TAIL EQUIVALENCE OF UNICORN PATHS

PIOTR PRZYTYCKI† AND MARCIN SABOK‡

Abstract. Let $S$ be an orientable surface of finite type. Using Pho-On’s infinite unicorn paths, we prove the hyperfiniteness of orbit equivalence relations induced by the actions of the mapping class group of $S$ on the Gromov boundaries of the arc graph and the curve graph of $S$. In the curve graph case, this strengthens the results of Hamenstädt and Kida that this action is universally amenable.

1. Introduction

An equivalence relation $E$ on a standard Borel space $X$ is Borel if $E$ is a Borel subset of $X \times X$. An equivalence relation is countable (resp. finite) if every equivalence class is countable (resp. finite). Given a Borel action of a countable group on a standard Borel space $X$, the induced orbit equivalence relation is a countable Borel equivalence relation. A Borel equivalence relation $E$ is hyperfinite if $E$ can be written as an increasing union of a sequence of finite Borel equivalence relations.

Let $S$ be an oriented surface of genus $g \geq 0$ with $n \geq 0$ punctures, of negative Euler characteristic. We denote by $\mathcal{A}(S)$ (for $n \geq 1$) and $\mathcal{C}(S)$ its arc graph and its curve graph, which are Gromov hyperbolic (see Section 2). The actions of the mapping class group $\text{Mod}(S)$ on $\mathcal{A}(S)$ and $\mathcal{C}(S)$ by automorphisms extend to actions on their Gromov boundaries by homeomorphisms. Our main result is:

Theorem 1.1. The orbit equivalence relation on $\partial \mathcal{A}(S)$ induced by the action of $\text{Mod}(S)$ is hyperfinite.

As a consequence we will derive:

Corollary 1.2. The orbit equivalence relation on $\partial \mathcal{C}(S)$ induced by the action of $\text{Mod}(S)$ is hyperfinite.

This strengthens the results of Hamenstädt [Ham09, Cor 2] and Kida [Kid08, Thm 1.4(ii)] that this equivalence relation is universally amenable.

Universal amenability. Let $\mu$ be a Borel probability measure on a standard Borel space $X$. A countable Borel equivalence relation

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on $X$ is $\mu$-amenable if there is a map assigning to each equivalence class $[x]$ a finitely additive probability measure $\Phi_{[x]}$, defined on all subsets of $[x]$, such that for each bounded Borel map $F : X^2 \to \mathbb{R}$, the function $f : X \to \mathbb{R}$ given by $f(x) = \int F(x, y) d\Phi_{[x]}(y)$ is $\mu$-measurable (see [Kec93, §3]). An equivalence relation is *universally amenable* if it is $\mu$-amenable for every Borel probability measure $\mu$.

Suppose now that we have a Borel action of a countable group $G$ on $X$ with amenable stabilisers. This action is universally amenable, if its induced orbit equivalence relation is universally amenable. (See [AEG94, Thm 5.1], for the original definition of universally amenable action see e.g. [Zim84].) In particular, the orbit equivalence relation induced by a Borel action of an amenable group is universally amenable.

Boundary actions have been studied extensively from the point of view of amenability. Connes, Feldman, and Weiss [CFW81] and, independently, Vershik [Ver78], showed that the induced action of the finitely generated free group $F_n$ on its Gromov boundary $\partial F_n$ is universally amenable. This was later generalised by Adams [Ada94] to all hyperbolic groups. Bestvina, Horbez, and Guirardel proved that the action of $\text{Out}(F_n)$ on the Gromov boundary of its free factor complex is universally amenable, see [BGH17, Thm 6.4] and [GHL20, Prop 7.2] (which uses the description of the Gromov boundary of the free factor complex in [BR15] and [Ham14]).

**Hyperfiniteness.** As shown independently by Weiss and Slaman–Steel [Gao09, Thm 7.2.4], a Borel equivalence relation $E$ is hyperfinite if and only if there is a Borel action of $\mathbb{Z}$ inducing $E$ as its orbit equivalence relation. Since $\mathbb{Z}$ is amenable, the hyperfiniteness of an equivalence relation implies its universal amenability. It is a well-known open problem, whether the converse holds, i.e. whether a universally amenable equivalence relation is always hyperfinite.

