block $B \subset \hat{M}$ is both P-vertical and S-vertical, finding convenient B^* will involve several steps. First of all, there is a cover B^* of B satisfying Step 1. Since we still want to replace B^* by a particular finite cover, in order to simplify the notation, we will assume that B already has the properties from Step 1.

By Lemma 4.6, for sufficiently large n we get a finite quotient B^* of B^S of degree $n!$ on all boundary components over B. By Remark 4.7, the cover B^* satisfies the second assertion in Step 1. As before, to get a characteristic cover over the boundary tori, we pass to a cover B' of B^*, determined by an arbitrary element of $H^1(B^*, \mathbf{Z}_{n!})$ dual to a degree 1 horizontal surface in B^*.

The last case is where a block B is P-vertical and S-horizontal. Here n is arbitrary. We first take the degree $n!$ cyclic cover B^* of B determined by $[S_0] \in H^1(B^*, \mathbf{Z}_{n!})$, where $S_0 = S \cap B$. Next we pass to a cover B' induced by any cover $S_0' \to S_0$ of degree $n!$ on each boundary component (use Lemma 4.6).

We take n sufficiently large for both the construction in Step 1 and various applications of Lemma 4.6 with the above data to work. Then all blocks B have covers B' that are $n!$-characteristic on the boundary. Now we will take several copies of each B'. In each of these copies we distinguish one elevation Σ_c of (connected) $S \cap B$. In the case where B is S-horizontal, the surface Σ_c is of degree 1 over the base surface of B'. Hence the intersection of Σ_c with each boundary torus is at most a single curve. We take the right number of copies of each B' so that the degree of the disconnected cover from the union of the copies of Σ_c to $S \cap B$ does not depend on B. We match up these blocks along boundary tori intersected by Σ_c, also matching the Σ_c, to form a connected semicover $\hat{M}'$ of $\hat{M}$. We pick any of the maps $\bar{M} \to \hat{M}'$ mapping $\bar{S}$ to the union Σ of the Σ_c. We will now verify that all the required properties of $\hat{M}'$ hold.

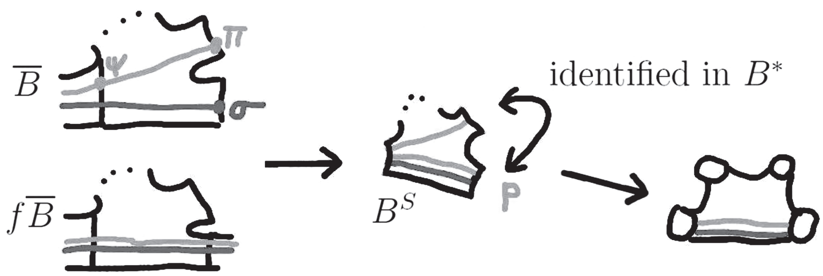

FIGURE 9. *Finding p.*

Property (ii) is clear from the construction. Note that $\hat{M}'$ is $\hat{S}'$-injective, since $\hat{S}' = \Sigma$. Moreover, obviously, for each copy B_c in $\hat{M}'$ of a block B' that covers the quotient B^* of B^S in $\bar{M}^S$, we have the following. Under the identification of B_c with B', the projection to B^* of the intersection $\hat{S}' \cap B_c$ is contained in the projection to B^* of $\hat{S} \cap B^S$. We now claim the same for $\hat{P}'$: the projection to B^* of the intersection $\hat{P}' \cap B_c$ is contained in the projection to B^* of $P^S \cap B^S$.

Before we justify the claim, we note that although the map $\bar{M} \to \hat{M}$ factors through $\hat{M}'$, the image of $\bar{P}$ in $\hat{M}'$ might be smaller than $\hat{P}'$. It is a priori unclear where the extension $\hat{P}'$ is located. We look for a surface Π for $\hat{P}'$ which replaces Σ for $\hat{S}'$ in the argument above.

Property (i) follows from the claim since B^* was chosen as in Step 1.

To justify the claim, let H be the set of elements $h \in \pi_1 S$ preserving the elevation of $\hat{S}$ to $\tilde{M}$ containing $\bar{S}$. In other words, $H = \pi_1\hat{S}$. Let $\Pi \subset \hat{M}'$ be the union of the projections of $h\bar{P}$, over $h \in H$. Obviously Π has the property that, for each $B_c \simeq B'$ as above, the projection to B^* of the intersection $\Pi \cap B_c$ is contained in the projection of $P^S \cap B^S$. Since the projection of $\bar{P}$ is contained in Π, in order to justify the claim, it remains to prove that Π is a surface properly embedded in $\hat{M}'$:

The quotient in $\hat{M}'$ of a translate $h\bar{P}$ with $h \in H$ might fail to be proper only at a quotient of a boundary line $h\pi$ of $h\bar{P}$. Then π lies in the boundary of an $\bar{S}$-vertical block $\bar{B}$. In one of the two planes of $\partial\bar{B}$ crossed by $\bar{S}$ there is a boundary line ψ of the piece containing π. Denote by σ the boundary line of $\bar{S} \cap \bar{B}$ in the plane not containing ψ. The only boundary component of the quotient of $h\bar{B}$ in $\hat{M}'$ that is possibly in the interior of $\hat{M}'$, besides the one containing the quotient of $h\psi$, is the one containing the quotient of $h\sigma$. Assume then that $h\pi$ and $h\sigma$, hence also π and σ are mapped into the same JSJ torus of $\hat{M}'$. Let $K^S \subset \partial B^S$ be the cylinder containing the quotient of σ. By the second assertion in Step 1, there is in B^S a piece p of P^S with a component of $p \cap K^S$ identified with π upon passing to B^*; see Figure 9. In the piece of $\bar{P}$ mapped to p let π_p be the boundary component identified with π in B^*. Note that we chose $B' \to B^*$ so that $\pi_1 B^* = \pi_1 B' \pi_1 S_0$ where S_0 is the lift of $S \cap B$ to B^*. Hence there is $f \in H$ such that $f\pi$ and π_p are identified in one of the copies of $B' \subset \hat{M}'$ as well. By the definition of the core, the line π_p is in the interior of $\bar{P}$. Hence the quotient in $\hat{M}'$ of $h\pi$ is in the interior of the quotient of $hf^{-1}\bar{P} \subset \Pi$. This completes the argument for the claim and hence for property (i).

As for property (iii), we also need to use the second assertion in Step 1. Let K' be a halt torus of $\hat{M}'$ in a copy of a block B'. Let $B^S \subset \bar{M}^S$ be the block mapped to the same B^* as B' and let K^S be that elevation from B^* to B^S of the quotient of K', which crosses $\hat{S}$. Then K^S lies also in the boundary of $\bar{M}^S$. Hence P^S is disjoint from K^S by the definition of the core. In view of Step 1, the quotient of $\bar{P}$ in $\hat{P}'$ is disjoint from K'. The same is true for $h\bar{P}$ over $h \in H$ (H as in the proof of the claim above), and hence for the whole $\hat{P}'$. Thus we have proved property (iii), that $\hat{S}'$ and $\hat{P}'$ do not cross K' simultaneously. This completes the argument for Step 2.

□

The graph manifold $\hat{M}'$ is a semicover of M. By Step 2(ii), we can complete it to a cover M' by taking an appropriate number of disjoint copies of any finite covers of blocks in M that are $n!$-characteristic on the boundary. We require that $\hat{M}'$ embed in M' as a closed submanifold: we do not allow accidental matching of boundary components of open $\hat{M}'$. By choosing those covers correctly, we keep M' to be S'-injective, where $S' \subset M'$ is the appropriate elevation of S. It remains to perform the following:

STEP 3: *There is a finite cover M'' of M', whose blocks B'' intersecting simultaneously the quotients S'', P'' of $\tilde{S}, \tilde{P}$ project to $B' \subset \hat{M}'$ so that $S'' \cap B''$ maps into $\hat{S}'$ and $P'' \cap B''$ maps into $\hat{P}'$.*

Proof of Step 3. Let $P' \subset M'$ be the quotient of $\tilde{P}$. Let $\check{M}$ be a P'-injective cover of M'. We keep the notation P' for the lift of P' to $\check{M}$ (quotient of $\tilde{P}$) and denote by $\check{S}$ the appropriate elevation of S (quotient of $\tilde{S}$). Let τ be the union of the JSJ tori of $\check{M}$ containing the boundary components of $\hat{P}'$ which are not in the boundary of P'.

We consider the degree 2 cover M'' of $\check{M}$ defined by the $\mathbf{Z}_2$-cohomology class $[\tau]$. The union of tori τ is disjoint from $\check{S}$, by Step 2(iii). On the other hand, by P'-injectivity, τ separates P' into $\hat{P}'$ and its complement. Hence both $\check{S}$ and P' lift to M'' and any piece of the lifted $P' \setminus \hat{P}'$ is in an $\check{S}$-empty block of M''. This implies that M'' satisfies Step 3. □

Conclusion. By Step 3, if surfaces S'' and P'' intersect in a block B'' of M'', then they project to surfaces $\hat{S}'$ and $\hat{P}'$ in a block B' of $\hat{M}'$. By Step 2(i), the surfaces $\hat{S}'$ and $\hat{P}'$ do not intersect in a basepoint-translate. Then the same is true for S'' and P''.

This concludes the proof of the main theorem except for the case where the core is empty, which we shall now discuss. First, applying Corollary 4.5, we pass to a cover M, where the path γ representing g does not go through the same block twice. By possibly passing to a further cover, we also assume that M is S-injective. Then, instead of the core we consider the minimal connected graph submanifold of $\tilde{M}$ crossed by both $\tilde{S}$ and $\tilde{P}$. Its blocks are in correspondence with some of the blocks of M crossed by γ. Steps 1 and 2 are now immediate. The surface $\hat{S}'$ is contained in a single block of the semicover $\hat{M}'$. We extend $\hat{M}'$ to a cover M' that is S'-injective and such that $S' \cap \hat{M}' = \hat{S}'$. We finally perform Step 3 as in the main argument. □

REMARK 6.1. Theorems 1.1 and 1.2 also hold when S and P are allowed to be ∂-parallel annuli. Theorem 1.1 follows directly from Corollary 4.3.

