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Ends of hyperbolic 3–manifolds, chapters IV–VI

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Note from the translators

With some exceptions in Chapter IV the translation follows the Annals version of the article. We included and updated only these references, on which Chapters IV–VI rely. We added some comments as footnotes. We tried rather to preserve the structure of the sentences then to make them sound more English. All translation comments are welcome; we will correct any errors or misprints found by the reader.

4 Geodesic currents on surfaces

Applying the intersection number lemma to the sequence of curves $\alpha_j \subset S_b$ given by Proposition 2.4, where the associated geodesics α_j^* are located arbitrarily far from S_b, we deduce that the self–intersection number $i(\alpha_j, \alpha_j)$ becomes arbitrarily small with respect to the square of the length $l(\alpha_j)$. In other words, α_j become "more and more like simple curves". The purpose of this section is to give a meaning to this assertion, to create a certain topological space in which α_j will converge to something that is a limit of simple curves. Apart from this motivation, this section is independent from the remaining part of the article.

4.1 Definitions

Let S be a surface of finite type without boundary, with negative Euler characteristic. We are interested in the set of homotopy classes of non–oriented homotopically nontrivial closed curves on S. To give a specific representative in each such homotopy class, it is convenient to equip S with an arbitrary hyperbolic metric without cusps. Each free homotopy class is then represented by a unique closed geodesic on S.

In [11], [12], Thurston studied the subset consisting of homotopy classes of simple curves. Equipping it with a topology, he defined a certain completion by measured geodesic laminations. A measured geodesic lamination is a partial foliation of S by simple (without self–intersections) disjoint geodesics with a foliation–invariant transverse measure (see §4.3). We are going to extend this construction to non–simple curves.

The difference between a simple curve and a non–simple curve comes obviously from … the self–intersections. To remove these, it is rather natural to lift the closed geodesics to the unit tangent bundle $T_1(S)$ of S. Since we are not interested in the orientation of the curves, we will look at the tangent line bundle $\mathbb{P}(S)$, the quotient of $T_1(S)$ by the involution, which is the antipodal map on each fiber.

Since we have fixed a hyperbolic metric on S, the unit tangent bundle is equipped with a *geodesic flow*, whose trajectories are defined as lifts of geodesics on S to $T_1(S)$ by considering their tangent vectors at each point. Passing to the quotient, the geodesic flow on $T_1(S)$ induces a 1–dimensional foliation $\mathcal{F}$ on $\mathbb{P}(S)$, called the *geodesic foliation*. There is therefore a natural correspondence between closed (non–oriented) geodesics on S and compact leaves of $\mathcal{F}$.

The convex core $C(S)$ of S, in the sense of §1.5, is the compact part of S delimited by the closed geodesics corresponding to the ends of S. Denote by $\mathbb{P}C(S)$ the compact set in $\mathbb{P}(S)$ formed by the lifts of geodesics on S, which are completely contained in this convex core. In particular $\mathbb{P}C(S)$ contains all lifts of closed geodesics on S, and the union of these lifts is dense in $\mathbb{P}C(S)$[1].

By definition, a *geodesic current* α on S is a positive transverse invariant measure for the geodesic foliation $\mathcal{F}$, whose support is contained in $\mathbb{P}C(S)$. This means that α defines a positive measure supported on $V \cap \mathbb{P}C(S)$ for each submanifold V of codimension 1 in $\mathbb{P}(S)$ transverse to $\mathcal{F}$, and that α is invariant under holonomy in the following sense: if $x_1 \in V_1$ and $x_2 \in V_2$ are two points on such transverse submanifolds located on the same leaf of $\mathcal{F}$, and if $\phi\colon U_1 \to U_2$ is a holonomy diffeomorphism between neighborhoods of x_1 and x_2 in V_1 and V_2 (defined by following the leaves of $\mathcal{F}$), then ϕ respects the measure induced by α on U_1 and U_2. Geodesic currents are hence particular cases of geometric currents introduced by D. Ruelle and D. Sullivan in [8].

We give now a fundamental example of a geodesic current. To a given closed geodesic α on S corresponds a compact leaf $\tilde{\alpha}$ of $\mathcal{F}$. We associate to it the geodesic current, which induces on each transverse manifold V the Dirac measure at the point $V \cap \tilde{\alpha}$; invariance under holonomy is then immediate.

We equip the set $\mathcal{C}(S)$ of currents on S with the unique weak topology, in which two currents α and β are close if there exists a finite family of continuous functions $f_i\colon V_i \to \mathbb{R}$ with compact support defined on transverse

[1] see Lemma 1.19 of the preprint

submanifolds V_i such that each $\alpha(f_i)$ is close to $\beta(f_i)$ (see e.g. [2], chapter III, §1 n° 9). We can even equip $\mathcal{C}(S)$ with a uniform space structure taking the entourages basis

$$\{(\alpha,\beta) \in \mathcal{C}(S) \times \mathcal{C}(S); \forall i = 1 \ldots n,\ |\alpha(f_i) - \beta(f_i)| < \varepsilon\}$$

for all $\varepsilon > 0$ and all finite families $f_i \colon V_i \to \mathbb{R}$ as before. We get then the following classical result in functional analysis (c.f. [2], chapter III, §1 n° 9):

Proposition 4.1. *The uniform space $\mathcal{C}(S)$ is complete.*

To understand well currents on S and their topology it is first necessary to understand well $\mathbb{P}(S)$ equipped with $\mathcal{F}$. A *flow box* B for $\mathcal{F}$ is defined by an elongated H–shape configuration on S, where the horizontal bar is a geodesic arc and where the two vertical bars are arcs transverse to the previous one and sufficiently small so that each geodesic arc joining the vertical bars which is homotopic to a path in the H, meets the vertical bars transversely (we will actually impose one more condition, see Lemma 4.3 below). The box $B \subset \mathbb{P}(S)$ consists of the lifts of all geodesic arcs in S joining the two vertical bars that are homotopic to a path in H. Barycentric coordinates on each geodesic arc give a diffeomorphism $B \simeq Q \times [0,1]$ for which the leaves of $B \cap \mathcal{F}$ correspond to $\{\star\} \times [0,1]$. We point out that Q can be lifted to B as a square transverse to the foliation and that this lift is unique up to holonomy; given a geodesic current $\alpha \in \mathcal{C}(S)$, we can therefore speak of the measure $\alpha(B) \in \mathbb{R}^+$, defined as the measure w.r.t. α of this transverse square. Likewise, if $\partial_{\mathcal{F}} B$ is the part of B corresponding to $\partial Q \times [0,1]$ (formed by the geodesic arcs meeting one of the extremities of the H), we define $\alpha(\partial_{\mathcal{F}} B)$ as the measure w.r.t. α of the boundary of the transverse square, which is the lift of Q.

To illustrate this, let us investigate what this means if the geodesic current is defined by a closed geodesic α on S (usually closed geodesics will be identified with the geodesic currents they define). If B is a flow box, $\alpha(B)$ is clearly the number of sub–arcs of α whose lifts are leaves of $B \cap \mathcal{F}$. In other words, $\alpha(B)$ is the number of sub–arcs of α that join the two vertical bars of the H defining B and are homotopic relative the endpoints to an arc in the H.

Proposition 4.2. *A neighborhood basis for a current $\alpha \in \mathcal{C}(S)$ consists of the open sets $\mathcal{U}(\alpha, B_1, \ldots, B_n; \varepsilon) = \{\beta \in \mathcal{C}(S) \colon \forall i\ |\alpha(B_i) - \beta(B_i)| < \varepsilon\}$, where $\varepsilon \in \mathbb{R}^+_*$ and the B_i are taken among all the flow boxes B such that $\alpha(\partial_{\mathcal{F}}(B)) = 0$.*

Proof. To explain the restriction $\alpha(\partial_{\mathcal{F}}(B)) = 0$, we recall a small subtlety of the weak convergence topology: if μ_i is a sequence of measures on a locally compact space X that converges weakly to a measure μ, and if A is a measurable subset of X, then $\mu_i(A)$ converge to $\mu(A)$ if $\mu(\partial A) = 0$, where ∂A is the boundary of A (see [2], chapter IV, §5 n° 12); the example of the Dirac measure shows that this condition is necessary.

From this it is clear that each $\mathcal{U}(\alpha, B_1, \ldots, B_n; \varepsilon)$ contains an open set in the weak topology.

The converse is only a question of approximating of continuous functions with compact support by stair–like functions. Note that each flow box B can be approximated by a box B' with $\alpha(\partial_{\mathcal{F}}(B')) = 0$. Indeed, it is sufficient to shorten slightly one of the bars of the H used to define B; as $\alpha(B)$ is finite, at most countable number of such shortenings give B' with $\alpha(\partial_{\mathcal{F}}(B')) = 0$. □

To apply Proposition 4.2, the following observation will be useful.

Lemma 4.3. *If $\partial_{\mathcal{F}}(B)$ does not meet any compact leaf of $\mathcal{F}$, then $\alpha(\partial_{\mathcal{F}}(B)) = 0$ for all $\alpha \in \mathcal{C}(S)$.*

Proof. Suppose $\alpha(\partial_{\mathcal{F}}(B)) > 0$. Then, for at least one point x of the four extreme points of H defining B, the arc k transverse to $\mathcal{F}$ formed by the directions in x pointing towards the other bar of the H has non–zero measure. Observe that there is only a countable number of leaves of $\mathcal{F}$ meeting k in at least two points, since there is at most one per element of $\pi_1(M, x)$.

We cover the compact set $\mathbb{P}C(S)$ by a finite number of flow boxes $B_1, \ldots, B_n$. We get from the finiteness of the $\alpha(B_i)$[2] that the part of k formed by the leaves meeting k in exactly one point has zero measure w.r.t. α. Hence the measure defined by α on k must have at least one atom, corresponding to a leaf passing more than once through k. Once more because of the finiteness of the $\alpha(B_i)$, this leaf is compact. □

From now on, unless the opposite is explicitly stated, we will require that the boundary of a flow box B does not meet any compact leaf of $\mathcal{F}$, which means that no closed geodesic on S goes through an extremity of the H defining B before hitting the opposite vertical bar of the H. Therefore, $\partial_{\mathcal{F}}(B)$ will have zero measure for all geodesic currents $\alpha \in \mathcal{C}(S)$.

If the reader is allergic to geodesic flows and transverse measures, we can give a definition of geodesic currents completely independent from these notions. Indeed, let $G(\tilde{S})$ be the manifold of non–oriented geodesics on the universal cover $\tilde{S}$ of S. If $\tilde{S}$ is identified with the hyperbolic plane $\mathbb{H}^2$, then $G(\tilde{S})$ is diffeomorphic to the Moebius strip $(S^1_\infty \times S^1_\infty \setminus \Delta)/(\mathbb{Z}/2)$, where Δ is the diagonal and $\mathbb{Z}/2$ acts by switching the two factors. The subset $GC(\tilde{S})$ formed by geodesics contained in the preimage of the convex core $C(S)$ corresponds to $(\Lambda \times \Lambda \setminus \Delta)/(\mathbb{Z}/2)$, where Λ is the limit set of $\pi_1(S)$ acting on

[2] and from the Poincaré recurrence

$\tilde{S} \cong \mathbb{H}^2$. Observing that the projection $\mathbb{P}(\tilde{S}) \to G(\tilde{S})$ is a submersion, we easily get that a geodesic current is the same thing as a $\pi_1(S)$–invariant measure on $G(\tilde{S})$ whose support is contained in $GC(\tilde{S})$. Moreover, the topology on $\mathcal{C}(S)$ corresponds to the classical weak topology on measures on a locally compact space.

Actually, these two points of view are complementary. The one of $\pi_1(S)$–invariant measures seems quite appropriate to do analysis. For instance, it is in this context that Sullivan shows in [10] that the Hausdorff measure on Λ induces a privileged current α on S, ergodic in its measure class (i.e. any $\pi_1(S)$–invariant function on $GC(\tilde{S})$ is constant α–almost everywhere). On the other hand, the point of view of measures transverse to the geodesic foliation is more convenient when geometric intuition is needed.

The proof of the following result, which will not be necessary *stricto–sensu* to prove Theorem A, but which justifies a bit the introduced definitions, gives an illustration.

Proposition 4.4. *The homotheties of the geodesic currents defined by closed geodesics on S are dense in $\mathcal{C}(S)$.*

Proof. The argument we are going to use is closely related to the train tracks used by Thurston in [12]. Bill Veech made me point out that this proposition is actually a specific case of a more general result by K. Sigmund [9].

We will first show that linear combinations (with positive coefficients) of closed geodesics are dense in $\mathcal{C}(S)$.

Let $\alpha \in \mathcal{C}(S)$ and let $\mathcal{U}(\alpha, B_1, \ldots, B_n; \varepsilon)$ be a neighborhood of α as in Proposition 4.2. Since $\mathbb{P}C(S)$ is compact, we can assume without loss of generality that flow boxes B_i cover $\mathbb{P}C(S)$. Moreover, up to cutting and perturbing these boxes a little, we can assume the interiors of the B_i to be pairwise disjoint – in the first step, by perturbing, we can make the vertical bars of the H's defining B's pairwise disjoint, then we observe that each $B_i \cap B_j$ and $\overline{B_i \setminus B_j}$ is a union of a finite number of flow boxes with disjoint interiors, which concludes the argument.

Having fixed this decomposition into boxes, there exists $l_h > 0$ such that each leaf of $B_i \cap \mathcal{F}$ projects in S onto a geodesic arc of length $> l_h$. Subdivide then the vertical bars of the H's to obtain a new decomposition into flow boxes B'_j with the same properties as the B_i (each B'_j is contained in a B_i), but which are narrow, meaning that the vertical bars of the H's defining the boxes B'_j have length $\leq l_v$ for a given "small" constant l_v, which will be specified later (while horizontal bars have length $> l_h$).

Choose a section Q_j of $B'_j \cap \mathcal{F}$ for each box B'_j consisting, for example, of the centers of all leaves. Let Q_{j_+} and Q_{j_-} be its two sides. If p and q are two symbols of the form j_+ or j_-, consider the subset of the side Q_p consisting of the points x such that Q_q is the first of the sides Q_r met by the half leaf of $\mathcal{F}$ issuing from x (on the side Q_p). We denote by $a(p,q)$ the measure of this subset for the current α.

Positive reals $a(p,q)$ clearly satisfy the equations:

$$\forall p, q, \quad a(p,q) = a(q,p)$$

$$\forall j, \quad \sum_p a(j_+, p) = \sum_q a(j_-, q) \quad (= \alpha(B'_j)).$$

As these are linear equations with integer coefficients, the $a(p,q)$ can be arbitrarily approximated by positive rationals $b(p,q)$ which satisfy the same family of equations, and such that $b(p,q) = 0$ iff $a(p,q) = 0$.

Let $D \in \mathbb{N}$ be a common denominator for the $b(p,q)$, so that $Db(p,q)$ are integers.

Each time $a(j_+, k_-) \neq 0$, there exists at least one leaf in $\mathcal{F}$ issuing from B'_j in the direction of Q_{j_+} and entering B'_k in the direction of Q_{k_-}. Let then $\kappa(j_+, k_-)$ be the geodesic arc joining the centers of the horizontal bars of the H's defining B'_j and B'_k in this specified homotopy class. Note that if the B'_j have been chosen sufficiently narrow, the length of $\kappa(j_+, j_-)$ is $\geq l_h$ and the angles between $\kappa(j_+, k_-)$ and the horizontal bars are arbitrarily small.

We define in the same way the arc $\kappa(p,q)$ for any pair of symbols p and q having the form j_- or j_+ such that $a(p,q) \neq 0$.

Take now $Db(p,q)$ copies of the arc $\kappa(p,q)$ for all p, q. As $\sum_p b(j_+, p) = \sum_q b(j_-, q)$ for all j, we can paste these copies and form a family γ of piecewise geodesic closed curves (there are of course a lot of possible choices for γ).

Each curve γ_1 in γ is nearly a geodesic. Indeed, it is made of geodesic arcs of length $\geq l_h$, while its angles are arbitrarily small if we took from the beginning the boxes B'_i sufficiently narrow, i.e. the constant l_v sufficiently small. An easy and classical argument shows then that there is an arbitrarily short homotopy between γ_1 and a closed geodesic γ_1^* in S. Let us recall briefly the argument: if we know already that γ_1 is homotopic to a geodesic γ_1^*, we consider a point of γ_1 which is the farthest possible from γ_1^* in the cover $\tilde{S}$ of S, such that $\pi_1(\tilde{S}) = \pi_1(\gamma_1) \subset \pi_1(S)$; if this point is actually far, a little of hyperbolic trigonometry shows that γ_1 has necessarily large angle at this point, which is impossible. A similar argument shows that γ_1 is not homotopic to 0: lift γ_1 to the universal cover and check that γ_1 must have a large angle at the farthest point from a given base point.

Let us return to our neighborhood $\mathcal{U}(\alpha, B_1, \ldots, B_n; \varepsilon)$ of $\alpha \in \mathcal{C}(S)$. Let $\beta \in \mathcal{C}(S)$ be obtained by dividing by D the Dirac current defined by the union γ^* of the geodesics γ_i^* homotopic to the curves in γ. If the constant l_v

was chosen sufficiently small, so that the homotopy between γ and γ^* sends each geodesic arc in γ onto an arc in γ^* whose lift in $\mathbb{P}(S)$ is at distance $\leq \eta$, then $\beta(B_i)$ is close to the sum of all $b(j_+, p)$ with $B'_j \subset B_i$, where the absolute value of the difference is bounded by the sum of the $b(j_+, p)$, whose B'_j is at distance $\leq \eta$ from $\partial_{\mathcal{F}} B_i$. Moreover, this bound is close to the α measure of the η–neighborhood of $\partial_{\mathcal{F}} B_i$ in $\mathbb{P}(S)$, which itself is small if η is small enough as $\alpha(\partial_{\mathcal{F}} B_i) = 0$. Therefore, $\beta(B_i)$ is close to the sum of $a(j_+, p)$ with $B'_j \subset B_i$, that is to $\alpha(B_i)$. We have hence shown that β is contained in $\mathcal{U}(\alpha, B_1, \ldots, B_n; \varepsilon)$ if B'_j are chosen sufficiently narrow and $b(p, q)$ close enough to $a(p, q)$.

As β was by construction a linear combination of closed geodesics, we have proved that these combinations are dense in $\mathcal{C}(S)$.

We still have to show that each linear combination of closed geodesics can be approximated by homotheties of closed geodesics. By induction, it is enough to show this for linear combinations $\lambda\alpha + \mu\beta$ of two closed geodesics α and β.

Choose two geodesic arcs k_+ and k_- joining α and β. We can choose k_+ and k_- such that the angles $(\alpha, k_+), (\alpha^{-1}, k_-), (\beta, k_+)$ and (β^{-1}, k_-) at those extremities are arbitrarily small (it is an easy exercise in hyperbolic plane geometry, consider the universal cover $\mathbb{H}^2$). We consider then 3 positive integers a, b and D such that $\frac{a}{D}$ and $\frac{b}{D}$ are arbitrarily close to λ and μ. If γ is the closed curve $\alpha^a k_+ \beta^b k_-$, we check as before that it is homotopic to a closed geodesic γ^* under a "short" homotopy if the various angles between k_+, k_-, α and β have been chosen sufficiently small. Therefore, if those angles are sufficiently small and if $\frac{a}{D}$ and $\frac{b}{D}$ are close enough to λ and μ, then the geodesic current $D^{-1}\gamma^*$ is close to $\lambda\alpha + \mu\beta$ in the topology of $\mathcal{C}(S)$, by the same argument as above.