The relative complexity of Borel equivalence relations is measured by Borel reducibility. Given two equivalence relations $E$ and $F$ on standard Borel spaces $X$ and $Y$, respectively, a function $f : X \to Y$ is a *Borel reduction* from $E$ to $F$ if $f$ is a Borel function and for every $a, b \in X$ we have $a \sim_E b$ if and only if $f(a) \sim_F f(b)$. A relation $E$ is *Borel reducible* to $F$, if there exists a Borel reduction from $E$ to $F$. The relation $E_0$ is defined on $\{0, 1\}^\mathbb{N}$ (with the product topology) as $(a_i)_{i=0}^\infty \sim_{E_0} (b_i)_{i=0}^\infty$ if $a_i = b_i$ for all $i$ sufficiently large. It is easy to see that $E_0$ is hyperfinite. In fact, a countable Borel equivalence relation is hyperfinite if and only if it is Borel reducible to $E_0$ [Gao09, Thm 7.2.2].

Let $\Omega$ be a countable set with discrete topology. The *tail equivalence relation* $E_t$ on $\Omega^\mathbb{N}$ is defined as $(a_i)_{i=0}^\infty \sim_{E_t} (b_i)_{i=0}^\infty$ if there exists $k \in \mathbb{Z}$ such that $a_i = b_{i+k}$ for all $i$ sufficiently large. Dougherty, Jackson, and Kechris showed that $E_t$ is Borel reducible to $E_0$, and so it is hyperfinite [DJK94, Cor 8.2]. It is not hard to see that the orbit equivalence relation induced by the action of $F_n$ on $\partial F_n$ is Borel reducible to $E_t$ with
finite $\Omega$. Hence that orbit equivalence relation on $\partial F_n$ is hyperfinite, which we will shortly express by saying that the boundary action of $F_n$ is hyperfinite.

More recently, Huang, Sabok, and Shinko [HSS19] showed that for cocompactly cubulated hyperbolic groups, their boundary actions are hyperfinite. The proof relied on a study of geodesic ray bundles in hyperbolic groups. While Touikan [Tou18] showed that that approach does not work for arbitrary hyperbolic groups, Marquis [Mar19] used it to prove the hyperfiniteness of boundary actions of groups acting cocompactly on locally compact hyperbolic buildings with trivial chamber stabilisers. Very recently, Marquis and Sabok [MS20] showed the hyperfiniteness of the boundary action of an arbitrary hyperbolic group.

Organisation. In Section 2 we recall the basics on arcs, laminations, and unicorn paths. In Section 3 we prove a pair of key lemmas: the local characterisation of Pho-On’s infinite unicorn paths, and the tail equivalence for asymptotic infinite unicorn paths. This allows for the proofs for Theorem 1.1 and Corollary 1.2 in Section 4.

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2. Unicorn paths

2.1. Arcs and laminations. As in the introduction, $S$ is obtained from a closed oriented surface of genus $g$ by removing $n$ points. Thus $S$ has $n$ topological ends, which are called punctures. An oriented arc on $S$ is a map from $(0, 1)$ to $S$ that is proper. A proper map induces a map between topological ends of spaces, and in this sense each endpoint of $(0, 1)$ is sent to a puncture of $S$. We will say that the oriented arc starts and ends at these punctures. A homotopy between oriented arcs $a$ and $b$ is a proper map $(0, 1) \times [0, 1] \to S$ whose restriction to $(0, 1) \times \{0\}$ equals $a$ and whose restriction to $(0, 1) \times \{1\}$ equals $b$. In particular, $a$ and $b$ start at the same puncture and end at the same puncture. A curve on $S$ is a map from a circle $S^1$ to $S$.

An oriented arc or a curve is simple if it is an embedding. In that case we can and will identify the oriented arc or the curve with its image in $S$. We record, however, the orientation of the arc, while for the curve we discard it. A curve is essential if it is not homotopically trivial. A curve $c: S^1 \to S$ is non-peripheral if it cannot be homotoped into the puncture in the sense that there is no proper map $S^1 \times [0, 1) \to S$ whose restriction to $S^1 \times \{0\}$ is $c$. An oriented arc $a: (0, 1) \to S$ is essential if there is no proper map $(0, 1) \times [0, 1) \to S$ whose restriction to $(0, 1) \times \{0\}$ is $a$. Unless otherwise stated, all oriented arcs in the article are simple and essential, and all curves are simple, essential and non-peripheral.
Suppose that the Euler characteristic \( \chi = 2 - 2g - n \) of \( S \) is negative. If \( n \geq 1 \), the arc graph \( \mathcal{A}(S) \) is the graph whose vertex set \( \mathcal{A} \) is the set of homotopy classes of oriented arcs on \( S \). Two vertices in \( \mathcal{A} \) are connected by an edge if they can be realised disjointly. Note that since our arcs are oriented, our \( \mathcal{A}(S) \) differs from the usual arc graph by replacing each vertex by two.