For Theorem 1.2, if P or S are ∂-parallel annuli, we homotope them into the boundary before we determine if their elevations $\tilde{S}, \tilde{P}^0$ intersect. Without loss of generality we can assume that S is a ∂-parallel annulus, parallel to a boundary torus T. We identify $\mathrm{Stab}(\tilde{S}), \mathrm{Stab}(\tilde{P}^0)$ with $\pi_1 S, \pi_1 P$ for an appropriate basepoint.

If P is also a ∂-parallel annulus, then it is also parallel to T, and it suffices to use Corollary 4.2 for the separability of the finite index subgroup $\pi_1 S\ \pi_1 P$ of $\pi_1 T$ in $\pi_1 M$.

If P is not a ∂-parallel annulus, we can assume that $\pi_1 S$ is not contained in $\pi_1 P \cap \pi_1 T$. Let $H \subset \pi_1 T$ be the finite index subgroup generated by $\pi_1 S$ and $\pi_1 P \cap \pi_1 T$. Then $H\pi_1 P = \pi_1 S\ \pi_1 P$. By the separability of H in $\pi_1 M$ (Corollary 4.2), there is a finite cover M' of M with boundary torus T' with fundamental group H. By Theorem 1.2 applied to T' and an appropriate elevation P' of P to M', we have that $H\pi_1 P'$ is separable in $\pi_1 M'$. Hence $H\pi_1 P$ is separable in $\pi_1 M$, as desired.

Acknowledgements. We thank Yi Liu for providing feedback on our preprint leading to many improvements.

References

1. I. Agol, 'Criteria for virtual fibering', *J. Topol.* 1 (2008) 269–284.
2. I. Agol, 'The virtual Haken conjecture', with an Appendix by Ian Agol and Daniel Groves, and Jason Manning, Preprint, 2012, arXiv:1204.2810.
3. S. V. Buyalo and P. V. Svetlov, 'Topological and geometric properties of graph manifolds', *Algebra i Analiz* 16 (2004) 3–68 (Russian, with Russian summary); English translation, *St. Petersburg Math. J.* 16 (2005) 297–340.
4. I. Chatterji and G. Niblo, 'From wall spaces to CAT(0) cube complexes', *Internat. J. Algebra Comput.* 15 (2005) 875–885.
5. G. Duchamp and D. Krob, 'The lower central series of the free partially commutative group', *Semigroup Forum* 45 (1992) 385–394.
6. F. Haglund and D. T. Wise, 'Special cube complexes', *Geom. Funct. Anal.* 17 (2008) 1551–1620.
7. F. Haglund and D. T. Wise, 'Coxeter groups are virtually special', *Adv. Math.* 224 (2010) 1890–1903.
8. E. Hamilton, 'Abelian subgroup separability of Haken 3-manifolds and closed hyperbolic n-orbifolds', *Proc. London Math. Soc.* 83 (2001) 626–646.
9. J. Hempel, 'Residual finiteness for 3-manifolds', *Combinatorial group theory and topology (Alta, UT, 1984)*, Annals of Mathematics Studies 111 (Princeton University Press, Princeton, NJ, 1987) 379–396.
10. S. P. Humphries, 'On representations of Artin groups and the Tits conjecture', *J. Algebra* 169 (1994) 847–862.
11. B. Leeb, '3-manifolds with(out) metrics of nonpositive curvature', *Invent. Math.* 122 (1995) 277–289.
12. Y. Liu, 'Virtual cubulation of nonpositively curved graph manifolds', *J. Topol.* (2013), to appear.
13. J. Luecke and Y.-Q. Wu, 'Relative Euler number and finite covers of graph manifolds', *Geometric topology (Athens, GA, 1993)*, AMS/IP Studies in Advanced Mathematics 2 (American Mathematical Society, Providence, RI, 1997) 80–103.
14. B. Nica, 'Cubulating spaces with walls', *Algebr. Geom. Topol.* 4 (2004) 297–309 (electronic).
15. P. Przytycki and D. T. Wise, 'Mixed 3-manifolds are virtually special' Preprint, 2012, arXiv:1205.6742.
16. P. Przytycki and D. T. Wise, 'Separability of embedded surfaces in 3-manifolds', Preprint, 2012, arXiv:1205.6742.
17. S. Wang and F. Yu, 'Graph manifolds with non-empty boundary are covered by surface bundles', *Math. Proc. Cambridge Philos. Soc.* 122 (1997) 447–455.
18. D. T. Wise, 'The structure of groups with quasiconvex hierarchy', Preprint, 2011, http://www.math.mcgill.ca/wise/papers.html.

Piotr Przytycki
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warsaw
Poland

pprzytyc@mimuw.edu.pl

Daniel T. Wise
Department of Mathematics and Statistics
McGill University
Montreal
Quebec
Canada H3A 2K6

wise@math.mcgill.ca