This ends the proof of Proposition 4.4. $\square$

4.2 Intersection number and length of geodesic currents

Proposition 4.5. *The function "intersection number" defined on the set of closed geodesics on S extends to a continuous symmetric bilinear mapping $i\colon \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}^+$.*

Proof. The geometric intersection number of two closed geodesics α_1, α_2 is equal to the number of triples $(x, \lambda_1, \lambda_2)$, where $x \in \alpha_1 \cap \alpha_2$ and λ_1, λ_2 are two distinct lines in $T_x(S)$ tangent to α_1, α_2 respectively. The advantage of this definition is that it is expressed only in terms of geodesic currents, and we will exploit this observation to define the function i on $\mathcal{C}(S) \times \mathcal{C}(S)$.

Starting from the bundle $\mathbb{P}(S) \to S$, we can consider the Whitney sum $\mathbb{P}(S) \oplus \mathbb{P}(S) \to S$. In other words, $\mathbb{P}(S) \oplus \mathbb{P}(S)$ is the 4–dimensional manifold of triples $(x, \lambda_1, \lambda_2)$, where $x \in S$ and λ_1 and λ_2 are two lines in the tangent space $T_x(S)$. Forgetting the first or the second line defines two projections p_1 and p_2 from $\mathbb{P}(S) \oplus \mathbb{P}(S)$ to $\mathbb{P}(S)$. We consider the two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of codimension 2 in $\mathbb{P}(S) \oplus \mathbb{P}(S)$, whose leaves are the preimages of the leaves of $\mathcal{F}$ by, respectively, p_1 and p_2. One easily checks that these foliations are transverse outside the diagonal Δ of $\mathbb{P}(S) \oplus \mathbb{P}(S)$.

Let α_1 and α_2 be two geodesic currents. Through p_1, α_1 induces a transverse invariant measure $p_1^*(\alpha_1)$ on $\mathcal{F}_1$, which, by transversality of $\mathcal{F}_1$ and $\mathcal{F}_2$, gives outside Δ a measure on each leaf of $\mathcal{F}_2$. Similarly, α_2 induces outside Δ a measure $p_2^*(\alpha_2)$ on each leaf of $\mathcal{F}_1$. Consider then the product measure $p_1^*(\alpha_1) \times p_2^*(\alpha_2)$ on $\mathbb{P}(S) \oplus \mathbb{P}(S) \setminus \Delta$. We define the intersection number $i(\alpha_1, \alpha_2)$ as the volume of this measure.

By the observation we made at the beginning of this proof, we see immediately that $i(\alpha_1, \alpha_2)$ is the usual intersection number when α_1, α_2 are closed geodesics. We still have to check that this number $i \colon \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}^+$ is continuous. Bilinearity and symmetry of i are immediate.

Fix two currents α_1 and α_2 in $\mathcal{C}(S)$. If B_1 and B_2 are two flow boxes for $\mathcal{F}$, let $B_1 \oplus B_2$ be the intersection of $p_1^{-1}(B_1)$ and $p_2^{-1}(B_2)$ in $\mathbb{P}(S) \oplus \mathbb{P}(S)$. If B_1 and B_2 are small enough (w.r.t. the injectivity radius of S), two leaves of $B_1 \cap \mathcal{F}$ and $B_2 \cap \mathcal{F}$ project to S onto simple geodesic arcs, meeting at most in one point. We get then from the definitions that the measure of $B_1 \oplus B_2$ for $p_1^*(\alpha_1) \times p_2^*(\alpha_2)$ is bounded above by the product $\alpha_1(B_1)\alpha_2(B_2)$. Therefore, if we cover the compact set $\mathbb{P}C(S)$ by a finite number of small flow boxes B_j, the volume $i(\alpha_1, \alpha_2)$ is bounded above by the sum of $\alpha_1(B_j)\alpha_2(B_k)$, and is therefore finite.

The continuity of i at $(\alpha_1, \alpha_2) \in \mathcal{C}(S) \times \mathcal{C}(S)$ will be more tricky. Indeed, as $\mathbb{P}C(S) \oplus \mathbb{P}C(S) \setminus \Delta$ is not compact, the weak convergence of measures $p_2^*(\beta_2) \times p_1^*(\beta_1)$ does not necessarily imply the convergence of their volumes $i(\beta_1, \beta_2)$. Hence the strategy will be the following. Fix $\epsilon > 0$. We will construct a neighborhood U of the diagonal Δ in $\mathbb{P}C(S) \oplus \mathbb{P}C(S)$, whose measure w.r.t. $p_2^*(\beta_2) \times p_1^*(\beta_1)$ is smaller than 2ϵ, if the currents (β_1, β_2) are close enough to (α_1, α_2). We will use then the weak convergence of $p_2^*(\beta_2) \times p_1^*(\beta_1)$ to $p_2^*(\alpha_2) \times p_1^*(\alpha_1)$ on the compact complement of $\mathrm{int}(U)$ in $\mathbb{P}C(S) \oplus \mathbb{P}C(S)$ to show that $i(\beta_1, \beta_2)$ differs by at most 5ϵ from $i(\alpha_1, \alpha_2)$, if (β_1, β_2) is close enough to (α_1, α_2).

The main technical difficulty will arise from the atoms of the geodesic currents considered. We recall that an atom of a measure μ is a point x such that $\mu(\{x\}) \neq 0$. An *atom* of a geodesic current α is of course a leaf of $\mathcal{F}$ passing through an atom of the measure given by α on a submanifold transverse to $\mathcal{F}$. An atomic leaf is necessarily closed: indeed, it can go only a finite number of times through a flow box, for the latter has finite measure, and the compact set $\mathbb{P}C(S)$ may be covered by a finite number of flow boxes. The same argument shows that for any $\eta > 0$, a geodesic current can have only a finite number of atoms of measure $\geq \eta$.

Take $(\alpha_1, \alpha_2) \in \mathcal{C}(S) \times \mathcal{C}(S)$ and $\epsilon > 0$. Choose also a finite covering of $\mathbb{P}C(S)$ by flow boxes B_j, sufficiently small so that two leaves of $B_j \cap \mathcal{F}$ and $B_k \cap \mathcal{F}$ project to two simple arcs in S meeting in at most one point.

Up to replacing S by its oriented double cover, which doubles the intersection number, we can suppose S to be orientable.

Fix two numbers ϵ' and ϵ'' sufficiently small, in a sense that will be made precise later; at the moment we can say that the value of ϵ' will depend on the $\alpha_1(B_j)$ and $\alpha_2(B_j)$ above, while the value of ϵ'' will depend additionally on ϵ'. Subdividing the vertical bars of the H defining B_j, we can cut this flow box into small boxes with disjoint interiors such that:

(i) either $\alpha_1(B'_k) < \epsilon'$

(ii) or the horizontal bar λ_k of the H defining B'_k is contained in an atomic leaf of α_1 of transverse measure $\geq \epsilon'$ and $\alpha_1(B'_k \setminus \lambda_k) < \epsilon''$, analogically for α_2.

Given two geodesic currents β_1 and β_2 we have seen that the contribution of $B'_k \oplus B'_k$ to $i(\beta_1, \beta_2)$ is bounded above by $\beta_1(B'_k)\beta_2(B'_k)$. For (β_1, β_2) sufficiently close to (α_1, α_2), the total contribution to $i(\beta_1, \beta_2)$ of $B'_k \oplus B'_k$ with $\alpha_1(B'_k) < \epsilon'$ is then bounded by $\epsilon' \sum \beta_2(B'_k)$, which equals $\epsilon' \sum \beta_2(B_j)$. Symmetrically for the B'_k with $\alpha_2(B'_k) < \epsilon'$. Therefore, if ϵ' was chosen small enough w.r.t. $\alpha_1(B_j)$ and $\alpha_2(B_j)$, the contribution to $i(\beta_1, \beta_2)$ of the $B'_k \oplus B'_k$, where B'_k avoids the common atoms of volume $\geq \epsilon'$ of α_1 and α_2, will be smaller than ϵ if (β_1, β_2) is close enough to (α_1, α_2).

We still need to deal with the B'_k for which the leaf λ_k of $B'_k \cap \mathcal{F}$ has measure $\geq \epsilon'$ for both α_1 and α_2. Consider the cover $\widetilde{S}_k$ of S, whose fundamental group is the one of the closed geodesic containing λ_k, and lift B'_k to the cover. Each non–compact geodesic on $\widetilde{S}_k$ crosses the box B'_k a finite number of times. By convention, we say that the closed geodesic on $\widetilde{S}_k$ crosses B'_k an infinite number of times. Denote by A_k^p the part of B'_k consisting of the arcs of $B'_k \cap \mathcal{F}$ located on a (non–compact) geodesic on $\widetilde{S}_k$ crossing the box B'_k exactly p times. Then B'_k is the union of the $A_k^p, p \in \mathbb{N}$, and of λ_k.

We use now an essential geometric observation: if g and h are two geodesics on the hyperbolic annulus $\widetilde{S}_k$ crossing the box B'_k p and q times respectively, the projections to S of $B'_k \cap g$ and $B'_k \cap h$ meet in at most $(p+q)$ points. Moreover, the projection of $B'_k \cap g$ meets the arc of λ_k in at most one point. We deduce that the contribution of $B'_k \oplus B'_k$ to $i(\beta_1, \beta_2)$ is bounded above by

$$\sum_{p,q} \frac{p+q}{pq} \beta_1(A_k^p)\beta_2(A_k^q) + \beta_1(\lambda_k) \sum_q \frac{\beta_2(A_k^q)}{q} + \beta_2(\lambda_k) \sum_p \frac{\beta_1(A_k^p)}{p}.$$

Moreover, $\beta(B'_k) = \beta(\lambda_k) + \sum_p \beta(A_k^p)$ for any geodesic current β. Therefore,

$$\sum_{p \geq q} \frac{p+q}{pq} \beta_1(A_k^p)\beta_2(A_k^q) \leq 2\beta_1(B'_k) \sum_q \frac{\beta_2(A_k^q)}{q},$$

and

$$\sum_{p < q} \frac{p+q}{pq} \beta_1(A_k^p)\beta_2(A_k^q) \leq 2\beta_2(B'_k) \sum_p \frac{\beta_1(A_k^p)}{p}.$$

Decomposing the sums, we get

$$\sum_p \frac{\beta(A_k^p)}{p} \leq \sum_{p<p_0} \frac{\beta(A_k^p)}{p} + \sum_{p \geq p_0} \frac{\beta(A_k^p)}{p_0} \leq \sum_{p<p_0} \frac{\beta(A_k^p)}{p} + \frac{\beta(B'_k)}{p_0} \leq \sum_{p<p_0} \frac{\beta(A_k^p)}{p} + \frac{\kappa}{p_0}$$

for any geodesic current β sufficiently close to α_1 or α_2 so that all $\beta(B_j)$ are bounded by a constant κ (which only depends on $\alpha_1(B_j)$ and $\alpha_2(B_j)$).

Fix p_0 such that $\frac{\kappa}{p_0} < \epsilon'$. Observe that any geodesic in the boundary of A_k^q in B'_k meets necessarily $\partial_{\mathcal{F}} B'_k$ (possibly farther). By Lemma 4.3[3], we deduce that the boundary of A_k^p has zero measure for any geodesic current. Therefore, since $\alpha_1(A_k^p)$ and $\alpha_2(A_k^p)$ are $< \epsilon''$, there exist neighborhoods of α_1 and α_2 in $\mathcal{C}(S)$ such that, for any geodesic current β in these neighborhoods,

$$\sum_p \frac{\beta(A_k^p)}{p} \leq p_0\epsilon'' + \epsilon' \leq 2\epsilon',$$

if we have chosen $\epsilon'' \leq \frac{\epsilon'}{p_0}$ at the beginning (recall that the choice of p_0 depended only on κ and ϵ').

Hence, if (β_1, β_2) is sufficiently close to (α_1, α_2), the contribution to $i(\beta_1, \beta_2)$ of all the $B'_k \oplus B'_k$ which meet the atoms of measure $\geq \epsilon'$ common to α_1 and α_2 is at most

$$6\epsilon' \big(\sum_k \beta_1(B'_k) + \sum_k \beta_2(B'_k) \big) \leq 12\epsilon' n\kappa,$$

where n is the number of boxes B_j we started with. If ϵ' was chosen sufficiently small at the beginning, this contribution is then $\leq \epsilon$.

[3]Lemma 4.4 in Annals

Considering the two possibilities for the B'_k, we have built then a neighborhood U of Δ, consisting of the $B'_k \oplus B'_k$, whose contribution to $i(\beta_1, \beta_2)$ is $\leq 2\epsilon$ for (β_1, β_2) close enough to (α_1, α_2). As the complement of $\mathrm{int}(U)$ in $\mathbb{P}C(S) \oplus \mathbb{P}C(S)$ is compact, by the weak convergence its contribution to $i(\beta_1, \beta_2)$ is ϵ–close to its contribution to $i(\alpha_1, \alpha_2)$, and then $i(\beta_1, \beta_2)$ differs at most by $5\epsilon(= 2\epsilon + 2\epsilon + \epsilon)$ from $i(\alpha_1, \alpha_2)$, if (β_1, β_2) is close enough to (α_1, α_2).

This ends the proof of the continuity of i, hence of Proposition 4.5. $\square$

Suppose now that the hyperbolic surface S comes with another *path semi–metric* m. Recall that m associates to each path in S a positive number (possibly zero or infinite), that m is invariant under change of parametrization of paths, additive with respect to the concatenation of paths, and continuous in the compact open topology. If moreover for any $x \neq y$ the numbers associated to the paths from x to y are greater then a constant > 0, then m is a *path–metric* (see [4]).

Proposition 4.6. *If the hyperbolic surface S is equipped with a path semi–metric m, then the function "length for m" defined for the closed (hyperbolic) geodesics extends to a linear continuous function $l_m \colon \mathcal{C}(S) \to \mathbb{R}^+$.*

Proof. Given a geodesic current $\alpha \in \mathcal{C}(S)$, we equip $\mathbb{P}(S)$ with the measure, which is locally the product of the transverse measure defined by α and of the measure induced by m on the leaves of $\mathcal{F}$. The length $l_m(\alpha)$ of α is then defined as the volume of this measure.

The continuity of $l_m(\alpha)$ is immediate by compactness of $\mathbb{P}C(S)$. $\square$

Note that the mapping l_m depends on the path metric m together with the hyperbolic structure on S, while all the notions introduced before were independent of this hyperbolic metric.

On the hyperbolic surface S, there is of course a privileged path metric defined by the hyperbolic metric, hence a privileged length function l on $\mathcal{S}$. The latter will be very often used to normalize the elements of $\mathcal{C}(S)$. Indeed, we will be dealing with geodesic currents which are only defined up to homothety, and it will be more convenient to consider the space of length 1 geodesic currents rather than the projectivized $\mathcal{PC}(S)$ of $\mathcal{C}(S)$, even if those two spaces are canonically isomorphic. In particular, we have the following property (valid for any path metric):

Proposition 4.7. *On the hyperbolic surface S the set of currents α of length $l(\alpha)$ smaller than or equal to 1 is compact.*

Proof. If B is a flow box, then let λ_B be the minimum of the length of the arcs of $B \cap \mathcal{F}$. Then $\alpha(B)$ is bounded above by $\frac{l(\alpha)}{\lambda_B}$ for any geodesic current α. In particular, for a fixed B, $\alpha(B)$ are uniformly bounded if α is a length ≤ 1 current. The compactness of the latter set follows immediately from a classical result in functional analysis (c.f. [2], Chapter III, §1 n° 9, for example) which is essentially the theorem of Tichonov. $\square$

4.3 Measured geodesic laminations

A *measured geodesic lamination* α on the hyperbolic surface S (of finite type, without cusps) is a partial foliation of the convex core $C(S)$ equipped with an invariant transverse measure. More precisely, it is given by a closed subset of $C(S)$ called the *support* $\mathrm{Supp}(\alpha)$ of α, which is a disjoint union of simple geodesics and by a measure defined on any arc k in S transverse to $\mathrm{Supp}(\alpha)$, whose support is $k \cap \mathrm{Supp}(\alpha)$ and which is invariant under any homotopy respecting $\mathrm{Supp}(\alpha)$.

A measured geodesic lamination α defines naturally a geodesic current which will be also called α. Indeed, $\mathbb{P}C(S)$ admits a basis of neighborhoods consisting of flow boxes B with the following property: either B does not meet the lifts of the geodesics of $\mathrm{Supp}(\alpha)$ in $\mathbb{P}C(S)$ or B is defined by an H on S whose horizontal bar is contained in $\mathrm{Supp}(\alpha)$ and whose vertical bars are transverse to $\mathrm{Supp}(\alpha)$ and isotopic via an isotopy which preserves $\mathrm{Supp}(\alpha)$. We define $\alpha(B)$ as 0 in the first case or the measure w.r.t. α of any of the two vertical bars in the second case. This clearly defines a geodesic current $\alpha \in \mathcal{C}(S)$.

Proposition 4.8. *The measured geodesic laminations are exactly the geodesic currents with zero self–intersection, that is $\alpha \in \mathcal{C}(S)$ such that $i(\alpha, \alpha) = 0$.*

Proof. We defined the intersection number of two geodesic currents as the volume of a measure on $\mathbb{P}(S) \oplus \mathbb{P}(S) \setminus \Delta$. If $\alpha \in \mathcal{C}(S)$ comes from a measured geodesic lamination, we see immediately that the measure used to define $i(\alpha, \alpha)$ has empty support, therefore $i(\alpha, \alpha) = 0$.

Conversely, fix $\alpha \in \mathcal{C}(S)$ such that $i(\alpha, \alpha) = 0$. Then its support $\mathrm{Supp}(\alpha) \subset \mathbb{P}C(S)$ projects to a union of disjoint simple geodesics on S. Indeed, otherwise we would get two arcs in $\mathrm{Supp}(\alpha)$ which project to two simple geodesic arcs k_1 and k_2 on S meeting (transversally) in a point. Adding two small bars at the ends of these arcs to form an H on S, we get two flow boxes B_1 and B_2 around the lifts if k_1 and k_2 in $\mathbb{P}S$. But then the contribution of $B_1 \oplus B_2$ to $i(\alpha, \alpha)$ is exactly $\alpha(B_1)\alpha(B_2)$, which is nonzero since k_1 and k_2 come from $\mathrm{Supp}(\alpha)$. As $i(\alpha, \alpha) = 0$, this cannot happen, and the image A of $\mathrm{Supp}(\alpha)$ in S is of the desired type.