Allow now \( n = 0 \), but suppose that we are not in one of the exceptional cases where \( g = 0 \) and \( n = 3 \) or 4, or \( g = 1 \) and \( n = 1 \). Then the curve graph \( \mathcal{C}(S) \) is the graph whose vertices are the homotopy classes of curves on \( S \). Again, two vertices are connected by an edge if they can be realised disjointly. In the exceptional cases the edges of \( \mathcal{C}(S) \) are defined differently, but we will not be appealing to that definition in our article. By [MM99] and [MS13], the graphs \( \mathcal{C}(S) \) and \( \mathcal{A}(S) \) are Gromov-hyperbolic.

We fix an arbitrary complete hyperbolic metric on \( S \). A geodesic lamination on \( S \) is a closed subset of \( S \) that is a disjoint union of leaves that are geodesic lines and circles in \( S \) that do not self-intersect. A geodesic lamination \( L \) is minimal if its every leaf is dense in \( L \). Let \( Y \subset S \) be a subsurface whose all boundary components are geodesic circles. We say that a geodesic lamination \( L \subset Y \) fills \( Y \) if every curve on \( Y \) intersects \( L \). Analogously, a pair of oriented arcs \( a, b \subset Y \) fills \( Y \) if every curve on \( Y \) intersects the geodesic representative of \( a \) or \( b \).

A peripherally ending lamination is a minimal geodesic lamination that fills a subsurface \( Y \) containing all the punctures of \( S \). An ending lamination is a minimal geodesic lamination that fills the entire \( S \). Let \( \mathcal{EL}(S) \subset \mathcal{EL}_0(S) \) denote the sets of ending, and peripherally ending laminations on \( S \), respectively, with the topology given by the following coarse Hausdorff convergence. Namely, \( L_n \rightharpoonup \text{CH} L \) if for any subsequence \( L_{n_i} \), Hausdorff converging to a geodesic lamination \( L' \), we have \( L \subset L' \) (see [Ham06]). By [Kla99] and [Sch13] (see also Theorem 3.2 in Section 3), the spaces \( \mathcal{EL}(S), \mathcal{EL}_0(S) \) can be equivariantly identified with the Gromov boundaries of \( \mathcal{C}(S) \) and \( \mathcal{A}(S) \).

### 2.2. Unicorns

**Definition 2.1.** Let \( a, b \in \mathcal{A} \), and keep the notation \( a, b \) for the geodesic oriented arcs representing them. A unicorn arc for \( a \) and \( b \) is the homotopy class of an oriented arc that is a concatenation \( a' \cup b' \) for \( a' \) an initial segment of \( a \), and \( b' \) a terminal segment of \( b \), possibly \( a' = a, b' = \emptyset \), or \( a' = \emptyset, b' = b \). Note that orienting the arcs replaces the choice of endpoints in [HPW15, Def 3.1].

The set of all oriented arcs that are such concatenations \( a' \cup b' \) can be ordered into a sequence \( (a'_i \cup b'_i)_{i=0}^{n-1} \) so that for all \( 0 \leq i < n \) we have \( a'_{i+1} \supset a'_i \) and \( b'_{i+1} \supset b'_i \). We denote by \( c_i \in \mathcal{A} \) the homotopy class of \( a'_i \cup b'_i \) and we call the sequence \( P(a, b) = (c_i)_{i=0}^{n} \in \mathcal{A}^{n+1} \) the unicorn path from \( a \) to \( b \).
Remark 2.2 ([HPW15, Rm 3.2]). For each $0 \leq i < n$, the unicorn arcs $c_i, c_{i+1}$ are adjacent in $A(S)$.

Let $L_0$ be a peripherally ending lamination. Let $l$ be a geodesic line on $S$ that does not self-intersect and ends at a puncture in the sense that $l$ contains a geodesic ray properly embedded in $S$. We say that $l$ is asymptotic to $L_0$, if $l \subset S \setminus L_0$. Since each puncture of $S$ lies in once-punctured ideal polygon of $S \setminus L_0$, the number of such $l$ is bounded by the total number of their ideal vertices, which is at most $2|\chi|$.

Definition 2.3 ([PO17, §3.1]). Let $a \in A$ and keep the notation $a$ for the geodesic oriented arc representing it. Let $l$ be a geodesic line asymptotic to $L_0 \in \mathcal{EL}_0(S)$. A unicorn arc for $a$ and $l$ is the homotopy class an oriented arc that is a concatenation $a' \cup l'$ for $a'$ an initial segment of $a$, and $l'$ a terminal segment of $l$, possibly $a' = a$ and $l' = \emptyset$.

The set of all oriented arcs that are such concatenations $a' \cup l'$ can be ordered into a sequence $(a'_i \cup l'_i)_{i=0}^{\infty}$ so that for all $i \geq 0$ we have $a'_{i+1} \subset a'_i$ and $l'_{i+1} \supset l'_i$. We denote by $c_i \in A$ the homotopy class of $a'_i \cup l'_i$ and we call the sequence $P(a,l) = (c_i)_{i=0}^{\infty} \in A^\mathbb{N}$ the infinite unicorn path from $a$ to $l$.

3. Key lemmas

Definition 3.1. Let $n \in \{3, 4, \ldots, \infty\}$. A sequence $(c_i)_{i=0}^{n} \in A^{n+1}$ is a locally unicorn path if for each $0 \leq j < k \leq n$ with $j + 3 \leq k < \infty$, the sequence $(c_i)_{i=j}^{k}$ is the unicorn path from $c_j$ to $c_k$.

By Remark 2.2, a locally unicorn path is an edge-path in $A(S)$. Moreover, by [HPW15, Lem 3.5] each finite unicorn path of length $\geq 3$ is a locally unicorn path. Furthermore, by [PO17, Lem 3.4] an infinite unicorn path is also locally unicorn.

By [HPW15, Prop 4.2] there is a universal constant $C$ such that each finite unicorn path $P(a,b)$ is at Hausdorff distance $\leq C$ from a geodesic edge-path in $A(S)$ from $a$ to $b$. Consequently, each locally unicorn path is bounded or converges w.r.t. the Gromov product (see [GdlH90, §7.2]) to a point in $\partial A(S)$. This leads to the following result of Pho-On (the existence of an equivariant homeomorphism was announced earlier by Schleimer [Sch13]).

Theorem 3.2 ([PO17, §3.2-3]). Let $a \in A$. Let $L_0 \in \mathcal{EL}_0(S)$ and let $l$ be a geodesic line asymptotic to $L_0$. Then $P(a,l)$ is not bounded and its limit $F(L_0) \in \partial A(S)$ w.r.t. the Gromov product depends only on $L_0$. Furthermore, $F: \mathcal{EL}_0(S) \rightarrow \partial A(S)$ is a $\text{Mod}(S)$-equivariant homeomorphism.

In fact, the local condition characterises infinite unicorn paths:
Lemma 3.3. Let $P$ be a locally unicorn path that is not bounded in $\mathcal{A}(S)$. Then $P$ is an infinite unicorn path.

Proof. Denote $P = (c_i)_{i=0}^{\infty} \in \mathbb{A}^{\mathbb{N}}$, and keep the notation $c_i$ for the geodesic oriented arcs representing them. Since $P$ is not bounded in $\mathcal{A}(S)$, it converges to some point $F(L_0) \in \partial \mathcal{A}(S)$. By [PO17, Lem 3.9], we have that $c_i$ coarse Hausdorff converge to $L_0 \in \mathcal{EL}_0(S)$. Denote $c_0 = c_0$. We claim that for each $n \geq 1$ there is a geodesic line $l$ asymptotic to $L_0$ such that for each $i \leq n$ the unicorn arc $a_i$ on the infinite unicorn path from $c$ to $l$ coincides with $c_i$. Since there are only finitely many $l$ asymptotic to $L_0$, the lemma follows from the claim.