We still have to build an invariant measure on each arc k transverse to A. We clearly can restrict ourselves to differentiable k and we define then $\alpha(k)$

as the measure w.r.t. α of the submanifold transverse to $\mathcal{F}$ consisting of the (x, λ), where $x \in k$ and the direction λ in x is not tangent to k. One easily checks that this defines a measured geodesic lamination whose associated geodesic current is precisely α. $\square$

The simplest measured geodesic laminations are obviously the closed simple geodesics equipped with the transverse Dirac measure. Thurston showed the following result [11],[12].

Proposition 4.9. *The subset $\mathcal{L}(S)$ of $\mathcal{C}(S)$ formed by measured geodesic laminations is the closure of the set of linear combinations of disjoint simple closed curves.*

The homotheties of simple closed geodesics are not dense in $\mathcal{L}(S)$ in general. Indeed, the components of $\partial C(S)$ and the simple closed geodesics reversing the orientation are isolated in the space of simple closed geodesics. Therefore, a measured geodesic lamination which contains such a leaf cannot be approximated by a multiple of a simple closed geodesic. Nevertheless, these are the only counterexamples.

By abuse of the language, we will often denote the support of the measured geodesic lamination α simply by $\alpha \subset S$.

5 Tightening the measured lamination

In this section we develop a technical tool which will play a fundamental role in the proof of Theorem A. The idea is roughly the following. Let φ be a mapping from a compact surface S into a hyperbolic manifold M, injective on fundamental groups. Given a simple curve γ on S, φ can be homotoped so that $\varphi(\gamma)$ is shorter and shorter. If $\varphi(\gamma)$ is parabolic in M, the length of $\varphi(\gamma)$ will tend to 0. Otherwise, $\varphi(\gamma)$ will converge to the closed geodesic γ^* in M homotopic to γ. We will proceed similarly for a measured geodesic lamination α on S, which will give a uniform estimate for all closed curves which are up to rescaling close to α in the space $\mathcal{C}(S)$ of geodesic currents.

To do this, we have to define the length $l_M(\varphi(\alpha))$ of $\varphi(\alpha)$ in M when α is a measured geodesic lamination on S, or more generally a current in $\mathcal{C}(S)$: on the surface S we consider the path semi–metric induced by φ and the metric on M (the length of a path is defined as the length of its image under φ). We define then $l_M(\varphi(\alpha))$ as the length of the geodesic current α as was done in Proposition 4.6. This length is finite as long as φ is reasonable, for example Lipschitz, which in practice will always be the case.

Usually each surface S will come with a privileged hyperbolic metric used to define $\mathcal{C}(S)$ and we will denote by $l_S(\alpha)$ the length of the geodesic current w.r.t. this metric.

Proposition 5.1. *Let S be a hyperbolic surface of finite type without cusps, $\varphi\colon S \to M$ a continuous mapping from S to a hyperbolic manifold M, injective on fundamental groups, and α a geodesic lamination on S. Then*

(i) either φ can be homotoped so that $l_M(\varphi(\alpha))$ is arbitrarily short,

(ii) or for any $\epsilon > 0$ and $t < 1$, φ can be homotoped so that: for any closed geodesic γ on S with $\frac{\gamma}{l_S(\gamma)}$ sufficiently close to $\frac{\alpha}{l_S(\alpha)}$ in $\mathcal{C}(S)$, $\varphi(\gamma)$ is homotopic to a closed geodesic γ^* in M which stays at distance $\leq \epsilon$ from $\varphi(\gamma)$ along a segment of length at least $tl_M(\varphi(\gamma))$.

Moreover, these two possibilities exclude each other.

In the case where the lamination α is connected, conclusion (i) (resp. (ii)) of Proposition 5.1 amounts precisely to saying that α is non–realizable (resp. realizable), in the sense of Thurston [12], §8.

We can prove a statement similar to 5.1 when α is any geodesic current. However, this requires us to modify the framework in the following fashion: consider not only mappings $S \to M$, but also multivalued mappings $\mathbb{P}(S) \to M$, which factor through manifolds obtained by cutting $\mathbb{P}(S)$ along parts of leaves of the geodesic foliation (these manifolds give approximation of geodesic currents, similar to the train tracks defined in §5.1). There is a natural notion of homotopy between such multivalued mappings and we can prove a statement similar to Proposition 5.1 starting from the mapping $\mathbb{P}(S) \to M$ defined by the composition of the projection $\mathbb{P}(S) \to S$ and $\varphi\colon S \to M$. After having suffered through the "simple" case, the reader should understand easily why we did not dare to inflict this one.

The main idea of the proof of Proposition 5.1 is the following. Consider a closed curve in a hyperbolic manifold: if its total angular variation (i.e. the curvature integral) is large, it can be shortened significantly through a homotopy; if its total angular variation is small w.r.t. its length, the curve is homotopic to a closed geodesic which lies relatively close to it. Even if this rough statement is not completely true, due to the "shortcuts" discussed in §5.3, it suggests quite well the strategy of the proof, which we are going to adopt. Using train tracks (c.f. §5.1) we are going to homotope φ so that $\varphi(\alpha)$ becomes a graph in M, and therefore behaves more or less like a closed curve. In §5.2 we will introduce a length reducing algorithm whose efficiency increases with the total angular variation of $\varphi(\alpha)$ (conveniently defined). If the length of $\varphi(\alpha)$ tends to 0, then conclusion (i) of the proposition holds.

Otherwise, the total angular variation of $\varphi(\alpha)$ becomes small w.r.t. its length and we show in §5.4, provided all "shortcuts" were conveniently eliminated in §5.3, that conclusion (ii) holds.

5.1 Train tracks

Train tracks are a useful tool introduced by Thurston to study measured geodesic laminations on a hyperbolic surface S. The reader can consult [12] §§8–9, [3], [5], [7] for more details on train tracks.

A *train track* τ on the surface S is a finite family of "long" rectangles R_i on S, foliated by arcs parallel to the "short" sides, glued along a family of disjoint arcs in the two short sides (together with a condition on $S \setminus \tau$, which will be stated further). Two rectangles meet only along their short sides and each short side is contained in an arc of gluing. A rectangle may "bite its own tail", which means that an arc in one of its short sides can glue onto an arc in the other short side.

If not yet, the origins of the terminology will be even clearer if we add the following definitions: the leaves of the rectangles are called the *ties* of the train track, and the gluing arcs along which more than two rectangles are glued are called the *switches*. The closures of the connected components of the complement of the switches are the *branches* of the train track.

We note that this definition is slightly different from the definition of train tracks given by Thurston, whose train tracks are dimension 1 objects. Nevertheless, one can pass easily from one framework to the other, even if the two definitions are not completely equivalent (ours contains a bit more information at the switches).

It is usual to require, and we will, one more condition on train tracks τ, the closure of each component of $S \setminus \tau$ is never a disc with ≤ 2 corners at its boundary. This condition is required for τ to satisfy the following, which is a uniqueness property on the route trains can follow.

Lemma 5.2. *In a train track τ on the surface S, two paths transverse to the ties which are homotopic in S (relative the endpoints) are homotopic in τ via a homotopy preserving the ties.*

Proof. Consider the preimage $\widetilde{\tau}$ of τ in the universal cover $\widetilde{S}$ of S. Due to the condition on the components of $S \setminus \tau$, an Euler characteristic argument shows that $\widetilde{\tau}$ does not contain any simple closed curve transverse to the ties except possibly at one point. Similarly, a simple curve in $\widetilde{\tau}$, which is transverse to the ties except at two points, bounds in $\widetilde{\tau}$ a disc foliated in the obvious way. Lemma 5.2 follows easily from these observations. $\square$

A lamination α is *carried* by the train track τ if it is contained in its interior and is transverse to the ties. It is easy to show that a lamination α is always carried by at least one train track: one easy way is to locally construct a foliation transverse to α in a neighborhood of α, then to observe that the set of points, which are connected to α by a path of length $\leq \epsilon$ along the leaves of the foliation, forms a train track, if ϵ is sufficiently small, and if the foliation is sufficiently regular ([12], §§8–9 can be also consulted).

A train track τ' is *contained* in τ if it is contained in τ as a subset of S, and if moreover any tie of τ' is contained in a tie of τ. It is then clear that any lamination carried by τ' is also carried by τ, although the converse is false in general.

To prove Proposition 5.1, we are going to use mappings $\varphi\colon S \to M$ of a particular type on a train track τ carrying α: first of all, φ sends each tie of τ to a single point; secondly φ sends each branch e of τ monotonically to a piecewise geodesic segment $\varphi(e)$ in M, the condition of monotonicity meaning that the restriction $k \to \varphi(e)$ of φ to any path k in e transverse to the ties is monotone w.r.t. the uniform parametrizations of these curves. We will say then that φ is *adapted* to τ. It is immediate that, for any train track τ on S, any mapping $S \to M$ is homotopic to a mapping adapted to τ.

Adapted mappings are particularly useful, because of the control we get over the whole train track. For instance, if φ is adapted to the train track τ carrying the measured lamination α, then the length of $\varphi(\alpha)$ in M is just

$$l_M(\varphi(\alpha)) = \sum_e \alpha(e) l_M(\varphi(e)),$$

where e varies over the set of all the branches of τ and where $\alpha(e)$ is the measure w.r.t. α of any tie of e.

Similarly, when φ is adapted to τ, we can easily compute the total angular variation of $\varphi(\alpha)$ for a measured geodesic lamination α carried by τ. Let us stop for a moment to define this notion.

For a path $\gamma \subset M$, which is a piecewise immersion, its *total angular variation*, or *total curvature*, is the sum of the integral of its curvature function (defined at the points that have a neighborhood in which γ is immersed) and of the external angles at the corners of γ. For the latter, we fix a convention: at a point $x \in \gamma$ where γ admits two tangent vectors v_+ and v_- in a given orientation class on γ, the *external angle* of γ at x is the angle between v_+ and v_- in M, while its *internal angle* at x is the angle between v_+ and $-v_-$. Notice that these two angles, external and internal, are contained in the interval $[0, \pi]$, that their sum equals π, and that they are independent of the orientation chosen on γ. Avoid confusing the external angle $\in [0, \pi]$ of γ in M with the external oriented angle $\in] -\infty, \pi]$ of the boundary of a riemannian surface with piecewise smooth boundary, which appears for example in the Gauss–Bonnet formula.

Suppose now that φ is reasonable enough so that it sends each geodesic g of S to a curve $\varphi(g)$ piecewise immersed (even if $\varphi|_g$ is not necessarily an immersion). We could require, for instance, that φ is an immersion except for a finite number of points, but this condition is of course too strong for φ to be adapted to a train track; an ad hoc condition would rather be that φ is adapted to a train track τ, whose ties are all geodesics, and that φ is an immersion on $S \setminus \tau$ except for a finite number of points. Then φ induces a measure of curvature on each geodesic on S and hence on each leaf of the geodesic foliation $\mathcal{F}$ of $\mathbb{P}(S)$. If α is a geodesic current, we define the *total angular variation* or *total curvature*, $K_M(\varphi(\alpha))$ of $\varphi(\alpha)$ in M as the volume of the measure on $\mathbb{P}(S)$ which is locally the product of α and of this curvature measure along the leaves. We immediately verify that this definition coincides with the one given previously when α is a closed geodesic, and that the mapping which assigns to $\alpha \in \mathcal{C}(S)$ the value $K_M(\varphi(\alpha))$ is continuous.

When α is a measured geodesic lamination carried by the train track τ to which φ is adapted, its total curvature $K_M(\varphi(\alpha))$ can be computed in a completely combinatorial way. For a tie s of τ, define a *local route* as a germ of an arc transverse to s, defined modulo homotopy which preserves the ties. Hence, at a switch, a local route specifies two branches of τ which meet along an arc at this switch; all other ties admit only one local route. For such a local route t for a tie s, we can define its *volume* $\alpha(t)$, which is the measure w.r.t. α of the subarc of s consisting of the points through which an arc transverse to s representing t passes. Similarly, if φ is adapted to τ, we can define the angle $\theta(\varphi(t))$ as the external angle at the point $\varphi(s)$ of the broken geodesic $\varphi(t)$ in M. Then

$$K_M(\varphi(\alpha)) = \sum_t \alpha(t)\theta(\varphi(t)),$$

the sum taken over all local routes in τ. (There is of course only a finite number of nonzero terms in this sum).

In particular, for a measured geodesic lamination α, the formula above defines the total curvature $K_M(\varphi(\alpha))$ as soon as φ is adapted to a train track τ carrying α, without worrying about the regularity of φ and τ which are needed to define the total curvature of $\varphi(\beta)$ for any geodesic current β. We do not need to worry about these issues for the moment.

The following lemma is essentially a preparation for more significant modifications of φ.

Lemma 5.3. *Under the hypothesis of Proposition* 5.1*, suppose additionally that the lamination α has no compact leaves. Fix a number $\eta > 0$. If $\varphi\colon S \to$*

M is adapted to a train track τ carrying the measured lamination α, we can homotope φ to φ', adapted to a train track τ' contained in τ and still carrying α, such that $\varphi'(\tau')$ is formed of geodesic arcs in M, all of length $\geq \eta$, and such that $l_M(\varphi'(\alpha)) \leq l_M(\varphi(\alpha))$.

Proof. First observe that there exists $\eta' > 0$ with the following property: for any geodesic arc k on S contained in the convex core $C(S)$ of length $\geq \eta'$ w.r.t. the hyperbolic metric on S, the geodesic arc in M homotopic to $\varphi(k)$ (relative the endpoints) has length $\geq \eta$. To see this, consider the universal covers $\widetilde{S}$ and $\widetilde{M}$ of S and M, and lift φ to $\widetilde{\varphi}\colon \widetilde{S} \to \widetilde{M}$. As φ induces an injection on fundamental groups, the intersection of $\widetilde{\varphi}(\widetilde{S})$ with each ball of radius η in $\widetilde{M}$ is compact. We deduce that such a constant exists locally for all the arcs k starting at the same point $x \in C(S)$, hence globally by compactness of $C(S)$.

The idea will be to build τ' contained in τ and carrying α, so that each branch e' of τ' contains a geodesic of length $\geq \eta'$ crossing each tie of e' transversally. It will then be sufficient to apply the preceding observation and to tighten each $\varphi(e')$ through a homotopy of φ, so that e' is sent to the geodesic arc in M (of length $\geq \eta$) homotopic to $\varphi(e')$ relative the endpoints. Clearly, this does not increase the length of the image of α in M.

It will be convenient to define the length of a branch e as the minimal length of the arcs in e which are transverse to the ties and join its two ends (i.e. the two sides of the rectangle which are ties).

We start with the case where τ has no branch which "bites its own tail", which means that its two ends meet along an arc. From each corner of τ, we draw in τ an arc transverse to the ties until we meet a switch. We can clearly choose those arcs disjoint from α and from each other. Let τ' be a train track carrying α obtained by cutting τ along these arcs. As no branch of τ bites its own tail, each branch of τ' is made of two rectangles obtained by cutting lengthwise two distinct branches of τ. In particular, each branch of τ' has length at least twice the minimum of the lengths of the branches of τ[4].

If there is a branch e of τ which bites its own tail, we will use the hypothesis that α does not have any compact leaf to reduce to the previous case. Indeed, this implies that no half leaf of α can stay forever in e. Starting from one of the corners in the intersection of the two ends of e, we can then construct an arc in e transverse to the ties, disjoint from α, which ends outside the intersection of the two ends: if α meets e, we just have to go along a leaf of α, otherwise this is immediate. If we cut τ along this arc, and if we do this

[4]Even if this is not strictly correct, the reader will easily fix the proof.

for all the branches which bite their tails, we get a train track without any branches which bite its own tail[5]. We can then apply the previous process to obtain again a train track carrying α whose branches all have length greater than twice the minimum of the length of the branches of τ.

Iterating this process, we get a train track τ' contained in τ which carries α, whose branches all have length $\geq \eta'$. Up to deleting the branches which do not meet α and rounding off the "scars" created, we can suppose that α passes through any branch of τ'. In particular, each branch e' of τ' contains a geodesic of length $\geq \eta'$ which meets transversely each tie of e' at one point. We saw before that this ends the proof of Lemma 5.3. □

Lemma 5.4. *Under the hypothesis and conclusions of Lemma* 5.3*, we can require that*

$$K_M(\varphi'(\alpha)) \leq K_M(\varphi(\alpha)).$$

Proof. Even if this will not be absolutely necessary, it will be convenient to choose the train track τ' used in the proof of Lemma 5.3 without branches biting their tail. We saw that this is always possible.

To check that the modification in Lemma 5.3 does not increase the total curvature, we decompose the homotopy from φ to φ' into a sequence of mappings $\varphi = \varphi_0, \varphi_1, \ldots, \varphi_n = \varphi'$, where we pass from φ_i to φ_{i+1} by a homotopy which tightens the image of a single branch e_i and fixes $\tau' \setminus e_i$. We just need then to check that at each step

$$K_M(\varphi_{i+1}(\alpha)) \leq K_M(\varphi_i(\alpha)).$$

As φ_i and φ_{i+1} coincide on $\tau' \setminus e_i$, we just have to show that for any arc a which is a small neighborhood of a component of $\alpha \cap e_i$ in a leaf of α, the sum of the external angles of $\varphi_{i+1}(a)$ in M is less than or equal to the one for $\varphi_i(a)$. We note that $\varphi_i(a \setminus e_i) = \varphi_{i+1}(a \setminus e_i)$ as e_i does not bite its own tail.

We realize the homotopy between the piecewise geodesic $\varphi_i(e_i)$ and the geodesic $\varphi_{i+1}(e_i)$ by a mapping $\Delta\colon D^2 \to M$ sending ∂D^2 onto $\varphi_i(e_i) \cup \varphi_{i+1}(e_i)$. As usual we can choose Δ to be hyperbolically simplicial for a triangulation of the disc D^2 whose vertices are located on the boundary and correspond to the corners of $\varphi_i(e_i)$ and to its endpoints (which are also the endpoints of $\varphi_{i+1}(e_i)$). Identifying Δ with its graph, the former inherits a hyperbolic metric with piecewise geodesic boundary, induced from the metric on M.

Let $x_1, \ldots, x_p$ be the vertices of the triangulation Δ, in this order on $\partial\Delta$. such that x_1 and x_p correspond to the two endpoints of $\varphi_i(e_i)$. Denote by θ_k

[5] This also requires fixing.

the oriented external angle (possibly negative) of Δ at x_k, which is π minus the internal angle of $\partial\Delta$ in Δ at this point. Let also θ_k^i be the external angle (non–oriented) of $\varphi_i(a)$ at $\Delta(x_k)$ in M and let θ_1^{i+1} and θ_p^{i+1} be the external angles in M of $\varphi_{i+1}(a)$ at $\Delta(x_1)$ and $\Delta(x_p)$. Then

$$\theta_1 + \theta_2 + \ldots + \theta_p \geq 2\pi,$$
$$\forall k \neq 1, p, \quad \theta_k^i \geq \theta_k,$$
$$(\pi - \theta_1^i) \leq (\pi - \theta_1^{i+1}) + (\pi - \theta_1),$$
$$(\pi - \theta_p^i) \leq (\pi - \theta_p^{i+1}) + (\pi - \theta_p).$$

Indeed, the first inequality comes from the Gauss–Bonnet formula (the difference between those terms is the area of Δ). The other inequalities come from considering the way various angular sectors around $\Delta(x_k)$ combine in M; it may be useful for this to draw a sphere in the tangent space to M at $\Delta(x_k)$, so that the various angles are expressed as the lengths of paths drawn on this sphere.