To justify the claim, note that since $P$ is a locally unicorn path, all $c_i$ with $i \geq 1$ end at a common puncture $p$. Let $D$ be the ideal polygon of $S \setminus L_0$ containing $p$. Let $l$ be a geodesic line asymptotic to $L_0$ ending at $p$. Let $c' \cup l'$ represent the $n$-th unicorn arc on the unicorn path from $c$ to $l$, let $x' = c' \cap l'$, and let $d'$ be the segment of $c$ that is the component of $D \cap c$ containing $x'$. See Figure 1. Let $D_p$ be the component of $D - \bigcup d'$ containing $p$, where the union is taken over all the geodesic lines $l$ asymptotic to $L_0$ and ending at $p$. Let $\alpha > 0$ be the minimum possible angle that makes with $L_0$ a geodesic ray in $D_p$ starting on $L_0$ and ending at $p$. Since $(c_i)_{i=0}^{\infty} \overset{\text{CH}}{\rightarrow} L_0$, there is $N \geq n$ such that $c_N$ does not intersect $L_0$ at angle $\geq \alpha$. Consequently, the component $c'_N$ of $c_N \cap D_p$ ending at $p$ starts on $d'$ for some $l$ (see Figure 1).

Let $q$ be the puncture at which $c$ starts. We have a bijection $h: l' \cap c \rightarrow c'_N \cap c$ such that each pair $x, h(x)$ lies in the same component of $D_p \cap c$. Furthermore, for each $x \in l' \cap c$, the segments $q x \subset c$ and $xp \subset l$ intersect only at $x$ if and only if the segments $qh(x) \subset c$ and $h(x)p \subset c_N$ intersect only at $h(x)$. In other words, the concatenation $qx \cup xp$ represents a unicorn arc for $c$ and $l$ if and only if the concatenation $qh(x) \cup h(x)p$ represents a unicorn arc for $c$ and $c_N$. Moreover, these two oriented arcs are homotopic. Finally, this correspondence preserves the
order of unicorn arcs. Thus, for $0 \leq i \leq n$, we have $a_i = c_i$, justifying the claim. □

Let $L_0 \in \mathcal{EL}_0(S)$. We define an equivalence relation $\sim_{L_0}$ on $A$, by declaring $a \sim_{L_0} b$ if the geodesic representatives of $a, b$ start at the same puncture and their first points in $L_0$ lie on the same side of the ideal polygon of $S \setminus L_0$ containing that puncture. Note that $\sim_{L_0}$ has at most $2|\chi|$ equivalence classes.

We now prove a tail equivalence lemma that will later allow us to reduce the orbit equivalence on $\partial A(S)$ to $E_t$.

Lemma 3.4. Let $L_0 \in \mathcal{EL}_0(S)$, and let $a, b \in A$ with $a \sim_{L_0} b$. Then for each geodesic line $l$ asymptotic to $L_0$, the unicorn path $(a_i)_{i=0}^{\infty}$ from $a$ to $l$ and the unicorn path $(b_i)_{i=0}^{\infty}$ from $b$ to $l$ satisfy $a_i = b_{i+k}$ for some $k \in \mathbb{Z}$ and all $i$ sufficiently large.

Proof. Let $a, b \in A$ with $a \sim_{L_0} b$ and keep the notation $a, b$ for the geodesic oriented arcs representing them. Let $x, y$ be the first points on $a, b$ in $L_0$. Since $a \sim_{L_0} b$, there is a geodesic segment $xy \subset L_0$. Furthermore, since $L_0$ is minimal, there are segments $xx', yy'$ in $a, b$ such that $x'y'$ is a geodesic segment in $L_0$, and $xx'y'y$ bounds a topological disc $B$ embedded in $S$. See Figure 2.

Consequently, the components of the intersection $B \cap l$ are geodesic segments joining $xx'$ to $yy'$, which yields a bijection $h: xx' \cap l \to yy' \cap l$. Let $a' \subset a$ be an initial segment of $a$ ending in $z \in xx' \cap l$, and let $b'$ be the initial segment of $b$ ending in $h(z)$. Furthermore, let $l', l''$ be the terminal segments of $l$ starting in $z, h(z)$, respectively. Assume without loss of generality $l'' \subset l'$, as in Figure 2.

Note that $xz$ intersects $zh(z) \subset l'$ only at $z$. Furthermore, $xz$ is disjoint from $l''$ if and only if $yh(z)$ intersects $l''$ only at $h(z)$. Consequently, the concatenation $a' \cup l'$ represents a unicorn arc if and only if $b' \cup l''$ represents a unicorn arc. Moreover, these two oriented arcs are homotopic. Finally, this correspondence preserves the order of unicorn arcs, and all $a_i$ for $i$ sufficiently large are accounted for in this way. □

![Figure 2. Rectangle B](image)
4. Hyperfiniteness

We fix a basepoint \( a_0 \in A \).