Combining these inequalities, we get:

$$\theta_1^{i+1} + \theta_p^{i+1} \leq \theta_1^i + \theta_2^i + \ldots + \theta_p^i.$$

This means exactly that the sum of external angles of $\varphi_{i+1}(a)$ is less than or equal to the one for $\varphi_i(a)$. This ends the proof of Lemma 5.4. □

5.2 Tightening through curvature

A curve which has significant corners can be tightened through a homotopy. We are going to apply this idea to the images of measured laminations under adapted mappings. The starting point is the following observation.

Lemma 5.5. *Given $\eta > 0$, there exists a constant $c(\eta)$ with the following property: for any path k in a hyperbolic manifold M formed by two geodesic arcs whose length is between η and $\frac{\eta}{2}$, the length of the geodesic arc k' in M homotopic to k (relative the endpoints) is bounded above by*

$$l_M(k) - c(\eta)\theta^2,$$

where θ is the external angle of k at the only corner.

Proof. If l, l', l_1 and l_2 denote respectively the lengths of k, k' and the two geodesic arcs of k, we have the following formula of the hyperbolic plane trigonometry (c.f. [1], §7.12):

$$\cosh l' = \cosh l_1 \cosh l_2 + \sinh l_1 \sinh l_2 \cos\theta = \cosh l - \sinh l_1 \sinh l_2 (1 - \cos\theta).$$

Hence when θ tends to 0, the difference $(\cosh l - \cosh l')$ is of the order of θ^2, as l_1 and l_2 are between η and $\frac{\eta}{2}$. Moreover,

$$\cosh l - \cosh l' = 2 \sinh \frac{l + l'}{2} \sinh \frac{l - l'}{2}$$

is also of the order of $l - l'$. The quotient $\frac{l-l'}{\theta^2}$ stays bounded when θ tends to 0, which gives the lemma. □

In the setting of Lemma 5.3 we are going to use the estimate of Lemma 5.5 to obtain a significant reduction of the length $l_M(\varphi(\alpha))$ depending on the *quadratic angular variation* $Q_M(\varphi(\alpha))$ of $\varphi(\alpha)$ in M. The latter is defined as follows. Remember that we saw in §5.1 that the total angular variation $K_M(\varphi(\alpha))$ is

$$K_M(\varphi(\alpha)) = \sum_t \alpha(t)\theta(\varphi(t)),$$

where t varies over the set of local routes in τ, $\alpha(t)$ is the measure of t w.r.t. α, and $\theta(\varphi(t))$ is the external angle of $\varphi(t)$ in M. We define similarly

$$Q_M(\varphi(\alpha)) = \sum_t \alpha(t)\theta(\varphi(t))^2.$$

Hence, when α is a closed curve, $Q_M(\varphi(\alpha))$ is exactly the sum of the squares of the external angles at the corners of the broken geodesic $\varphi(\alpha)$ in M.

Lemma 5.6. *Under the hypothesis of Proposition* 5.1*, suppose that φ is adapted to a train track τ carrying α, such that $\varphi(\tau)$ consists of geodesic arcs all of length $\geq \eta$. Then φ can be homotoped to φ' adapted to a train track τ' contained in τ and carrying α, such that*

$$l_M(\varphi'(\alpha)) \leq l_M(\varphi(\alpha)) - c(\eta)Q_M(\varphi(\alpha))$$

and

$$K_M(\varphi'(\alpha)) \leq K_M(\varphi(\alpha)),$$

where the constant $c(\eta)$ is the one of Lemma 5.5 *and $Q_M(\varphi(\alpha))$ is the quadratic angular variation of $\varphi(\alpha)$ in M.*

Proof. As each branch e of τ is sent by φ onto a broken line consisting of geodesic arcs of length $\geq \eta$, we can cut e along some ties to get rectangles R such that each $\varphi(R)$ is a geodesic arc with length between η and $\frac{\eta}{2}$. For each of these rectangles R, let s_R be the middle tie, which is sent to the center of the arc $\varphi(R)$.

The train track τ' is defined as follows. Starting from each corner of τ, we draw an arc transverse to the ties, disjoint from α, until we meet the middle tie s_R. We choose these arcs disjoint from each other and we obtain τ' by cutting τ along these arcs. By construction, each switch of τ' is contained in a middle tie s_R, and the s_R cut τ' into rectangles R' with $\varphi(R')$ formed of two geodesics arcs and of length between η and $\frac{\eta}{2}$.

We choose now $\varphi' \colon S \to M$ adapted to τ' and homotopic to φ, such that φ' sends each rectangle R', cut by s_R from τ', onto the geodesic arc in M homotopic to the arc $\varphi(R')$. From Lemma 5.5, we get immediately that

$$l_M(\varphi'(\alpha)) \leq l_M(\varphi(\alpha)) - c(\eta) Q_M(\varphi(\alpha)).$$

Moreover, if the original train track τ had no branch biting its tail, which we may always assume, we verify that

$$K_M(\varphi'(\alpha)) \leq K_M(\varphi(\alpha))$$

in the same way as in the proof of Lemma 5.4. □

5.3 Shortcuts

Still in the setting of Proposition 5.1, we suppose $\varphi \colon S \to M$ is adapted to a train track τ. Given two numbers ϵ and η with $0 < \epsilon < \eta$, an (ϵ, η)*–shortcut* for $\varphi|_\tau$ is an arc k contained in τ transverse to the ties, such that $\varphi(k)$ has length $\geq \eta$, while the geodesic arc k' in M homotopic to $\varphi(k)$ has length $\leq \epsilon$. (In meta–mathematical sense, the shortcut would rather be k', but since it is completely determined by $k \ldots$)

(ϵ, η)–shortcuts will disturb certain estimates we are going to obtain in the next paragraph. We would like therefore to eliminate them.

Lemma 5.7. *For ϵ small enough w.r.t. η, we have the following: let $\varphi \colon S \to M$ be adapted to a train track τ carrying a lamination α without compact leaves, such that the induced mapping $\varphi_* \colon \pi_1(S) \to \pi_1(M)$ is injective. Then τ contains a train track τ' carrying α, and we can homotope φ to φ' adapted to τ' so that $\varphi'|_{\tau'}$ does not admit any (ϵ, η)–shortcuts and*

$$l_M(\varphi'(\alpha)) \leq l_M(\varphi(\alpha))$$

and

$$K_M(\varphi'(\alpha)) \leq K_M(\varphi(\alpha)).$$

(The result is still valid if α has compact leaves, and follows easily from this special case, but we will not need it.)

Proof. We fix two numbers ϵ' and η', with $0 < \epsilon' < \eta'$. We will specify them later when it will have become clear what is meant by the assertion that ϵ is small enough w.r.t. η.

The idea of the proof will be, up to modifying τ, to find for $\varphi|_\tau$ a finite family $k_1, \ldots, k_n$ of (ϵ', η')–shortcuts, which do not cross any switches, and which is maximal in the sense that any tie containing the endpoint of another (ϵ', η')–shortcut necessarily meets the interior of one of k_i. We will modify φ by replacing each $\varphi(k_i)$ with the homotopic geodesic arc in M. If ϵ' and η' were chosen conveniently w.r.t. ϵ and η, we will show that there are no more (ϵ, η)–shortcuts.

To start with, we show that there is a bound on the number of times an (ϵ', η')–shortcut can cross the branches of τ. Indeed, given two branches e_1 and e_2 of τ, there is only a finite number of homotopy classes of paths of length $\leq \epsilon'$ joining $\varphi(e_1)$ to $\varphi(e_2)$ in M. As $\varphi_* \colon \pi_1(S) \to \pi_1(M)$ is injective, there is only a finite number of homotopy classes of paths joining e_1 to e_2 on S which are realized by (ϵ', η')–shortcuts. By Lemma 5.2, there is a bound on the number (with multiplicity) of branches crossed by an (ϵ', η')–shortcut.

In particular, the length in M of the image under φ of an (ϵ', η')–shortcut is bounded above by a constant A. By the argument used in Lemma 5.3 and since we assumed the lamination α is without compact leaves, it is carried by a train track $\tau_0 \subset \tau$, whose branches all have an image in M of length $> 2A$. Hence, an (ϵ', η')–shortcut of $\varphi|_{\tau_0}$ meets each tie of τ_0 in at most one point, and meets at most one switch.

It will be convenient to use the following notation: if X is a subset of τ_0, then $R_0(X)$ is the union of the ties of τ_0 meeting X.

We consider first the shortcuts which do not cross any switches. We choose a finite family of such shortcuts $\{k_1, \ldots, k_p\}$, such that the rectangles $R_0(k_i)$ have disjoint interiors, and the union of the interiors of $R_0(k_i)$ is maximal among the families of this type. The existence of such a maximal family is immediate, as well as the fact that any other (ϵ', η')–shortcut k which does not cross any switch has at least one endpoint in the interior of some $R_0(k_i)$.

Now we focus on the union U_0 of $R_0(k)$, where k is an (ϵ', η')–shortcut such that ∂k avoids the interior of $R_0(k_i)$ (which implies that k crosses a switch). Note that U_0 cannot contain a complete branch of τ_0, since the images of the latter under φ have length $> 2A$. On the other hand, U_0 can contain some k_i. For each corner of τ_0 contained in U_0, we join this corner to one of the ties of ∂U_0 by a path in U_0 disjoint from α and transverse to the ties; we choose these paths disjoint from each other. Let τ' be the train track obtained by cutting τ_0 along these paths.

We denote by $R'(X)$ the union of the ties of τ', which meet the set X, and let U' be the part of τ' corresponding to U_0. By construction, U' does not contain any switch of τ' in its interior.

We re–index k_i so that those contained in U_0 are exactly those corresponding to the indices $i < m$. Then k_j with $m \leq j \leq p$ are also (ϵ', η')–shortcuts for $\varphi|_{\tau'}$. We complete the latter with a family $\{k_{p+1}, \ldots, k_n\}$ of (ϵ', η')–shortcuts for $\varphi|_{\tau'}$ contained in U' such that

(i) $R'(k_j)$ for $j > p$ (and hence for $j \geq m$) have disjoint interiors;

(ii) for any $i < m$, each component of $R_0(k_i) \cap U'$ is contained in $R'(k_j)$ for some $j > p$;

(iii) the union of the interiors of $R'(k_j)$ is maximal among families $\{k_{p+1}, \ldots, k_n\}$ with the above properties.

We claim that the family $\{k_m, \ldots k_p, k_{p+1}, \ldots, k_n\}$ of (ϵ', η')–shortcuts for $\varphi|_{\tau'}$ has the following property: any other (ϵ', η')–shortcut k for $\varphi|_{\tau'}$ has at least one end in the interior of one of $R'(k_i)$, with $m \leq i \leq n$. Indeed, every such k is also a shortcut for $\varphi|_{\tau_0}$. If ∂k avoids the interior of $R'(k_i) = R_0(k_i)$ for all $m \leq i \leq p$, this means that either ∂k meets the interior of $R_0(k_i)$ for some $i < m$, or k crosses a switch of τ_0 and is therefore contained in $U_0 \cap \tau' = U'$. In the first case, ∂k meets the interior of $R'(k_j)$ for some $j > p$ by condition (ii); the same happens in the second case due to the maximality condition (iii).

Now that we have defined τ' and $k_m, \ldots, k_n$, we homotope φ to φ' adapted to τ', which sends each $R'(k_i)$ with $i \geq m$ to the geodesic arc in M homotopic to $\varphi(k_i)$, and which coincides with φ on the remaining part of τ'. (This is of course possible only because the k_i avoid the switches of τ'.) Clearly, passing from φ to φ' increases neither the length of the image of α in M, nor its total curvature by the argument of Lemma 5.4.

It is finally time to specify how small ϵ' and η' were chosen at the beginning, if we want to show that $\varphi'|_{\tau'}$ does not have any (ϵ, η)–shortcuts. For fixed η, we chose ϵ' and η' so that $\epsilon' < \eta' < \eta - 2\epsilon'$. We require then in the hypotheses of the lemma that $\epsilon > 0$ is sufficiently small so as to satisfy the following property: for any geodesic arcs k_1 and k_2 of length $\leq \epsilon'$ in M with $d(k_1, k_2) \leq \epsilon$ we have $d(\partial k_1, \partial k_2) < \epsilon'$ (where $(d(X, Y)$ is the minimum of the distances between x and y for $x \in X$ and $y \in Y$). Indeed, such a property holds for $\epsilon = 0$ by the triangle inequality, and then for $\epsilon > 0$ sufficiently small by continuity (we could of course give explicit estimates).

Under these conditions, $\varphi'|_{\tau'}$ will not admit any (ϵ, η)–shortcuts. Indeed, applying the hypothesis on ϵ to the arcs $\varphi'(k_i)$, we could otherwise shorten or elongate our segment k near its endpoints in order to get an arc $k' \subset \tau'$ which would be transverse to the ties, with its endpoints outside the interiors of $R'(k_i)$, such that the length of $\varphi(k')$ in M is $\geq \eta - 2\epsilon' \geq \eta'$ and the length of the geodesic arc in M homotopic to $\varphi(k')$ is $\leq \epsilon'$. But then k' would be an (ϵ', η')–shortcut for $\varphi'|_{\tau'}$ of the type excluded by the definition of k_i. $\square$

Complement 5.8. *Under the conditions of Lemma 5.7, we assume additionally that $\varphi(\tau)$ consists of geodesic arcs of length $\geq \eta$. We then reach the conclusion that there exists a subset X of τ', which is a union of ties, such that $\varphi'(X)$ consists of geodesic arcs of length $\geq \eta$, and that*

$$l_M(\varphi'(\alpha \setminus X)) \leq 5[l_M(\varphi(\alpha)) - l_M(\varphi'(\alpha))],$$

and for any arc k transverse to the ties in τ', whose endpoints are in $\tau' \setminus X$ but which meets X, the geodesic arc in M homotopic to $\varphi'(k)$ has length $> \epsilon$.

(The 5 in the inequality is not optimal, we could for instance replace it by any constant > 4.)

Proof. In the above proof of Lemma 5.7, we cut τ_0 into rectangles R such that any $\varphi(R)$ is a geodesic arc with length between η and 2η. We then modify slightly the construction of τ' by opening the corners of $S \setminus \tau_0$ farther than ∂U_0 until we meet either $R_0(k_i)$ for some $m \leq i \leq p$, or a tie which separates two such rectangles R. We define φ' as before.

We consider one of the rectangles R of τ_0 above. If R does not meet the interior of one of k_i with $m \leq i \leq p$, its trace $R \cap \tau'$ consists of rectangles of τ' meeting at at most one corner. If R meets the interior of some k_i with $m \leq i \leq p$, observe that $R \cap \tau'$ stays connected.

We take for X the union of the components of $R \cap \tau'$ which do not meet the interior of any k_j, with $m \leq j \leq n$. By construction, $\varphi'(X) = \varphi(X)$ and consists of geodesic arcs of length $\geq \eta$. Moreover, if α_i is the measure w.r.t. α of any tie of τ' meeting the interior of k_i,

$$l_M(\varphi'(\alpha \setminus X)) \leq (4\eta + \epsilon') \sum_{i=m}^{n} \alpha_i.$$

Indeed, ϵ' bounds the lengths of $\varphi'(k_i)$ and the contribution from a component of $R \cap \tau'$, whose interior meets ∂k_i is bounded by $2\eta\alpha_i$.

We also have

$$l_M(\varphi(\alpha)) - l_M(\varphi'(\alpha)) \geq (\eta' - \epsilon') \sum_{i=m}^{n} \alpha_i.$$

As ϵ' and η' are restricted only by the condition $0 < \epsilon' < \eta' < \eta - 2\epsilon'$, we can choose them so that $\frac{(4\eta+\epsilon')}{(\eta'-\epsilon')}$ is ≤ 5, which yields the required inequality. The last property follows directly from the construction of k_i. $\square$

5.4 The proof of Proposition 5.1

Recall that we have a finite type hyperbolic surface S, a mapping $\varphi\colon S \to M$ into a hyperbolic manifold M, injective on fundamental groups, and a measured geodesic lamination α on S. Essentially, we will apply alternatively the process of tightening through curvature and through taking shortcuts. Then either the length of $\varphi(\alpha)$ will become arbitrarily small, or we will reach a situation where "$\varphi(\alpha)$ is not far from being formed by geodesics", and where conclusion (ii) holds.

We are going to alternatively use the process of tightening through curvature and the process of tightening via shortcuts to construct a sequence of mappings $\varphi_n\colon S \to M$ homotopic to φ and adapted to some train tracks τ_n carrying α, such that the length $l_M(\varphi_n(\alpha))$ decreases.

To do this, we start with φ and a train track τ carrying α. The compact leaves of α must be isolated; indeed, a simple geodesic which passes close to a simple geodesic without meeting it must spiral around it, which is not compatible with the existence of an invariant transverse measure. Therefore, up to cutting τ, we can suppose that each compact leaf of α is contained in a component of τ which is just one of its collar neighborhoods. Let τ^c be the union of these collars.

We can already say that the components of τ^c will also be components of each τ_n. Hence we can already determine $\varphi_n|_{\tau^c}$; since φ_n is adapted to τ_n, we just have to specify the images of the compact leaves γ of α: If $\varphi(\gamma)$ is loxodromic, φ_n sends γ to the corresponding closed geodesic γ^* of M. Otherwise, $\varphi_n(\gamma)$ is parabolic and we just require that the length and the curvature of $\varphi_n(\gamma)$ decreases with n and tends to 0; for example it suffices to take $\varphi_n(\gamma)$ to be a piecewise geodesic with only one corner, and have this corner moving towards the corresponding cusp when n tends to infinity.

We consider the compact leaves γ of α such that $\varphi(\gamma)$ is loxodromic and fix a number η larger than the lengths of the closed geodesics in M homotopic to these $\varphi(\gamma)$. Then we fix $\epsilon_0 > 0$ small enough to apply the tightening Lemma 5.7 with (ϵ_0, η)–shortcuts.

We start with $\tau_0 = \tau$ and homotope φ to φ_0 adapted to τ_0, such that its restriction $\varphi_0|_{\tau^c}$ is of the type required above. We then define τ_n and φ_n recurrently. Suppose τ_n and φ_n are constructed. We define τ_{n+1} and φ_{n+1} in two steps.

First we apply Lemma 5.3 to φ_n and to the train track $\tau_n \setminus \tau^c$, where we recall that τ^c is formed by the collars around the compacts leaves of α. This gives a train track τ_n' contained in τ_n and carrying α and φ_n' adapted to τ_n' and homotopic to φ_n such that $\varphi_n'(\tau_n' \setminus \tau^c)$ is formed by geodesic arcs of length $\geq \eta$ and $\varphi_n'|_{\tau^c} = \varphi_n|_{\tau^c}$.

The second step depends on the parity of n. If n is even, we apply Lemma 5.6 on tightening through curvature to φ'_n and $\tau'_n \setminus \tau^c$ to define φ_{n+1} and τ_{n+1} (without changing the conditions required for $\varphi_{n+1}|_{\tau^c}$). If n is odd, we use Lemma 5.7 on tightening through (ϵ_0, η)–shortcuts and its Complement 5.8.

Having defined τ_n and φ_n, we note that by Lemmas 5.4, 5.6 and 5.7, and by the construction of $\varphi_n|_{\tau^c}$, the sequences $l_M(\varphi_n(\alpha))$ and $K_M(\varphi_n(\alpha))$ are decreasing. In particular, they converge.

Claim 5.9. *The sequence $K_M(\varphi_n(\alpha))$ tends to 0.*

Proof. Because of the conditions imposed on $\varphi_n|_{\tau^c}$, we can assume that α has no compact leaves.

Let τ'_{2m} and φ'_{2m} be the train track and the mapping defined in the first step of the construction of τ_{2m+1} and φ_{2m+1} from τ_{2m} and φ_{2m}. Recall that $\varphi'_{2m}(\tau'_{2m})$ is formed by geodesic arcs of length $\geq \eta$.

By Lemma 5.6,

$$l_M(\varphi_{2m+1}(\alpha)) \leq l_M(\varphi'_{2m}(\alpha)) - c(\eta) Q_M(\varphi'_{2m}(\alpha)).$$

As the sequence $l_M(\varphi_n(\alpha))$ converges, we deduce that the quadratic angular variation tends to 0 when n tends to infinity.

Recall moreover that

$$K_M(\varphi'_{2m}(\alpha)) = \sum_t \alpha(t)\theta(\varphi'_{2m}(t))$$

and

$$Q_M(\varphi'_{2m}(\alpha)) = \sum_t \alpha(t)\theta(\varphi'_{2m}(t))^2,$$

where t varies over all local routes of τ'_{2m}, $\alpha(t)$ is the measure of t w.r.t. α, and $\theta(\varphi'_{2m}(t))$ is the external angle of $\varphi'_{2m}(t)$ in M. In particular, if we fix a number $\theta_0 > 0$,

$$K_M(\varphi'_{2m}(\alpha)) \leq \theta_0 \sum_{t\in A} \alpha(t) + \pi \sum_{t\in B} \alpha(t) \leq \theta_0 \sum_{t\in A} \alpha(t) + \pi \frac{Q_M(\varphi'_{2m}(\alpha))}{\theta_0^2},$$

where A is the set of local routes t such that $0 < \theta(\varphi'_{2m}(t)) \leq \theta_0$ and B is the set of t such that $\theta(\varphi'_{2m}(t)) > \theta_0$.

Moreover, since $\varphi'_{2m}(\tau'_{2m})$ is formed by geodesic arcs of length $\geq \eta$, each local route t with $\theta(\varphi'_{2m}(t)) > 0$ (i.e. corresponding to a corner) contributes at least $\eta\alpha(t)$ to the length of $\varphi_{2m}(\alpha)$. In particular,

$$\sum_{t\in A} \alpha(t) \leq \frac{l_M(\varphi'_{2m}(\alpha))}{\eta} \leq \frac{l_M(\varphi_0(\alpha))}{\eta}$$

and therefore

$$K_M(\varphi'_{2m}(\alpha)) \leq \theta_0 \frac{l_M(\varphi_0(\alpha))}{\eta} + \pi \frac{Q_M(\varphi'_{2m}(\alpha))}{\theta_0^2}.$$

For any $\epsilon > 0$, fix θ_0 such that $\theta_0 \frac{l_M(\varphi_0(\alpha))}{\eta} \leq \frac{\epsilon}{2}$. For n large enough, $\pi \frac{Q_M(\varphi'_{2m}(\alpha))}{\theta_0^2} \leq \frac{\epsilon}{2}$ and so $K_M(\varphi'_{2m}(\alpha)) \leq \epsilon$. Hence we have shown that the sequence $K_M(\varphi'_{2m}(\alpha))$ tends to 0. As

$$K_M(\varphi_{2m+2}(\alpha)) \leq K_M(\varphi_{2m+1}(\alpha)) \leq K_M(\varphi'_{2m}(\alpha)) \leq K_M(\varphi_{2m}(\alpha))$$

by Lemmas 5.4, 5.6 and 5.7, we deduce that $K_M(\varphi_n(\alpha))$ tends to 0. $\square$

Before finishing the proof of Proposition 5.1, for future reference, we mention the following easy results.

Lemma 5.10. *Given φ_n and $\epsilon > 0$, there exists a constant $c_1(\varphi_n, \epsilon)$ such that, for any geodesic arc k in the convex core $C(S)$, for which the geodesic arc in M homotopic to $\varphi_n(k)$ has length $\leq \epsilon$, the length of $\varphi_n(k)$ in M is $\leq c_1(\varphi_n, \epsilon)$.*

Proof. Since φ_n induces an injection on fundamental groups, the lift to the universal covers $\widetilde{\varphi}_n \colon \widetilde{S} \to \widetilde{M}$ is proper. In particular, the preimage under $\widetilde{\varphi}_n$ of a ball of radius ϵ in $\widetilde{M}$ is compact. Hence such a constant exists locally for the arcs k issuing from some point $x \in C(S)$, hence globally by compactness of $C(S)$. $\square$

Lemma 5.11. *Given $\epsilon > 0$ and $\eta > 0$, there is a constant $c_2(\epsilon, \eta)$ with the following property: in the hyperbolic plane $\mathbb{H}^2$ consider a quadrilateral with vertices z_1, x, z_2, x', whose oriented external angles are respectively $\frac{\pi}{2}, \theta, \frac{\pi}{2}, \theta'$, and such that the distances $d(x', z_1)$ and $d(x', z_2)$ are $\leq \epsilon$, while $d(x, z_1)$ and $d(x, z_2)$ are $\leq \eta$. Then $d(x, z_1)$ and $d(x, z_2)$ are $\leq c_2(\epsilon, \eta)\theta$.*

Proof. It is enough to show that the quotients $\frac{d(x,z_1)}{\theta}$ and $\frac{d(x,z_2)}{\theta}$ stay bounded when θ tends to 0. Let θ_1 and θ_2 be the internal angles between the arc xx' and, respectively, $x'z_1$ and $x'z_2$. Then, by Gauss–Bonnet,

$$\theta_1 + \theta_2 \leq \theta,$$

while by elementary hyperbolic trigonometry (see [1], §7.11)

$$\begin{aligned} \tanh d(x, z_1) &= \sinh d(x', z_1) \tan \theta_1, \\ \tanh d(x, z_2) &= \sinh d(x', z_2) \tan \theta_2, \end{aligned}$$

which gives the result. $\square$

Claim 5.12. *If the limit of the sequence $l_M(\varphi_n(\alpha))$ is nonzero, then conclusion (ii) of Proposition 5.1 holds: given $\epsilon > 0$ and $t < 1$, for any m sufficiently large and any closed geodesic γ on S with $\frac{\gamma}{l_S(\gamma)}$ close enough (w.r.t. m) to $\frac{\alpha}{l_S(\alpha)}$ in $\mathcal{C}(S)$, the curve $\varphi(\gamma)$ is homotopic to a closed geodesic γ^* in M, which is at distance $\leq \epsilon$ from $\varphi_{2m}(\gamma)$ on a segment of length at least $tl_M(\varphi_{2m}(\gamma))$.*

Proof. Fix $\epsilon > 0$ and $t < 1$. W.l.o.g., we can suppose ϵ is less than or equal to the number ϵ_0 used in the construction of φ_n.

At this point, we will have to consider the total curvature $K_M(\varphi_n(\beta))$ of $\varphi_n(\beta)$ for a geodesic current $\beta \in \mathcal{C}(S)$, which will require that φ_n and τ_n have certain regularity properties (see §5.1). For instance, we can require that all the ties are geodesics and φ_n is an immersion on $S \setminus \tau_n$ except at a finite number of points: since τ_n is contained in τ_0, the first condition will be immediately satisfied if τ_0 had from the beginning geodesic ties, which we can always assume; since the only conditions required until now for φ_n were about its restriction to τ_n, we can obviously assume it satisfies the required condition. Then the total curvature $K_M(\varphi_n(\beta))$ is defined for any geodesic current β, and depends continuously on $\beta \in \mathcal{C}(S)$.

Since τ_{2m} and φ_{2m} are constructed by tightening through (ϵ_0, η)–shortcuts φ'_{2m-1} and $\tau'_{2m-1} \setminus \tau^c$, let $X_{2m} \subset \tau_{2m}$ be the union of the part X of $\tau_{2m} \setminus \tau^c$ given by Complement 5.8 and of the components of τ^c which are collars around compact leaves of α whose image under φ is loxodromic. Hence, since the closed geodesics in $\varphi_{2m}(\tau^c \cap X_{2m})$ all have length $\geq \eta$, $\varphi_{2m}(X_{2m})$ is formed by geodesic arcs in M of length $\geq \eta$.

With the notation and definitions of Section 4, we cover the compact set $\mathbb{P}C(S)$ with a finite number of flow boxes $B_1, \ldots, B_r$ in $\mathbb{P}(S)$ with disjoint interiors such that any arc of $B_i \cap \mathcal{F}$ projects onto an arc transverse to the ties contained in X_{2m} if $1 \leq i \leq p$, onto an arc transverse to the ties contained in $\tau_{2m} \setminus X_{2m}$ if $p < i \leq q$, or is an arc disjoint from α if $q < i \leq r$. To do this, we start with covering the support of α with flow boxes $B_1, \ldots, B_q$ having the required properties, defined by some H whose horizontal bar is contained in $\mathrm{Supp}(\alpha)$ and whose vertical bars are ties of τ_{2m}. Then we find a completion of this covering to a covering of $\mathbb{P}C(S)$ by boxes $B'_{q+1}, \ldots, B'_r$ such that each arc of $B'_j \cap \mathcal{F}$ is an arc disjoint (in $\mathbb{P}C(S)$) from α. Moreover, we can suppose that the vertical bars of the H defining B'_j are disjoint from the H defining B_i with $i \leq q$, and are disjoint from each other. We note that then each $B'_j \cap B'_k$, $\overline{B'_j \setminus B'_k}$ and $\overline{B'_k \setminus B_j}$ is a union of a finite number of flow boxes with disjoint interiors, which allows us to finish.

Note that the boxes B_i with $i \leq p$ are exactly the ones over which we have the best control, by all our constructions.

First we restrict ourselves to the closed geodesics γ on S such that $\varphi(\gamma)$ is loxodromic, and hence homotopic to a closed geodesic γ^* in M. Let γ_{2m} be the curve obtained in M from $\varphi_{2m}(\gamma)$ by replacing the image of each component of $\gamma \cap B_i$ with $i > p$ by its homotopic geodesic arc in M. We realize then the homotopy between γ^* and the piecewise geodesic γ_{2m} by a mapping $A\colon S^1 \times [0,1] \to M$ which is hyperbolically simplicial for a drum triangulation of the annulus $S^1 \times [0,1]$, whose vertices are on the boundary ∂A, as in the proof of Lemma 2.1. Identifying as usual A with its graph, the metric in M induces a hyperbolic metric on A with piecewise geodesic boundary, each of the two components of ∂A is naturally identified with γ_{2m} or γ^*.

In γ_{2m}, we consider the image under φ_{2m} of $\gamma \cap B_i$, with $1 \leq i \leq p$. By construction, it is formed by geodesic arcs of length $\geq \eta$ (in both M and A). We denote by γ_{2m}^0 its complement in γ_{2m} (that is γ_{2m}^0 is essentially the part where γ_{2m} differs from $\varphi_{2m}(\gamma)$).

Through any point $z \in \gamma_{2m} \setminus \gamma_{2m}^0$ which is not a corner, we trace in A a geodesic arc λ_z orthogonal to the boundary of length $\frac{\epsilon}{2}$ (if possible).

If $\lambda_z \cap \lambda_{z'} \neq \emptyset$ but $z \neq z'$, we can join z with z' by an arc k in γ_{2m} which is homotopic relative the endpoints to an arc of length $\leq \epsilon$ in A, hence in M. By Lemma 5.10, the length of k in M is bounded above by certain constant $c_1(\varphi_{2m}, \epsilon)$.

If k is contained in the closure of $\gamma_{2m} \setminus \gamma_{2m}^0$, it must have length $\leq \eta$, otherwise it would give an (ϵ_0, η)–shortcut for $\varphi_{2m}|_{\tau_{2m}}$, which is not allowed. Hence it contains at most one corner of $\gamma_{2m} \setminus \gamma_{2m}^0$. By Gauss–Bonnet, k has exactly one corner of oriented external angle $\theta_A > 0$ in A, and hence of external angle $\theta \geq \theta_A$ in M. By Lemma 5.11, the length of k is bounded above by $c_2(\epsilon, \eta)\theta$.

If k meets γ_{2m}^0, recall that there exists an arc $k' \subset \gamma$ such that k is homotopic to $\varphi_{2m}(k')$ (relative the endpoints) in M. From the definition of X_{2m} in Complement 5.8 (and since k is homotopic to an arc of length $\leq \epsilon$ in M), k' cannot be contained in the train track τ_{2m} and transverse to the ties, hence contains an arc of $\gamma \cap B_i$ for some $i > q$.

Let γ_{2m}^1 be the part of $\gamma_{2m} \setminus \gamma_{2m}^0$ consisting of the points which are distance $\leq c_1(\varphi_{2m}, \epsilon)$ from the part of γ_{2m} corresponding to an arc of $\varphi_{2m}(\gamma \cap B_i)$ with $i > q$ or at distance $\leq c_2(\epsilon, \eta)\theta$ from a corner of $\gamma_{2m} \setminus \gamma_{2m}^0$ of external angle θ in M. Then, for any z and z' in $\gamma_{2m} \setminus (\gamma_{2m}^0 \cup \gamma_{2m}^1)$, the considerations above show that the arcs λ_z and $\lambda_{z'}$ of A are disjoint; we can similarly show that they are embedded.

Given $z \in \gamma_{2m} \setminus \gamma_{2m}^0$, there are four possibilities:

(1) $z \in \gamma_{2m}^1$;

(2) $z \notin \gamma_{2m}^1$ and the arc λ_z has really length $\frac{\epsilon}{2}$ in A;

(3) $z \notin \gamma_{2m}^1$ and the arc λ_z terminates on γ_{2m}^0 in A;

(4) $z \notin \gamma_{2m}^1$ and the arc λ_z terminates on γ^* in A.

(The arc λ_z cannot meet $\gamma_{2m} \setminus \gamma_{2m}^0$ by the definition of γ_{2m}^1). These four possibilities define the partition of γ_{2m} into $\gamma_{2m}^0, \gamma_{2m}^1, \gamma_{2m}^2, \gamma_{2m}^3$ and γ_{2m}^4. We are going to show that γ_{2m}^4 is "large", by estimating the lengths of the other pieces.

For $i \leq q$, all the arcs of $B_i \cap \mathcal{F}$ have the same image under φ_{2m}, which we will denote by $\varphi_{2m}(B_i)$. On the other hand, let c_3 be an upper bound on the length of the images under φ_{2m} of the arcs of $B_j \cap \mathcal{F}$ with $j > q$. Then:

$$l_M(\gamma_{2m}^0) \leq \sum_{i=p+1}^{q} \gamma(B_i) l_M(\varphi_{2m}(B_i)) + c_3 \sum_{j=q+1}^{r} \gamma(B_j)$$

and

$$l_M(\gamma_{2m} \setminus \gamma_{2m}^0) \geq l_M(\gamma_{2m}) - \sum_{i=p+1}^{q} \gamma(B_i) l_M(\varphi_{2m}(B_i)) - c_3 \sum_{j=q+1}^{r} \gamma(B_j),$$

where $\gamma(B_i)$ is the measure of the box B_i w.r.t. the current defined by γ, which is simply the number of times γ crosses B_i.

By the definition of γ_{2m}^1,

$$\begin{aligned} l_M(\gamma_{2m}^1) &\leq 2c_1(\varphi_{2m}, \epsilon) \sum_{j=q+1}^{r} \gamma(B_j) + 2c_2(\epsilon, \eta) K_M(\gamma_{2m}) \\ &\leq 2c_1(\varphi_{2m}, \epsilon) \sum_{j=q+1}^{r} \gamma(B_j) + 2c_2(\epsilon, \eta) K_M(\varphi_{2m}(\gamma)), \end{aligned}$$

where the second inequality comes from $K_M(\gamma_{2m}) \leq K_M(\varphi_{2m}(\gamma))$, which can be proved like Lemma 5.4.

Let U be the tube formed by λ_z, with $z \in \gamma_{2m}^2$. Then

$$\text{area}(A) \geq \text{area}(U) \geq \frac{\epsilon}{2} l_M(\gamma_{2m})$$

w.r.t. the hyperbolic metric with piecewise geodesic boundary on A. Moreover, by Gauss–Bonnet, the area of the annulus A is equal to the sum of its oriented external angles. Such an oriented external angle in A is less than or equal to the corresponding external angle of γ_{2m} or γ^* in M. Therefore,

$$\text{area}(A) \leq K_M(\gamma_{2m}) + K_M(\gamma^*) \leq K_M(\varphi_{2m}(\gamma))) + 0,$$

and consequently

$$l_M(\gamma_{2m}^2) \leq \frac{2}{\epsilon} K_M(\varphi_{2m}(\gamma)).$$

Some elementary hyperbolic trigonometry shows that the mapping which maps z to the other endpoint of λ_z locally increases distances (c.f. [1], §7.17). As it is also one–to–one for $z \notin \gamma^1_{2m}$, we deduce in particular that

$$l_M(\gamma^3_{2m}) \leq l_M(\gamma^0_{2m}) \leq \sum_{i=p+1}^{q} \gamma(B_i) l_M(\varphi_{2m}(B_i)) + c_3 \sum_{j=q+1}^{n} \gamma(B_j).$$

Combining all these inequalities, we get that

$$l_M(\gamma^4_{2m}) = l_M(\gamma_{2m} \setminus \gamma^0_{2m}) - l_M(\gamma^1_{2m}) - l_M(\gamma^2_{2m}) - l_M(\gamma^3_{2m})$$

is greater than or equal to

$$l_M(\varphi_{2m}(\gamma)) - 2\sum_{i=p+1}^{q} \gamma(B_i) l_M(\varphi_{2m}(B_i)) - (2c_1(\varphi_{2m}, \epsilon) + 2c_3) \sum_{j=q+1}^{n} \gamma(B_j)$$
$$- \Big(2c_2(\epsilon, \eta) + \frac{2}{\epsilon}\Big) K_M(\varphi_{2m}(\gamma)).$$

We denote now by γ^*_4 the part of γ^* consisting of the endpoints of λ_z for $z \in \gamma^4_{2m}$. By construction, each point of γ^*_4 is at distance $\leq \frac{\epsilon}{2}$ from $\varphi_{2m}(\gamma)$. Moreover, since the map which sends z to the other end of λ_z increases distances, the length of γ^*_4 is at least the length of γ^4_{2m}. In particular, $\frac{l_M(\gamma^*_4)}{l_M(\varphi_{2m}(\gamma))}$ is bounded below by

$$1 - 2\sum_{i=p+1}^{q} \gamma(B_i) \frac{l_M(\varphi_{2m}(B_i))}{l_M(\varphi_{2m}(\gamma))} - (2c_1(\varphi_{2m}, \epsilon) + 2c_3) \sum_{j=q+1}^{n} \frac{\gamma(B_j)}{l_M(\varphi_{2m}(\gamma))}$$
$$- \Big(2c_2(\epsilon, \eta) + \frac{2}{\epsilon}\Big) \frac{K_M(\varphi_{2m}(\gamma))}{l_M(\varphi_{2m}(\gamma))}.$$

As $\alpha(B_j) = 0$ for any $j > q$, by continuity we get that the lower limit of $\frac{l_M(\gamma^*_4)}{l_M(\varphi_{2m}(\gamma))}$ when $\frac{\gamma}{l_S(\gamma)}$ converges to $\frac{\alpha}{l_S(\alpha)}$ in $\mathcal{C}(S)$ is bounded below by

$$1 - 2\sum_{i=p+1}^{q} \alpha(B_i) \frac{l_M(\varphi_{2m}(B_i))}{l_M(\varphi_{2m}(\alpha))} - \Big(2c_2(\epsilon, \eta) + \frac{2}{\epsilon}\Big) \frac{K_M(\varphi_{2m}(\alpha))}{l_M(\varphi_{2m}(\alpha))} =$$
$$= 1 - 2\frac{l_M(\varphi_{2m}(\alpha \setminus X_{2m}))}{l_M(\varphi_{2m}(\alpha))} - \Big(2c_2(\epsilon, \eta) + \frac{2}{\epsilon}\Big) \frac{K_M(\varphi_{2m}(\alpha))}{l_M(\varphi_{2m}(\alpha))}.$$

Now, when m tends to infinity, $l_M(\varphi_{2m}(\alpha))$ tends to a nonzero constant by hypothesis, and $K_M(\varphi_{2m}(\alpha))$ tends to 0 by Claim 5.9. Moreover, recalling the definition of X_{2m}, by Complement 5.8 we see that $l_M(\varphi_{2m}(\alpha \setminus X_{2m}))$ tends to 0 thanks to the inequality in 5.8 and the convergence of $l_M(\varphi_{2m}(\alpha))$.