**Definition 4.1.** Let \( \mathcal{P} \subset A^\mathbb{N} \) be the set of infinite unicorn paths from \( a_0 \) to any geodesic line asymptotic to any \( L_0 \in \mathcal{E}L_0(S) \). Let \( f: \mathcal{P} \to \partial A(S) \) be the map assigning to such path its limit \( F(L_0) \in \partial A(S) \) w.r.t. the Gromov product (see Theorem 3.2).

Note that \( f \) is finite-to-one, since \( a_0 \) is fixed and there are finitely many geodesic lines asymptotic to a given \( L_0 \in \mathcal{E}L_0(S) \).

**Remark 4.2.** We equip the countable set \( A \) with the discrete topology and \( A^\mathbb{N} \) with the product topology. Then the set \( \mathcal{P} \subset A^\mathbb{N} \) is Borel. Indeed, by Lemma 3.3, \( \mathcal{P} \) is the set of locally unicorn paths that are not bounded. The set of locally unicorn paths is closed in \( A^\mathbb{N} \), since each of the conditions on \( (c_i)_{i=j}^k \) to be a unicorn path is closed. Furthermore, for each \( n \geq 0 \), the set of sequences in \( A^\mathbb{N} \) at distance \( \leq n \) from \( a_0 \) is closed, so the set of sequences in \( A^\mathbb{N} \) at bounded distance from \( a_0 \) is a countable union of closed sets. Consequently, \( \mathcal{P} \) is a countable intersection of open sets.

Since locally unicorn paths are uniformly Hausdorff close to geodesic edge-paths, and the function \( f \) assigns their limits in \( \partial A(S) \), we have that \( f \) continuous w.r.t. the metric on \( \partial A(S) \) defined using the Gromov product.

Let \( \mathcal{T} \) be the countable set of finite length edge-paths in \( A(S) \), up to the action of \( \text{Mod}(S) \), equipped with the discrete topology. For an infinite unicorn path \( P = (c_i)_{i=0}^\infty \), given \( i \geq 0 \) and \( j = i + 1, \ldots \), the subsurfaces \( \Sigma_{i,j} \subseteq S \) filled by \( c_i \) and \( c_j \) form an ascending sequence \( \Sigma_{i,j} \subseteq \Sigma_{i,j+2} \subseteq \cdots \) that stabilises with some subsurface which we call \( \Sigma_i \subseteq S \). For each \( i \geq 0 \), let \( m(i) > i + 1 \) be minimal satisfying \( \Sigma_{i+1,m(i)} = \Sigma_{i+1} \). Let \( T_i = (c_j)_{j=i}^{m(i)} \), and let \([T_i]\) be the equivalence class of \( T_i \) in \( \mathcal{T} \). Let \( g: \mathcal{P} \to \mathcal{T}^\mathbb{N} \) be the map defined by \( g(P) = ([T_i])_{i=0}^\infty \). Let \( E_i \) be the tail equivalence relation on \( \mathcal{T}^\mathbb{N} \) described in Section 1 (with \( \Omega = \mathcal{T} \)).

Note that the definition of \( g \) can be analogously extended to infinite unicorn paths \( P \notin \mathcal{P} \) (i.e. to infinite unicorn paths that start at points distinct from \( a_0 \)), which we will make use of later on.

**Remark 4.3.** We equip \( \mathcal{T}^\mathbb{N} \) with the product topology. Then the map \( g: \mathcal{P} \to \mathcal{T}^\mathbb{N} \) is Borel. Indeed, for all \( 0 \leq i < j \), the maps \( P \to \Sigma_{i,j} \) are continuous maps from \( \mathcal{P} \) to the countable discrete set of subsurfaces of \( S \), and hence their limits \( P \to \Sigma_i \) are Borel. Thus, for all \( 0 \leq i < j \), the subset of \( \mathcal{P} \) defined by the identity \( \Sigma_{i,j} = \Sigma_i \) is Borel, and so the maps \( m(i): \mathcal{P} \to \mathbb{N} \) are Borel. Consequently, all the maps \([T_i]: \mathcal{P} \to \mathcal{T} \) are Borel, as desired.
Lemma 4.4. Let \( P, P' \in \mathcal{P} \). If \( g(P) \sim_{E_t} g(P') \), then there is \( \psi \in \text{Mod}(S) \) satisfying \( \psi f(P) = f(P') \). Conversely, for each orbit \( \omega \) of the action of \( \text{Mod}(S) \) on \( \partial \mathcal{A}(S) \), there are finitely many equivalence classes of \( E_t \) on \( \mathcal{T}^N \) containing all \( g(P) \) for \( P \in \mathcal{P} \) with \( f(P) \in \omega \).