Hence, if m is sufficiently large, the lower limit of $\frac{l_M(\gamma_4^*)}{l_M(\varphi_{2m}(\gamma))}$ when $\frac{\gamma}{l_S(\gamma)}$ converges to $\frac{\alpha}{l_S(\alpha)}$ in $\mathcal{C}(S)$ is strictly greater than $t < 1$ fixed at the beginning. This shows that γ^* satisfies the conclusion of Claim 5.12: since the mapping from $\gamma_4^* \subset \gamma^*$ to $\gamma_{2m}^4 \subset \varphi_{2m}(\gamma)$ defined along λ_z moves each point by at most $\frac{\epsilon}{2}$, this shows precisely that γ^* is at distance at most $\leq \epsilon$ from $\varphi_{2m}(\gamma)$ along a segment of length at least $l_M(\gamma_4^*) \geq t l_M(\varphi_{2m}(\gamma))$. This would end the proof of Claim 5.12 if we knew in advance that $\varphi(\gamma)$ is loxodromic in M.

To end the proof of Claim 5.12, we still have to check that $\varphi(\gamma)$ is actually loxodromic in M if $\frac{\gamma}{l_S(\gamma)}$ is close enough to $\frac{\alpha}{l_S(\alpha)}$ in $\mathcal{C}(S)$. The proof is analogous to the previous one. Suppose, by way of contradiction, that $\varphi(\gamma)$ is parabolic, and define γ_{2m} from $\varphi_{2m}(\gamma)$ as before. Considering a homotopy $A\colon S^1 \times [0,\infty[\to M$ between γ_{2m} and the corresponding cusp which is hyperbolically simplicial for a triangulation of $S^1 \times [0,\infty[$ by rays $a \times [0,\infty[$, the same estimates as before (but now with $\gamma_{2m}^4 = \emptyset$) give the contradiction we are looking for. $\square$

To finish the proof of Proposition 5.1, it only remains to show that its two conclusions (i) and (ii) exclude each other. To do this, suppose conclusion (ii) holds and apply it to $t > \frac{1}{2}$. Thus we get φ' homotopic to φ such that, for any closed geodesic γ with $\frac{\gamma}{l_S(\gamma)}$ sufficiently close to $\frac{\alpha}{l_S(\alpha)}$ in $\mathcal{C}(S)$, the closed geodesic γ^* in M homotopic to $\varphi(\gamma)$ has length $\geq t l_M(\varphi'(\gamma))$. In particular, if γ is close enough to α, $\frac{l_M(\gamma^*)}{l_S(\gamma)} \geq \frac{l_M(\varphi'(\alpha))}{2 l_S(\alpha)}$, since $t > \frac{1}{2}$.

On the other hand, if condition (i) holds, for any $\epsilon > 0$ we can homotope φ to φ'' such that $\frac{l_M(\varphi''(\alpha))}{2 l_S(\alpha)} < \epsilon$.

In particular, for γ sufficiently close to α,

$$\frac{l_M(\gamma^*)}{l_S(\gamma)} \leq \frac{l_M(\varphi''(\gamma))}{l_S(\gamma)} < \epsilon,$$

which contradicts the previous conclusion for ϵ conveniently chosen. This ends the proof of Proposition 5.1. $\square$

5.5 The case of parabolic curves

We will need further a refinement of Proposition 5.1 if α consists of a finite number of disjoint simple curves whose images under φ correspond to parabolic elements in $\pi_1(M)$. In this case, obviously conclusion (i) of Proposition 5.1 is satisfied, but we want an estimate similar to the one of conclusion (ii).

Proposition 5.13. *Under the hypothesis of Proposition* 5.1*, we suppose additionally that α consists of closed (simple disjoint) geodesics $\alpha_1, \dots, \alpha_p$ such* *that each $\varphi(\alpha_i)$ is a parabolic curve in M. Then, for any closed geodesic γ on S with $\frac{\gamma}{l_S(\gamma)}$ sufficiently close to $\frac{\alpha}{l_S(\alpha)}$ in $\mathcal{C}(S)$, the curve $\varphi(\gamma)$ is homotopic to a closed geodesic γ^* which meets at least one of the cuspidal components of $M_{thin}(\mu)$, into which one of $\varphi(\alpha_i)$ can be homotoped (except of course if α is connected and $\gamma = \alpha$).*

Actually, the proof shows that an arbitrarily large fraction of γ^* is contained in those cuspidal components.

Proof. Up to replacing M by its cover $\widetilde{M}$ such that $\pi_1(\widetilde{M}) = \varphi_*(\pi_1(S))$, we can suppose w.l.o.g that the homomorphism $\varphi_*\colon \pi_1(M) \to \pi_1(S)$ induced by φ is an isomorphism. Even if not essential, this assumption will be quite convenient.

For each $i = 1, \ldots, p$, we begin with choosing a small collar neighborhood τ_i of α_i, such that the union of τ_i forms a train track τ carrying α. Then, still for each $i \leq p$, we define a flow box B_i by an H on S whose horizontal bar is α_i, while the two vertical bars merge in a tie of τ_i, and small enough for each leaf of $B_i \cap \mathcal{F}$ to project to S inside τ_i transversely to the ties. As in the proof of Claim 5.12, we complete this family with flow boxes $B_{p+1}, \ldots, B_q$ such that $B_1, \ldots, B_p, B_{p+1}, \ldots, B_q$ cover the compact set $\mathbb{P}C(S)$ and their interiors are mutually disjoint. We can also suppose that B_j for $j > p$ are small enough not to "bite their tail".

Since the box B_i, for $1 \leq i \leq p$, "bites its own tail", the leaves of $B_i \cap \mathcal{F}$ generally go around B_i several times. For $n \geq 1$, we denote by B_i^n the union of the leaves of $B_i \cap \mathcal{F}$ which make exactly n turns around B_i. Then B_i is the union of the lift of α_i and all B_i^n.

After homotoping, we can suppose that φ is adapted to the train track $\tau = \bigcup \tau_i$ and $\varphi(\tau)$ is contained in the interior of the thin part $M_{thin}(\mu)$.

Given a closed geodesic γ on S, let γ' be the piecewise geodesic in M obtained from $\varphi(\gamma)$ by replacing $\varphi(k)$ with the homotopic geodesic arc in M for each component k of $\gamma \cap B_j$. We denote by γ_1 the part of γ consisting of $\gamma \cap B_i$, for $1 \leq i \leq p$, and by γ_1' the corresponding part of γ'.

From now on, we suppose that γ is none of α_i.

If k is a component of γ_1, it is also a component of $\gamma \cap B_i^n$. Then, the corresponding component k' of γ_1' depends only on i and on n. Indeed, if $x_i \in \alpha_i$ is the intersection of α_i with the (merging) vertical bars of the H defining B_i, then k' is the geodesic arc in M joining $\varphi(x_i)$ with itself and homotopic to $\varphi(\alpha_i)^n$. Note that by convexity k' is completely contained in the cuspidal component of $M_{thin}(\mu)$ containing $\varphi(\alpha_i)$, hence γ_1' avoids the interior of $M_0(\mu)$.

Suppose we already know that γ' is homotopic to a closed geodesic γ^* of M through a homotopy A. As usual, we can suppose that A is hyperbolically simplicial for a drum triangulation of $S^1 \times [0,1]$, so that its graph, still denoted by A, inherits a hyperbolic metric with piecewise geodesic boundary. Moreover, the boundary ∂A is identified with the disjoint union of γ' and γ^*.

We have chosen B_j, for $j > p$, so that no component of $\gamma \cap B_j$ can wind around B_j several times, hence its length is uniformly bounded. Therefore, there is a compact set $K \subset M$ around $\varphi(C(S))$ which contains $\gamma' \setminus \gamma'_1$ for any closed geodesic $\gamma \subset S$.

From each point $z \in \gamma'_1$, we trace in A an arc λ_z orthogonal to ∂A at z, until it meets ∂A again or its image in M meets $K \cup M_0(\mu)$ (of course $\lambda_z = \{z\}$ if $z \in K$).

Such an arc λ_z must be embedded. Otherwise in the annulus A it would give a closed curve homotopic to γ' whose image in M is contained in the component of $M_{thin}(\mu)$ containing the image of z, which contradicts our hypothesis that γ is not α_i (since we assumed $\pi_1(M) = \pi_1(S)$).

We claim that two distinct λ_z cannot meet. Indeed, if there are two distinct points x' and y' on γ'_1 such that $\lambda_{x'} \cap \lambda_{y'} \neq \emptyset$, one of the arcs joining x' to y' in γ' is homotopic in M to an arc contained in $\lambda_{z'} \cup \lambda_{y'}$, hence in $M_{thin}(\mu)$. Moreover, due to Gauss–Bonnet, this arc $k' \subset \gamma'$ cannot be entirely contained in γ'_1. By convexity of the components of $M_{thin}(\mu)$, we can choose a homotopy from γ' to $\varphi(\gamma)$ which keeps the images of x' and y' in $M_{thin}(\mu)$, and we deduce that there are two points x and y on γ_1 which can be joined by an arc $k \subset \gamma$ meeting $\gamma \setminus \gamma_1$ such that $\varphi(k)$ is homotopic to an arc in $M_{thin}(\mu)$. In particular, x and y are in the same τ_i and $\varphi(k)$ is homotopic to an arc in the curve $\varphi(\tau_i)$ since the latter generates π_1 of the component of $M_{thin}(\mu)$ that contains it (since $\pi_1(M) = \pi_1(S)$). We conclude that the geodesic arc k is homotopic in S to an arc in τ_i. But this contradicts the fact that k is not completely contained in γ_1. Indeed, in the cover of S with the fundamental group $\pi_1(\alpha_i)$, a geodesic which leaves the lift of B_i never comes back, hence the arc k is forced to stay in B_i. This shows our claim that the λ_z are pairwise disjoint.

In particular, no λ_z returns to γ'_1. Moreover, since by definition λ_z avoid the compact set K, none of them terminates on $\gamma' \setminus \gamma'_1$. Since we want to show that γ^* goes through at least one cuspidal component of $M_{thin}(\mu)$, we just have to show that at least one λ_z ends on γ^*. We will suppose otherwise, by way of contradiction.

Let U be the union of λ_z. We want an estimate on the area of U. For this, we consider a component k of γ_1, and the corresponding component k' of γ'_1. We already saw that $k' \subset M$ depends only on i and n such that k is a component of $\gamma \cap B_i^n$. If U_k is the union of λ_z for $z \in k'$, we are going to find a lower bound a_{in} for the area of U_k, which depends only on i and n.

To do this, we consider the cusp of M corresponding to the parabolic subgroup of $\pi_1(M)$ generated by α_i. More precisely, we consider the neighborhoods of this cusp which are projections of horoballs in the universal cover $\mathbb{H}^3$ of M. Among those neighborhoods whose interior avoids $M_0(\mu)$, K and the basepoint $\varphi(x_i) \in \varphi(\alpha_i)$, there is a maximal one V_i.

The arc k' meets the convex set V_i in a subarc $k' \cap V_i$ (possibly empty). Let $k'' \subset \partial V_i$ be the image of $k' \cap V_i$ under the radial projection $V_i \to \partial V_i$ from the cusp. Then the union of k'' and $k' \cap V_i$ bounds a disc D_{in} immersed and totally geodesic (the "shadow" of $k' \cap V_i$ under the radial projection), whose area a_{in} bounds below the area of U_k. This last assertion can be verified as follows. In the universal cover $\mathbb{H}^3$ identified with the upper half–plane model so that the point at infinity corresponds to the cusp of V_i, we lift $U_k \cap V_i$ and D_{in} to the component of the preimage of V_i which is a horoball centered at infinity; then the Euclidean orthogonal projection of $U_k \cap V_i$ onto the hyperplane of $\mathbb{H}^3$ containing D_{in} decreases the hyperbolic area, and the area of its image is bounded below by the area of D_{in}.

Therefore, the area of U_k is bounded below by a_{in}. Before proceeding, we show that $\frac{a_{in}}{n}$ tends to a constant $a_i > 0$ when n tends to infinity. Indeed, as ∂V_i is horospherical, the exterior curvature of k'' in D_{in} is $+1$ at any point. The Gauss–Bonnet formula then yields

$$a_{in} = l_M(k'') - \theta_1 - \theta_2,$$

where θ_1 and θ_2 are the two internal angles of D_{in}. Namely, when n tends to infinity, the arc k enters deeper and deeper into the cusp, and θ_1 and θ_2 tend to $\frac{\pi}{2}$, while $l_M(k'')$ is close to the length of the radial projection of k' onto ∂V_i, which is asymptotically proportional to n (in the upper half–space model, the generator of the stabilizer in $\pi_1(M)$ of the point at infinity acts by translation or by reflection–translation).

Let us return to the area of U. We have seen that

$$\text{area}(U) = \sum_{k \subset \gamma_1} \text{area}(U_k) \geq \sum_{k \subset \gamma_1} a_{in} \geq \sum_{i=1}^{p} \sum_{n=1}^{\infty} a_{in} \frac{\gamma(B_i^n)}{n},$$

where $\gamma(B_i^n)$ is the measure w.r.t. γ of $Q \cap B_i^n$ for any transverse section Q of B_i (remember that each component of $\gamma \cap B_i^n$ contributes n to $\gamma(B_i^n)$).

When $\frac{\gamma}{l_S(\gamma)}$ converges to $\frac{\alpha}{l_S(\alpha)}$ in $\mathcal{C}(S)$, each $\frac{\gamma(B_i^n)}{l_S(\gamma)}$ tends to $\frac{\alpha(B_i^n)}{l_S(\alpha)} = 0$. Moreover, we saw that $\frac{a_{in}}{n}$ tends to $a_i > 0$ when n tends to infinity. Decomposing the sums into two pieces, we get that when $\frac{\gamma}{l_S(\gamma)}$ converges to $\frac{\alpha}{l_S(\alpha)}$, the limit of

$$\frac{\sum_{n=1}^{\infty} a_{in} \frac{\gamma(B_i^n)}{n}}{l_S(\gamma)}$$

is the same as the limit of

$$\frac{\sum_{n=1}^{\infty} a_i \gamma(B_i^n)}{l_S(\gamma)} = a_i \frac{\gamma(B_i)}{l_S(\gamma)},$$

which is $a_i \frac{\alpha(B_i)}{l_S(\alpha)}$. Then

$$\liminf \frac{\text{area}(U)}{l_S(\gamma)} \geq \sum_{i=1}^{p} a_i \frac{\alpha(B_i)}{l_S(\alpha)} > 0,$$

when $\frac{\gamma}{l_S(\gamma)}$ converges to $\frac{\alpha}{l_S(\alpha)}$ in $\mathcal{C}(S)$.

We also have

$$\text{area}(U) \leq \text{area}(A) \leq 2\pi \sum_{j=p+1}^{q} \gamma(B_j),$$

where the second inequality comes from the Gauss–Bonnet formula and from a rough estimate on the sum of the external oriented angles of γ' in A. Indeed, the oriented external angles of γ^* in A are all negative, each angle of γ' is $\leq \pi$, and each component of $\gamma \cap B_j$, for $j > p$, contributes 1 or 2 corners to γ'.

In particular, since $\alpha(B_j) = 0$ for all $j > p$, we get

$$\lim \frac{\text{area}(U)}{l_S(\gamma)} = 0$$

when $\frac{\gamma}{l_S(\gamma)}$ converges to $\frac{\alpha}{l_S(\alpha)}$ in $\mathcal{C}(S)$, which together with the previous inequality gives the contradiction we were looking for. Therefore, this shows that at least one λ_z terminates on γ^*, hence γ^* meets at least one component of $M_{thin}(\mu)$.

If we had not assumed from the beginning that $\varphi(\gamma)$ is homotopic to a geodesic γ^*, the same argument based on area estimates shows that actually, $\varphi(\gamma)$ cannot be parabolic if $\frac{\gamma}{l_S(\gamma)}$ is close enough to $\frac{\alpha}{l_S(\alpha)}$ (but $\gamma \neq \alpha$). $\square$

6 Proof of the main theorem

6.1 The setting

Recall the setting of Theorem A. M is a 3–dimensional hyperbolic manifold, whose fundamental group is finitely generated and satisfies condition (*). We consider an end b of the complement $M_0(\mu)$ of the interior of the cuspidal components of the thin part $M_{thin}(\mu)$, and we assume that b is geometrically infinite. We want to show that it is simply degenerate.

Since b is geometrically infinite, Proposition 2.3 gives us, up to decreasing the Margulis constant μ, a sequence of closed geodesics α_j^* of M, which *exit* the end b, which means that each neighborhood of b in $M_0(\mu)$ contains all α_j^* for j large enough. We can suppose also that the geodesics α_j^* behave well w.r.t. the non–cuspidal components of $M_{thin}(\mu)$ in the following sense: if α_j^* meets a Margulis tube, it is completely contained in it (and hence it is its core). Indeed, either b admits a neighborhood which does not meet any Margulis tube and the above property is obviously satisfied, or any neighborhood U of b in $M_0(\mu)$ meets a Margulis tube. As the action of $\pi_1(M)$ on the universal cover is properly discontinuous, the compact set ∂U can only meet a finite number of Margulis tubes, and we deduce that U contains a Margulis tube. In this second case, we can take for the geodesics α_j^* the cores of the Margulis tubes contained in the same component of $M_0(\mu) \setminus M_c(\mu)$ as b.