Proof. Denote \( P = (c_i)_{i=0}^\infty \), \( P' = (c'_i)_{i=0}^\infty \). Let \( T'_i \) be defined for \( P' \) analogously as \( T_i \) was for \( P \). If \( g(P) \sim_{E_t} g(P') \), then there are \( k \in \mathbb{Z}, j \in \mathbb{N} \) such that \( [T'_i] = [T'_{i+k}] \) for all \( i \geq j \). In particular, there is \( \psi \in \text{Mod}(S) \) with \( \psi T_j = T'_j \). We will show inductively that \( \psi T_i = T'_{i+k} \) for all \( i \geq j \), so in particular \( \psi c_i = c'_i + k \) implying \( \psi f(P) = f(P') \).

Suppose that we have established \( \psi T_i = T'_{i+k} \) for some \( i \geq j \). If \( m(i + 1) \leq m(i) \), then \( \psi T_{i+1} = T'_{i+1+k} \) is immediate, so we can assume \( m(i + 1) > m(i) \). Let \( \rho \in \text{Mod}(S) \) be such that \( \rho T_{i+1} = T'_{i+1+k} \). Then \( \rho^{-1} \psi \) fixes all \( c_{i+1}, \ldots, c_{m(i)} \). Thus the restriction of \( \rho^{-1} \psi \) to the subsurface \( \Sigma_{i+1} \subseteq S \), which \( c_{i+1} \) and \( c_{m(i)} \) fill, is the identity map. By the definition of \( \Sigma_{i+1} \), we have that \( c_{m(i)+1}, \ldots, c_{m(i)+1} \) all lie in \( \Sigma_{i+1} \). This implies that \( \rho^{-1} \psi \) fixes them and so \( \psi T_{i+1} = T'_{i+1+k} \), completing the induction.

For the converse, let \( \omega \) be the orbit under \( \text{Mod}(S) \) of some \( F(L_0) \in \partial \mathcal{A}(S) \). Let \( l_1, \ldots, l_n \) be the finitely many geodesic lines asymptotic to \( L_0 \). Choose \( a_1, \ldots, a_p \in A \) that are representatives of the equivalence classes of \( \sim_{L_0} \) distinct from the one containing \( a_0 \). For \( 0 \leq q \leq p \) and \( 1 \leq j \leq n \), let \( P_{aj} = P(a_j, l_j) \).

Let \( P = (c_i)_{i=0}^\infty \in \mathcal{P} \) with \( f(P) \in \omega \). By Theorem 3.2, we have \( P = P(a_0, l) \) where \( l \) is a geodesic line asymptotic to \( \psi L_0 \) for some \( \psi \in \text{Mod}(S) \). In particular, for some \( 1 \leq j \leq n \) we have \( l = \psi l_j \). Thus \( \psi^{-1} P = P(\psi^{-1} a_0, l_j) \). Choose \( 0 \leq q \leq p \) so that \( a_q \sim_{L_0} \psi^{-1} a_0 \). By Lemma 3.4, writing \( P_{aj} = (b_i)_{i=0}^\infty \), we have \( \psi^{-1} c_i = b_{i+k} \) for some \( k \in \mathbb{Z} \) and all \( i \) sufficiently large. We have then that \( g(P_{aj}) \) and \( g(\psi^{-1} P) \), hence also \( g(P) \), are tail equivalent.

Proof of Theorem 1.1. Write \( E \) for the equivalence relation on \( \partial \mathcal{A}(S) \) induced by the action of \( \text{Mod}(S) \) and write \( E^* \) for the equivalence relation on \( \mathcal{P} \) that is the pullback of \( E \) via \( f \), i.e. \( P \sim_{E^*} P' \) if \( f(P) \sim_E f(P') \). Since \( E \) is Borel and countable, and \( f \) is Borel and finite-to-one, we have that \( E^* \) is also Borel and countable.