We saw in Proposition 2.4 that, by definition of $M_c(\mu)$ and incompressibility of S_b, each α_j^* is homotopic to a curve $\alpha_j \subset S_b$ through a homotopy which does not meet S_b.

Since α_j^* exit the end b, the sequence of distances $d(\alpha_j^*, S_b)$ tends to infinity. Then, because of the good behavior of α_j^* w.r.t. the Margulis tubes, we can apply the intersection number lemma (Proposition 3.4), which gives that the quotient

$$\frac{i(\alpha_j, \alpha_k)}{l_M(\alpha_j) l_M(\alpha_k)}$$

tends to 0 when j and k tend to infinity. (Note that the sequence $l_M(\alpha_j)$ tends to ∞, since the action of $\pi_1(M)$ on $\widetilde{M}$ is properly discontinuous.)

Now we fix once and for all an identification between S_b and the convex core of a hyperbolic surface S. Also, until now the curves α_j were free on S_b, however from now on we require them to be geodesics on S w.r.t. this identification. We denote by l_S the length on S, while l_M denotes the length in M.

By compactness of $S_b = C(S)$, there is a constant, which bounds above the ratio $\frac{l_M(\alpha)}{l_S(\alpha)}$ for any closed geodesic on S. Therefore,

$$\frac{i(\alpha_j, \alpha_k)}{l_S(\alpha_j) l_S(\alpha_k)}$$

tends to 0 as j and k tend to infinity.

We consider now the geodesic currents $\frac{\alpha_j}{l_S(\alpha_j)}$ in $\mathcal{C}(S)$. Since they have length 1, by Proposition 4.7 we can pass to a subsequence such that these geodesic currents converge to $\alpha_\infty \in \mathcal{C}(S)$. By continuity of the intersection number,

$$i(\alpha_\infty, \alpha_\infty) = \lim i(\frac{\alpha_j}{l_S(\alpha_j)}, \frac{\alpha_k}{l_S(\alpha_k)}) = \lim \frac{i(\alpha_j, \alpha_k)}{l_S(\alpha_j) l_S(\alpha_k)} = 0,$$

hence α_∞ is a measured geodesic lamination (Proposition 4.8).

Consider now Proposition 5.1 applied to $\varphi\colon S \to M$, which extends the canonical embedding $S_b \to M$, and to the lamination α_∞. The existence of α_j, whose homotopic geodesics α_j^* are contained in arbitrarily small neighborhoods of the end b shows that conclusion (ii) of Proposition 5.1 cannot hold. Therefore, conclusion (i) holds, and for any $\epsilon > 0$ we can homotope φ, so that

$$l_M(\varphi(\alpha)) < \epsilon.$$

In particular, α_∞ does not contain any leaf which is homotopic to a closed geodesic in M. Moreover, since α_j^* avoid the cuspidal components of $M_{thin}(\mu)$, Proposition 5.13 yields that α_∞ cannot consist only of compact leaves parabolic in M. Therefore, α_∞ has at least one non–compact leaf. Let γ_∞ be the measured lamination obtained by restricting α_∞ to the (connected) closure of this non–compact leaf.

It can be shown a posteriori [6] that γ_∞ is nearly the whole of α_∞, namely that α_∞ is the union of γ_∞ and some components of ∂S_b. But we do not know this for the moment.

Without loss of generality, we can restrict ourselves to the case $\pi_1(S_b) = \pi_1(M)$. Indeed, if $\widetilde{M}$ is the cover of M such that $\pi_1(\widetilde{M}) = \pi_1(S_b)$, the component of $M_0(\mu) \setminus M_c(\mu)$ containing b can be lifted diffeomorphically to $\widetilde{M}_0(\mu)$ onto a neighborhood of $\widetilde{b}$. Moreover, $\widetilde{b}$ is geometrically finite if and only if b is (use for example Proposition 2.4), and simply degenerate if and only if b is (this is immediate). By replacing M with $\widetilde{M}$, we will now assume that $\pi_1(S_b) = \pi_1(M)$, which will simplify the statements.

In this case, if $M_b(\mu)$ is the union of the cuspidal components of $M_{thin}(\mu)$ meeting S_b, we get from [13] omitting the μ to simplify the expressions, that the pair $(M_c, \partial M_c \cap M_b)$ is homeomorphic to the product $(S_b \times [0,1], \partial S_b \times [0,1])$. In particular, the cuspidal components of $M_{thin}(\mu)$ correspond to disjoint simple closed curves in S_b, well defined up to isotopy (as it was in $\partial M_c(\mu)$).

Let $C(\gamma_\infty)$ be the convex hull of γ_∞ in S, i.e. $C(\gamma_\infty)$ is the minimal convex subsurface of S which contains γ_∞. We can describe $C(\gamma_\infty)$ as follows: consider the family of closed (simple) geodesics on S homotopic to the

[6] see the preprint

boundary components of $S \setminus \gamma_\infty$ which are not homotopically trivial, then $C(\gamma_\infty)$ is the relatively compact open set of S delimited by these geodesics. Applying Proposition 4.9, we get that the measured lamination γ_∞ is the limit of a sequence of measured laminations γ_k with all leaves compact and contained in $C(\gamma_\infty)$.

For each index j, choose a leaf β_j of γ_j, and let $\lambda_j \in \mathbb{R}^+$ be the coefficient at β_j in the decomposition of the measured lamination γ_j into linear combination of its leaves, i.e. λ_j is the measure w.r.t γ_j of a short arc intersecting β_j in one point. If the leaf β_j is chosen so that its contribution $\lambda_j l_S(\beta_j)$ to $l_S(\gamma_j)$ is maximal, then the sequence $\lambda_j\beta_j$ converges to a non–zero measured lamination β_∞, after passing to a subsequence; the non–vanishing of β_∞ comes from $l_S(\lambda_j\beta_j) \geq \frac{l_S(\gamma_j)}{n}$, where the constant n depends only on the topological type of S. Since the support of γ_∞ is connected, we note that γ_∞ and β_∞ have the same support.

Hence we have found a family of simple closed geodesics β_j in $C(\gamma_\infty) = C(\beta_\infty)$ such that the sequence $\frac{\beta_j}{l_S(\beta_j)}$ converges to $\frac{\beta_\infty}{l_S(\beta_\infty)}$ for a certain measured lamination β_∞ which is bounded above by α_∞, which means that α_∞ decomposes into $\beta_\infty + \beta'$ for a lamination $\beta' \in \mathcal{L}(S)$. The reader might guess that these β_j will show that the end is simply degenerate, that is that each neighborhood of b in $M_0(\mu)$ will contain the closed geodesics β_j^* in M homotopic to β_j for j large enough (after possibly decreasing the Margulis constant μ). Note that we can assume that each β_j is loxodromic in M, since there is only a finite number of geodesics on S_b which are parabolic in M.

6.2 β_j^* stay away from the cusps

We will show in this section that there exists a Margulis constant μ' such that all β_j^* are contained in $M_0(\mu')$. This is a consequence of the simplicity of β_j, and the main tool for its proof will be hyperbolically simplicial mappings, as those used in §1.3 for the bounded diameter lemma. This section is largely inspired by Proposition 9.7.1 of [12].

Recall that we have identified the surface S_b with the convex core $C(S)$ of the hyperbolic surface S, and that $\varphi\colon S \to M$ extends the canonical embedding $S_b \to M$.

Let S' be an open subset of S containing the convex hull $C(\beta_\infty)$ such that:

(i) the closure $\overline{S'}$ is a compact submanifold of S, and φ sends each component of $\partial\overline{S'}$ onto a parabolic curve in M,

(ii) S' is minimal up to isotopy satisfying the preceding property, which means that each simple closed curve in $S' \setminus C(\beta_\infty)$, whose image under φ is parabolic, is homotopic to a component of $\partial S'$.

The existence of such surface S' is immediate, and, even if we are not going to use it, we can check that S' is unique up to isotopy (consider the simple disjoint curves in S_b associated to the cusps of M).

As in Lemma 1.7, we construct a sequence of mappings $\varphi_j\colon S' \to M$, hyperbolically simplicial w.r.t. a triangulation of S' with only one vertex, homotopic to the restriction $\varphi|_{S'}\colon S' \to M$, such that φ_j sends the simple curve β_j to the homotopic closed geodesic β_j^* in M. Moreover, we can assume that β_j^* are not contained in $M_{thin}(\mu)$, otherwise our conclusions are immediate, and then we can have the vertex of the triangulation of S', for which φ_j is hyperbolically simplicial, sent by φ_j into the complement of the thin part $M_{thin}(\mu)$.

We will denote by S'_j the surface S' equipped with the path metric induced by φ_j. As in §1.3, let $(S'_j)_{thin}(\mu)$ be the thin part of S'_j, which consists of the points of S'_j through which there is a homotopically non–trivial loop of length $\leq \mu$.

The lemma below is part of the price we have to pay for having replaced pleated surfaces of [12]with hyperbolically simplicial mappings.

Lemma 6.1. *Each component of $(S'_i)_{thin}(\mu)$ is isometric to a component of the thin part of a hyperbolic surface, which is a quotient of a horoball in $\mathbb{H}^2$, or a neighborhood of a geodesic consisting of points at bounded distance, by a group of isometries.*

Proof. Consider $x \in (S'_j)_{thin}(\mu)$. By definition, there is a geodesic arc γ_x of length $\leq \mu$ joining x to itself. The loop γ_x is freely homotopic in S'_j to a closed geodesic or a cusp. Among the arcs joining x to this geodesic or this cusp, in the homotopy class specified by the homotopy of γ_x, there is one h_x, which has minimal length (say locally in the case of a cusp); the existence of h_x can be easily shown by a broken geodesics argument, taking into account the singularity of S'_j. Applying Gauss–Bonnet shows that, among the two angular sectors defined by γ_x in x, the one which locally contains h_x has angle $< \pi$. Therefore, h_x is locally contained in $(S'_j)_{thin}(\mu)$, hence globally by connectivity. In particular, h_x avoids the singular points of S'_j, since they are outside $(S'_j)_{thin}(\mu)$.

Moreover, since φ_j induces an injection between $\pi_1(S')$ and $\pi_1(M) = \pi_1(S)$, the subgroup of $\pi_1(M, x)$ generated by all the γ_x of this kind is cyclic, and the closed geodesic or the cusp associated to x is unique. We deduce that any component of $(S'_j)_{thin}(\mu)$ is of the desired type. $\square$

Lemma 6.2. *There exists $\mu_1 \leq \mu$, independent of S'_j, such that all simple closed geodesics γ in S'_j avoid the cuspidal components of $(S'_j)_{thin}(\mu_1)$,* *and each component of $\gamma \cap (S'_j)_{thin}(\mu_1)$ meets the core of the corresponding component of $(S'_j)_{thin}(\mu_1)$.*

Proof. We can show this through elementary hyperbolic trigonometry models for the components of $(S'_j)_{thin}(\mu)$ described in Lemma 6.1.

Another point of view is the following. Having fixed a component V of $(S'_j)_{thin}(\mu)$, the space of all simple geodesic arcs in V joining ∂V to itself and, if V is not cuspidal, avoiding its core, is compact. Indeed, it is homeomorphic to the space of subarcs of ∂V. Then there exists maximal $\mu_V > 0$ such that $V \cap (S'_j)_{thin}(\mu_V)$ do not meet any of these simple arcs. As the space of models we have described is compact in the geometric topology (c.f. [12], §5.11), μ_V admit a lower bound $\mu_1 > 0$. A more pragmatic way to say this is that, if V is not cuspidal, μ_V depends only on the length of the core of V and tends to μ_V of the unique cuspidal model when this length tends to 0. □

Let μ_1 satisfy the conclusion of Lemma 6.2. By the bounded diameter lemma (Lemma 1.10), there is a constant $c_1(\mu_1)$ such that any two points of S'_j can be joined by a path k in S'_j such that the length of $k \setminus (S'_j)_{thin}(\mu_1)$ is $\leq c_1(\mu_1)$ (the constant $c_1(\mu_1)$ also depends on the topological type of S'_j, but the latter is fixed).

Lemma 6.3. *Any point $x \in S'$ such that $\varphi_j(x)$ is at distance $> 2c_1(\mu_1)$ from $M_0(\mu_1)$ is contained in $(S'_j)_{thin}(\mu_1)$.*

Proof.

Since $\varphi_j(S')$ cannot be entirely contained in $M_{thin}(\mu_1)$, the bounded diameter lemma gives that x is at distance $\leq c_1(\mu_1)$ from a component V of $(S'_j)_{thin}(\mu_1)$. Suppose that x does not belong to V and draw the shortest path from x to V. We continue this geodesic arc in the other direction through x until a point y at distance $c_1(\mu_1)$ from x, and denote by k_x the geodesic arc of length $> c_1(\mu_1)$ created.

Since $\varphi_j(y)$ is still at distance $> c_1(\mu_1)$ from $M_0(\mu_1)$, the bounded diameter lemma gives again that y is at distance $\leq c_1(\mu_1)$ from a component V' of $(S'_j)_{thin}(\mu_1)$. We join y to V' by an arc k_y of minimal length (with $k_y = y$ if $y \in V'$). Then, since $\varphi(k_x \cup k_y)$ is contained in the cuspidal component of $M_{thin}(\mu_1)$ containing $\varphi_j(x)$, the subgroup of $\pi_1(S', y)$ generated by $\pi(V, y)$ and $\pi_1(V', y)$, linking V and V' to the base point y by respectively k_x and k_y, has a cyclic image in $\pi_1(M) = \pi_1(S_b)$, and is then cyclic. Therefore, $V' = V$ and the path $k_x \cup k_y$ can be homotoped into V relative the endpoints. In particular, we deduce by the convexity of V that $y \notin V$. Moreover, k_x and k_y are orthogonal to ∂V and Gauss–Bonnet formula shows that they must merge. As they have different lengths, this gives the contradiction we were looking for and shows that x must be in $(S'_j)_{thin}(\mu_j)$. $\square$

Choose μ' so that each cuspidal component of $M_{thin}(\mu')$ is at distance $> 2c_1(\mu_1)$ from $M_0(\mu_1)$. If β_j^* meets a cuspidal component W of $M_{thin}(\mu')$, Lemma 6.3 gives that β_j meets a component of $(S'_j)_{thin}(\mu_1)$ whose fundamental group corresponds to $\pi_1(W) \subset \pi_1(M) = \pi_1(S)$. By Lemma 6.2, this component of $(S'_j)_{thin}(\mu_1)$ cannot be cuspidal in S'_j, and β_j meets its core γ^j_W. Moreover, for any point of $\beta_j \cap \gamma^j_W$, the arc of $\beta_j \cap (S'_j)_{thin}(\mu)$ containing it is not homotopic to an arc in $\partial(S'_j)_{thin}(\mu)$ and is consequently of length bounded below by a constant $c_2(\mu, \mu_1)$, which tends to infinity when μ is fixed and μ_1 tends to 0. Indeed, if an explicit formula is needed, Theorem 7.35 of [1] can be used to choose

$$\cosh\Big(\frac{c_2(\mu,\mu_1)}{2}\Big) = \sinh\Big(\frac{\mu}{2}\Big)\sinh\Big(\frac{\mu_1}{2}\Big).$$

In particular,

$$l_M(\beta_j^*) \geq c_2(\mu,\mu_1) i(\beta_j, \gamma^j_W).$$

Now, γ^j_W is homotopic to the simple closed geodesic γ_W of S corresponding to the cusp of W. Moreover, since β_j^* is the closed geodesic in M homotopic to $\beta_j \subset S_b \subset M$,

$$l_M(\beta_j^*) \leq l_M(\beta_j) \leq c_3 l_S(\beta_j),$$

where c_3 bounds the quotient of the metric induced on the compact S_b by the metrics on S and in M. We deduce that

$$i\Big(\frac{\beta_j}{l_S(\beta_j)}, \gamma_W\Big) \leq \frac{c_3}{c_2(\mu,\mu_1)}$$

whenever β_j^* meets the cuspidal component W of $M_{thin}(\mu')$. Note that the right hand side term of the inequality tends to 0 when μ is fixed and μ_1 tends to 0.

Summary. For a fixed constant μ, we choose μ_1 small enough w.r.t. μ, and μ' small enough w.r.t. μ_1. Then the only cuspidal components W of $M_{thin}(\mu')$ that β_j^* can meet correspond homotopically to simple curves γ_W which satisfy the above and can be homotoped inside S', but are not homotopic into the components of $\partial\overline{S'}$.

Now, there is only a finite number of such closed geodesics γ_W on S which correspond to cusps of M and can be homotoped into S' but not into $\partial\overline{S'}$. Moreover, each of these curves has non–zero intersection number with $\frac{\beta_\infty}{l_S(\beta_\infty)}$ by the condition of minimality in the definition of the surface S'. Therefore, if at the beginning we had chosen μ_1 small enough, so that $\frac{c_3}{c_2(\mu,\mu_1)}$ is less than these intersection numbers, and if j is sufficiently large for $\frac{\beta_j}{l_S(\beta_j)}$ to be sufficiently close to $\frac{\beta_\infty}{l_S(\beta_\infty)}$ in $\mathcal{C}(S)$, the preceding inequality is never satisfied, hence β_j^* do not meet any cuspidal component W of $M_{thin}(\mu')$.

We have reached the goal of this section.

6.3 β_j^* exit an end of $M_0(\mu)$

We saw in the previous section that, up to decreasing the Margulis constant μ, all β_j^* are contained in $M_0(\mu)$. Actually, we even showed that $\varphi_j(S')$ do not meet any cuspidal component of $M_{thin}(\mu)$, which corresponds to a curve of S' that is not homotopic to the boundary. In particular, the components of $(S'_j)_{thin}(\mu)$ which are not cuspidal in S'_j are sent into non cuspidal components of $M_{thin}(\mu)$, i.e. Margulis tubes.

We are going to show that for any compact set K in $M_0(\mu)$, only a finite number of β_j^* meet K. Once this is done, we immediately pass to a subsequence of β_j^* which exits an end b' of $M_0(\mu)$, which means that each neighborhood of b' contains all β_j^* for j large enough.

Let K be a compact set in $M_0(\mu)$. We consider the sequence of hyperbolically simplicial mappings $\varphi_j\colon S' \to M$ used in the previous section. Recall that S'_j is the surface S' equipped with the path metric induced by φ_j and the metric in M.

The first step is to build a compact set K' such that any β_j^* meeting K is contained in K'. For this we use the bounded diameter lemma, to be precise its refinement given as Lemma 1.11. Indeed, if $(S'_j)_0(\mu)$ is the complement in S'_j of the interior of the cuspidal components of $(S'_j)_{thin}(\mu)$, Lemma 1.11 gives a constant $c(\mu)$ such that any two points on $(S'_j)_0(\mu)$ can be joined by a path whose length outside $(S'_j)_{thin}(\mu) \cap (S'_j)_0(\mu)$ is bounded by $c(\mu)$. As we have just noticed, these non–cuspidal components of $(S'_j)_{thin}(\mu)$ are sent by φ_j into Margulis tubes of $M_{thin}(\mu)$. The diameter of β_j^* modulo the Margulis tubes of $M_{thin}(\mu)$ is then bounded by $c(\mu)$. If μ was chosen such that all these tubes are at distance ≥ 1 from each other, there is only a finite number of Margulis tubes which can be joined to K by a path whose length outside of these tubes is $\leq c(\mu)$. We deduce that the $c(\mu)$–neighborhood K' of the union of K and this finite number of Margulis tubes has the property we were looking for: each β_j^* meeting K is contained in K'.

We suppose, by way of contradiction, that there is an infinite family of β_j^* meeting K, hence contained in K'. Without loss of generality, we can suppose all β_j^* satisfy this.

Then a classical result states that the length of $l_M(\beta_j^*)$ is comparable to the *homotopic length* $l_h(\beta_j)$ of its conjugacy class in $\pi_1(M) = \pi_1(S)$. To define the latter, we fix a generating set for $\pi_1(M)$. Then, $l_h(\beta_j)$ is the minimal length of a word in these generators representing the conjugacy class of β_j. Changing the generating set only modifies $l_h(\beta_j)$ by a bounded factor.

Since we are staying in the compact set K', a result usually attributed to Milnor [6] gives a constant c_1 depending only on K' such that

$$c_1^{-1} l_h(\beta_j) \leq l_M(\beta_j^*) \leq c_1 l_h(\beta_j)$$

for any j.

Similarly, in the compact subset $C(S)$ of S, there exists a constant c_2 such that

$$c_2^{-1} l_h(\beta_j) \leq l_S(\beta_j) \leq c_2 l_h(\beta_j).$$

We deduce that

$$\frac{l_M(\beta_j^*)}{l_S(\beta_j)} \geq \frac{1}{c_1 c_2},$$

and then

$$\frac{l_M(\beta_j')}{l_S(\beta_j)} \geq \frac{1}{c_1 c_2}$$

for any curve β_j' homotopic to β_j in M.

In particular, if the mapping $\varphi : S \to M$ is such that its restriction to S_b is homotopic to the canonical embedding $S_b \to M$,

$$\frac{l_M(\varphi(\beta_j))}{l_S(\beta_j)} \geq \frac{1}{c_1 c_2}$$

for any j, and passing to the limit,

$$\frac{l_M(\varphi(\beta_\infty))}{l_S(\beta_\infty)} \geq \frac{1}{c_1 c_2} > 0.$$

But we saw in §6.1 that the existence of α_j^* implies that the lamination α_∞ satisfies conclusion (i) of Proposition 5.1. This means that we can find such φ with $l_M(\varphi(\alpha_\infty))$, hence $l_M(\varphi(\beta_\infty))$, arbitrarily small. This gives the required contradiction and proves that only a finite number of $\varphi_j(S')$ can meet a given compact set K.

Hence from now on, by passing to a subsequence, we can suppose that β_j^* exit an end b' of $M_0(\mu)$. We still have to show that $b' = b$.

6.4 β_j^* exit the right end

This is maybe the most delicate part of the proof of Theorem A (and the author confesses being stuck on it for months). For instance, it is essentially for this part of the proof that we have introduced in Section 5 the shortening operations on measured laminations. Indeed, the use of Proposition 5.1 we have made in the previous section could have been replaced with rougher estimates. It is worth comparing the analysis of this paragraph with that of Thurston in §9.6 of [12] in the more restricted framework where α_j and β_k are simple curves, for which he has to introduce his delicate interpolations of pleated surfaces. The proof we are going to use is mainly based on the intersection number lemma proven in Section 3.

On the hyperbolic surface S, whose convex core is identified with S_b, we have two sequences (α_j) and (β_k) of closed geodesics. The sequence $\frac{\alpha_j}{l_S(\alpha_j)}$ converges to the measured geodesic lamination α_∞ in $\mathcal{C}(S)$, and the geodesics α_j^* of M homotopic to α_j exit the end b of $M_0(\mu)$, which means that each neighborhood of b contains all α_j^* except for a finite number. Similarly, the sequence $\frac{\beta_k}{l_S(\beta_k)}$ converges to $\frac{\beta_\infty}{l_S(\beta_\infty)}$, where β_∞ is bounded above by α_∞, and the geodesics β_k^* exit an end b' of $M_0(\mu)$ (the fact that β_k are simple will not be used in this section). We want to show that $b = b'$. Suppose otherwise, by way of contradiction.

Let γ be a closed geodesic in the interior of the convex hull $C(\beta_\infty)$. Then

$$i(\gamma, \alpha_\infty) \geq i(\gamma, \beta_\infty) \neq 0.$$

We saw in §6.1 that conclusion (i) of Proposition 5.1 holds for the lamination $\alpha_\infty \subset S_b \subset M$. In other words, we can build a mapping $\varphi\colon S \to M$, whose restriction to $S_b = C(S)$ is homotopic to the embedding $S_b \to M$, such that the length $l_M(\varphi(\alpha_\infty))$ is arbitrarily small.

We fix such a mapping φ. By transversality, we can suppose that the image under φ of the union of all closed geodesics on S avoids the curve $\gamma \subset S_b \subset M$. We choose a homotopy A_j between α_j^* and $\varphi(\alpha_j)$, and a homotopy B_k between β_k^* and $\varphi(\beta_k)$. Note that, since $\pi_1(M) = \pi_1(S_b)$, the homotopy A_j is unique up to homotopy respecting α_j^* and $\varphi(\alpha_j)$, the same holds for B_k.

Lemma 6.4. *There is a constant $c(\gamma)$ such that, in the notation of Section 3,*

$$i(A_j, \gamma) \leq c(\gamma) l_M(\varphi(\alpha_j)),$$
$$i(B_k, \gamma) \leq c(\gamma) l_M(\varphi(\beta_k)).$$

Proof. This is the intersection number lemma argument. We suppose first that $\varphi(\alpha_j)$ is piecewise geodesic and, since $i(A_j,\gamma)$ is invariant under homotopy of A_j by Lemma 3.1, we choose A_j hyperbolically simplicial for a drum triangulation of the annulus $S^1 \times [0,1]$. Then, if ϵ is the minimum of the injectivity radius of M on γ, we get as in the first case of Proposition 3.4:

$$\begin{aligned} i(A_j,\gamma) &\leq c_1(\epsilon) l_M(\gamma)\mathrm{area}(A_j) + c_2(\epsilon) l_M(\gamma)\mathrm{length}(\partial A_j) \\ &\leq \big(c_1(\epsilon)+c_2(\epsilon)\big) l_M(\gamma)\mathrm{length}(\partial A_j) \\ &\leq \big(c_1(\epsilon)+c_2(\epsilon)\big) l_M(\gamma)\big(l_M(\alpha_j^*)+l_M(\varphi(\alpha_j))\big) \\ &\leq 2\big(c_1(\epsilon)+c_2(\epsilon)\big) l_M(\gamma) l_M(\varphi(\alpha_j)) \end{aligned}$$

which is an inequality of the desired kind. When $\varphi(\alpha_j)$ is not a piecewise geodesic, we do the same by approximating with piecewise geodesics.

The proof is of course identical for $i(B_k,\gamma)$. □

Now that we have these estimates, we will try to link $i(A_j,\gamma)$ and $i(B_k,\gamma)$ to $i(\beta_\infty,\gamma)$. A convenient method to do this will be to use the duality formula of Lemma 3.3.

Let K be a compact set in M whose interior contains entirely the image of the homotopy between $\varphi|_{S_b}$ and the embedding $S_b \to M$. We can choose K such that its intersection K_0 with $M_0(\mu)$ is a codimension 0 submanifold of $M_0(\mu)$ containing the compact core $M_c(\mu)$. Moreover, we can do it so that no component of $M_0(\mu)\backslash K$ is relatively compact. Then, by Proposition 1.3, each component of $M_0(\mu)\backslash M_c(\mu)$ contains exactly one component of the boundary ∂K_0 and, for degree reasons, the image in $\pi_1(M)$ of each component of ∂K_0 is the same as the one of the component of $\partial M_c(\mu)$ it faces. In particular, there exist two homotopies C_b and $C_{b'}$ between γ and two curves located respectively in the component of ∂K_0 facing b and the one facing b'.

Lemma 6.5. *If j and k are large enough for α_j^* and β_k^* to avoid the compact set K and the images of C_b and $C_{b'}$, then*

$$i(A_j,\gamma) = i(\varphi(\alpha_j), C_{b'})$$

and

$$i(B_k,\gamma) = i(\varphi(\beta_k), C_b).$$

Proof. By Lemma 3.1, $i(A_j,\gamma)$ depends only on α_j^* and $\varphi(\alpha_j)$. By the construction of K, we can choose the homotopy A_j such that its image is contained in the union of K and the component of $M_0(\mu)\setminus M_c(\mu)$ containing b, by gluing together a homotopy between α_j^* and $\alpha_j \subset S_b$ and a homotopy between α_j and $\varphi(\alpha_j)$. In particular, the image of A_j avoids the curve $\partial C_{b'}\backslash\gamma$.

Then we just have to apply the duality formula of Lemma 3.3. The same argument works for $i(B_k, \gamma)$. $\square$

When j tends to infinity, the geodesic current $\frac{\alpha_j}{l_S(\alpha_j)}$ converges to $\frac{\alpha_\infty}{l_S(\alpha_\infty)}$, and we would like to say that $\frac{i(\varphi(\alpha_j), C_{b'})}{l_S(\alpha_j)}$ tends to $\frac{i(\varphi(\alpha_\infty), C_{b'})}{l_S(\alpha_\infty)}$. To do this, we have to give a meaning to this statement. That is what we are going to do now.

Let $\widetilde{S}$ be the cover of S whose fundamental group is generated by γ, we still denote by γ the lift of this curve to $\widetilde{S}$. The geodesics on $\widetilde{S}$ are of three types:

1) those which cross γ transversely in one point;

2) those which stay at positive distance from γ;

3) those spiraling towards γ (including γ).

The first ones are determined by their intersection point with γ and their direction at this point. The second ones are characterized by the unique point in $\widetilde{S} \setminus \gamma$ where they are parallel to γ, which means orthogonal to the perpendicular to γ through this point. The space $G_p(\widetilde{S})$ of the geodesics of type 1) and 2) on $\widetilde{S}$ (the p in $G_p(\widetilde{S})$ refers to the fact that these are exactly the proper geodesics on $\widetilde{S}$) can then be identified with two submanifolds of $\mathbb{P}(\widetilde{S})$, transverse to the geodesic foliation, parametrized respectively by $\gamma \times]0, \pi[$ and $\widetilde{S} \setminus \gamma$, and which meet each geodesic of $G_p(\widetilde{S})$ in exactly one point.

If α is a geodesic current on S, it lifts to a measure transverse to the geodesic foliation on $\mathbb{P}(\widetilde{S})$, which induces a measure on the two submanifolds identified with $G_p(\widetilde{S})$. Therefore we can speak about the measure induced by α on $G_p(\widetilde{S})$.

We consider now the cover $\widetilde{M}$ of M, whose fundamental group is generated by γ, and we lift φ to $\widetilde{\varphi}\colon \widetilde{S} \to \widetilde{M}$ and $C_{b'}$ to $\widetilde{C_{b'}}\colon S^1 \times [0,1] \to \widetilde{M}$. Let $\widetilde{S}_b$ be the preimage in $\widetilde{S}$ of the convex core of S_b of S. (We will avoid the notation $C(S)$ to avoid confusing with C_b and $C_{b'}$.) We perturb slightly φ and $C_{b'}$ so that, by transversality, $\widetilde{\varphi}^{-1}(\widetilde{C}_{b'})$ is a family of immersed arcs and curves on $\widetilde{S}$; this can clearly be done preserving the property that $\varphi^{-1}(\gamma)$ avoids the union of all closed geodesics on S. As φ induces an injection on fundamental groups, the restriction $\widetilde{\varphi}|_{\widetilde{S}_b} : \widetilde{S}_b \to \widetilde{M}$ is proper, and $\widetilde{\varphi}^{-1}(\widetilde{C}_{b'}) \cap \widetilde{S}_b$ is compact. Let $X \subset G_p(\widetilde{S})$ be the set of proper geodesics on $\widetilde{S}$ contained in $\widetilde{S}_b$ which do not intersect $\partial\widetilde{\varphi}^{-1}(\widetilde{C}_{b'}) = \widetilde{\varphi}^{-1}(\gamma)$ and have odd intersection number with $\widetilde{\varphi}^{-1}(\widetilde{C}_{b'})$; note that this condition makes sense only because $\widetilde{\varphi}^{-1}(\widetilde{C}_{b'}) \cap \widetilde{S}_b$ is compact and we restrict to proper geodesics avoiding $\partial\widetilde{\varphi}^{-1}(\widetilde{C}_{b'})$. If α is a geodesic current, we define $i(\varphi(\alpha), C_{b'})$ as the volume of this set X w.r.t. the measure induced by α on $G_p(\widetilde{S})$.

Lemma 6.6. *The mapping $\alpha \to i(\varphi(\alpha), C_{b'})$ defined above is continuous in any current of which γ is not an atom, and coincides with the standard notion of the geometric intersection number when α is a closed geodesic on S.*

Proof. Suppose γ is not an atom of $\alpha \in \mathcal{C}(S)$ and, to show the continuity in α, take $\epsilon > 0$. Since $\alpha(\gamma) = 0$, there is a flow box B in $\mathbb{P}(S)$ crossed by γ such that $\alpha(B) < \epsilon$. We lift B to $\widetilde{S}$.

Let X_B be the subset of X consisting of these closed geodesics, which cross B on $\widetilde{S}$. Then $X \setminus X_B$ is relatively compact in $G_p(\widetilde{S})$. Moreover, the boundary of X consists of geodesics passing through $\widetilde{\varphi}^{-1}(\gamma)$; since $\widetilde{\varphi}^{-1}(\gamma)$ does not meet any closed geodesic on S, the same reasoning as in Lemma 4.3[7] shows that the boundary of X has zero measure for any geodesic current on S. Therefore, if β converges to α in $\mathcal{C}(S)$, $\beta(X \setminus X_B)$ tends to $\alpha(X \setminus X_B)$ by the weak convergence of the measures induced on $G_p(\widetilde{S})$ (see [2], chapter 4, §5, n° 12, for a proof of this property of weak convergence that we have already used many times in Section 4). Moreover, $\alpha(X_B) \leq \alpha(B) < \epsilon$ and $\beta(X_B) \leq \beta(B) < \epsilon$ if β is close enough to α. Hence $|\alpha(X) - \beta(X)|$ is bounded above by 3ϵ if β is close enough to α.

This proves continuity of the mapping $\alpha \to i(\varphi(\alpha), C_{b'})$ for geodesic currents of which γ is not an atom. That this mapping coincides with the standard notion (defined in Section 3) when α is a closed geodesic is almost immediate. □

Combining Lemmas 6.4, 6.5 and 6.6, we get that

$$\begin{aligned} i(\varphi(\alpha_\infty), C_{b'}) &= \lim_{j\to\infty} \frac{i(\varphi(\alpha_j), C_{b'})}{l_S(\alpha_j)} = \lim_{j\to\infty} \frac{i(A_j, \gamma)}{l_S(\alpha_j)} \\ &\leq c(\gamma) \lim_{j\to\infty} \frac{l_M(\varphi(\alpha_j))}{l_S(\alpha_j)} \leq c(\gamma) l_M(\varphi(\alpha_\infty)). \end{aligned}$$

We define similarly $i(\varphi(\beta_\infty), C_b)$ and show using B_k that

$$i(\varphi(\beta_\infty), C_b) \leq c(\gamma) l_M(\varphi(\beta_\infty)).$$

We consider now the homotopy C between the curves $\partial C_b \setminus \gamma$ and $\partial C_{b'} \setminus \gamma$ in M defined by gluing together C_b and $C_{b'}$. To define $i(\varphi(\alpha_\infty), C_{b'})$, we used the subset X of $G_p(\widetilde{S})$ consisting of the geodesics contained in $\widetilde{S}_b$, avoiding

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$\widetilde{\varphi}^{-1}(\gamma)$, and having odd intersection number with $\widetilde{\varphi}^{-1}(\widetilde{C}_{b'})$. Let $Y \subset G_p(\widetilde{S})$ be defined similarly by replacing $C_{b'}$ with C_b. Then, by definition,

$$\begin{aligned} i(\varphi(\beta_\infty), C_{b'}) &= \beta_\infty(X) \\ i(\varphi(\beta_\infty), C_b) &= \beta_\infty(Y) \\ i(\varphi(\beta_\infty), C) &= \beta_\infty(X \cup Y \setminus X \cap Y). \end{aligned}$$

In particular,

$$\begin{aligned} i(\varphi(\beta_\infty), C) \leq i(\varphi(\beta_\infty), C_{b'}) + i(\varphi(\beta_\infty), C_b) \leq i(\varphi(\alpha_\infty), C_{b'}) + i(\varphi(\beta_\infty), C_b) \leq \\ \leq c(\gamma) l_M(\varphi(\alpha_\infty)) + c(\gamma) l_M(\varphi(\beta_\infty)) \leq 2c(\gamma) l_M(\varphi(\alpha_\infty)). \end{aligned}$$

Moreover, $\varphi(S_b)$ avoids ∂C by the construction of C_b and $C_{b'}$, and $i(\varphi(\beta_\infty), C)$ stays unchanged if we homotope φ among the mappings having this property (apply Lemma 3.1 approximating β_∞ in $\mathcal{C}(S)$ by multiples of closed geodesics). In particular, since the image of the homotopy between $\varphi|_{S_b}$ and the embedding $S_b \to M$ avoids ∂C, because it is completely contained in the interior of the compact set K, we deduce that

$$i(\varphi(\beta_\infty), C) = i(\beta_\infty, C) = i(\beta_\infty, \gamma),$$

where the second equality comes from Lemma 3.1 and from replacing C with C' such that $\partial C' = \partial C$ and $C' \cap S_b = \gamma$ (note that this is the first time we use the hypothesis $b' \neq b$). With the preceding estimates for $i(\varphi(\beta_\infty), C)$, we have shown that

$$i(\beta_\infty, \gamma) \leq 2c(\gamma) l_M(\varphi(\alpha_\infty)).$$

$i(\beta_\infty, \gamma)$ is non–zero by the construction of γ and we could choose at the beginning such φ that $l_M(\varphi(\alpha_\infty))$ is arbitrarily small (by conclusion (i) of Proposition 5.1). This gives us the contradiction we were looking for, and shows that $b = b'$.

Therefore the end b is simply degenerate, and here ends the proof of Theorem A.

We conclude with a remark. For our proof of Theorem A, we only used a weak form of the continuity of the intersection number on geodesic currents on a surface S, which is: if γ_j is a sequence of geodesic currents converging to $\gamma_\infty \in \mathcal{C}(S)$, and if $i(\gamma_j, \gamma_j)$ tends to 0, then $i(\gamma_\infty, \gamma_\infty) = 0$ and γ_∞ is a measured geodesic lamination. The proof of this statement is far simpler than the one of the continuity of the function i, but we thought it is conceptually more coherent to give the proof of the general property.

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