Since \( f \) is a Borel finite-to-one function, it has a Borel right inverse by the Lusin–Novikov uniformisation theorem [Kec95, Thm 18.10]. Consequently, \( E \) is Borel reducible to \( E^* \). Thus it is enough to show that \( E^* \) is hyperfinite.

Write \( E^*_t \) for the equivalence relation on \( \mathcal{P} \) that is the pullback of \( E_t \) via \( g \). Since \( E_t \) is Borel, and \( g \) is Borel, we have that \( E^*_t \) is Borel. By Lemma 4.4, we have \( E^*_t \subseteq E^* \) and every equivalence class of \( E^* \) contains finitely many equivalence classes of \( E^*_t \). (In particular, \( E^*_t \) is countable.) Thus by [JKL02, Prop 1.3(vii)] it is enough to show that \( E^*_t \) is hyperfinite.
Note that $g$ is a Borel reduction of $E_t^*$ to $E_t$. Thus since $E_t$ is hyperfinite [DJK94, Cor 8.2], we have that $E_t^*$ is hyperfinite as well. □

Proof of Corollary 1.2. Assume first that $S$ has $n \geq 1$ punctures. Then by [Kla99, Thm 1.3], Theorem 1.1, and [JKL02, Prop 1.3(iii)] it suffices to prove that $\mathcal{EL}(S)$ is a Borel subset of $\mathcal{EL}_0(S)$. Indeed, $L_0 \in \mathcal{EL}_0(S)$ is a minimal filling lamination if and only if each curve $c$ on $S$ intersects $L_0$ and does it transversally. Given $c$, this is an open condition, and so $\mathcal{EL}(S)$ is a countable intersection of open sets.

Secondly, assume $n = 0$ and let $S'$ be the surface obtained from $S$ by adding one puncture at a point outside the closure of the union of all embedded geodesic circles and lines, which exists by [BS85, Thm I]. This induces a closed embedding $e: \mathcal{EL}(S) \to \mathcal{EL}(S')$, which is a section for the map $r: \mathcal{EL}(S') \to \mathcal{EL}(S)$ defined by forgetting the puncture. See [PO17, §4.2] for details. Thus for each $L_1, L_2 \in \mathcal{EL}(S)$, with $\psi'(L_1) = e(L_2)$ for some $\psi' \in \text{Mod}(S')$, we have that the image $\psi \in \text{Mod}(S)$ of $\psi'$ under the puncture forgetting map $\text{Mod}(S') \to \text{Mod}(S)$ satisfies $\psi(L_1) = L_2$.

Conversely, let $L \in \mathcal{EL}(S)$ and let $R_1, \ldots, R_n \subset S$ be the components of $S \setminus L$. For $1 \leq j \leq n$, let $L_j$ be a lamination in $\mathcal{EL}(S')$ obtained from $L$ by adding a puncture in $R_j$, under an arbitrary identification with $S'$. All such identifications differ by $\text{Mod}(S')$, so the resulting orbit $[L_j]$ in $\mathcal{EL}(S')$ does not depend on our choice. Since $e$ is a section for $r$, we have $e(L) \in \bigcup_{j=1}^n [L_j]$. Analogously, for any $\psi \in \text{Mod}(S)$, we have $e(\psi(L)) \in \bigcup_{j=1}^n [L_j]$.

Consequently, under the identification of $\mathcal{EL}(S)$ with $e(\mathcal{EL}(S))$, each orbit of $\text{Mod}(S)$ on $\mathcal{EL}(S)$ consists of the intersections of finitely many orbits of $\text{Mod}(S')$ on $\mathcal{EL}(S')$ with $e(\mathcal{EL}(S))$. Thus by [JKL02, Prop 1.3 (iii,vii)], the hyperfiniteness of the action of $\text{Mod}(S)$ on $\mathcal{EL}(S)$ follows from the hyperfiniteness of the action of $\text{Mod}(S')$ on $\mathcal{EL}(S')$. □

References


Dep. of Math. & Stat., McGill University, Burnside Hall, 805 Sherbrooke St. W, Montreal, Quebec, Canada H3A 0B9

*Email address*: piotr.przytycki@mcgill.ca

*Email address*: marcin.sabok@mcgill.ca