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DISMANTLABLE CLASSIFYING SPACE FOR THE FAMILY OF PARABOLIC SUBGROUPS OF A RELATIVELY HYPERBOLIC GROUP

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Abstract Let G be a group hyperbolic relative to a finite collection of subgroups $\mathcal{P}$. Let $\mathcal{F}$ be the family of subgroups consisting of all the conjugates of subgroups in $\mathcal{P}$, all their subgroups, and all finite subgroups. Then there is a cocompact model for $E_{\mathcal{F}}G$. This result was known in the torsion-free case. In the presence of torsion, a new approach was necessary. Our method is to exploit the notion of dismantlability. A number of sample applications are discussed.

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1. Introduction

Let G be a finitely generated group hyperbolic relative to a finite collection $\mathcal{P} = \{P_\lambda\}_{\lambda\in\Lambda}$ of its subgroups (for a definition see § 2). Let $\mathcal{F}$ be the collection of all the conjugates of P_λ for $\lambda \in \Lambda$, all their subgroups, and all finite subgroups of G. *A model for* $E_{\mathcal{F}}G$ is a G-complex X such that all point stabilisers belong to $\mathcal{F}$, and for every $H \in \mathcal{F}$ the fixed-point set X^H is a (nonempty) contractible subcomplex of X. A model for $E_{\mathcal{F}}G$ is also called the *classifying space for the family* $\mathcal{F}$. In this article we describe a particular classifying space for the family $\mathcal{F}$. It admits the following simple description.

Let S be a finite generating set of G. Let $V = G$ and let W denote the set of cosets gP_λ for $g \in G$ and $\lambda \in \Lambda$. We consider the elements of W as subsets of the vertex set of the Cayley graph of G with respect to S. Then $|\cdot,\cdot|_S$, which denotes the distance in the Cayley graph, is defined on $V \cup W$. The *n-Rips graph* Γ_n is the graph with vertex set $V \cup W$ and edges between $u, u' \in V \cup W$ whenever $|u, u'|_S \leqslant n$. The *n-Rips complex* $\Gamma_n^{\blacktriangle}$ is obtained from Γ_n by spanning simplices on all cliques. It is easy to prove that Γ_n is a fine δ-hyperbolic connected graph (see § 2). Our main result is the following.

Theorem 1.1. *For n sufficiently large, the n-Rips complex $\Gamma_n^{\blacktriangle}$ is a cocompact model for $E_{\mathcal{F}}G$.*

Theorem 1.1 was known to hold if

- G is a torsion-free hyperbolic group and $\mathcal{P} = \emptyset$, since in that case the n-Rips complex $\Gamma_n^{\blacktriangle}$ is contractible for n sufficiently large [1, Theorem 4.11].
- G is a hyperbolic group and $\mathcal{P} = \emptyset$, hence $\mathcal{F}$ is the family of all finite subgroups, since in that case $\Gamma_n^{\blacktriangle}$ is $\underline{E}(G)$ [21, Theorem 1], see also [13, Theorem 1.5] and [15, Theorem 1.4].
- G is a torsion-free relatively hyperbolic group, but with different definitions of the n-Rips complex, see the work of Dahmani [8, Theorem 6.2], or Mineyev and Yaman [22, Theorem 19].

In the presence of torsion, a new approach was necessary. Our method is to exploit the notion of *dismantlability*. Dismantlability, a property of a graph guaranteeing strong fixed-point properties (see [25]) was brought to geometric group theory by Chepoi and Osajda [6]. Dismantlability was observed for hyperbolic groups in [13], following the usual proof of the contractibility of the Rips complex [5, Prop III.Γ 3.23].

While we discuss the n-Rips complex only for finitely generated relatively hyperbolic groups, Theorem 1.1 has the following extension.

Corollary 1.2. *If G is an infinitely generated group hyperbolic relative to a finite collection $\mathcal{P}$, then there is a cocompact model for $E_{\mathcal{F}}G$.*

Proof. By [23, Theorem 2.44], there is a finitely generated subgroup $G' \leqslant G$ such that G is isomorphic to G' amalgamated with all P_λ along $P'_\lambda = P_\lambda \cap G'$. Moreover, G' is hyperbolic relative to $\{P'_\lambda\}_{\lambda\in\Lambda}$. Let S be a finite generating set of G'. While S does not generate G, we can still use it in the construction of $X = \Gamma_n^{\blacktriangle}$. More explicitly, if X' is the n-Rips complex for S and G', then X is a tree of copies of X' amalgamated along vertices in W. Let $\mathcal{F}'$ be the collection of all the conjugates of P'_λ, all their subgroups, and all finite subgroups of G'. By Theorem 1.1, we have that X' is a cocompact model for $E_{\mathcal{F}'}G'$, and it is easy to deduce that X is a cocompact model for $E_{\mathcal{F}}G$. □

Applications

On our way towards Theorem 1.1 we establish the following, for the proof see § 2. We learned from François Dahmani that this corollary can also be obtained from one of Bowditch's approaches to relative hyperbolicity.

Corollary 1.3. *There is finite collection of finite subgroups $\{F_1, \ldots, F_k\}$ such that any finite subgroup of G is conjugate to a subgroup of some P_λ or some F_i.*

Note that by [23, Theorem 2.44], Corollary 1.3 holds also if G is infinitely generated, which we allow in the remaining part of the introduction.

Our next application regards the cohomological dimension of relatively hyperbolic groups in the framework of Bredon modules. Given a group G and a nonempty family

$\mathcal{F}$ of subgroups closed under conjugation and taking subgroups, the theory of (right) modules over the orbit category $\mathcal{O}_{\mathcal{F}}(G)$ was established by Bredon [4], Tom Dieck [27] and Lück [16]. In the case where $\mathcal{F}$ is the trivial family, the $\mathcal{O}_{\mathcal{F}}(G)$-modules are $\mathbb{Z}G$-modules. The notions of cohomological dimension $\mathsf{cd}_{\mathcal{F}}(G)$ and finiteness properties $FP_{n,\mathcal{F}}$ for the pair $(G, \mathcal{F})$ are defined analogously to their counterparts $\mathsf{cd}(G)$ and FP_n. The geometric dimension $\mathsf{gd}_{\mathcal{F}}(G)$ is defined as the smallest dimension of a model for $E_{\mathcal{F}}G$. A theorem of Lück and Meintrup [17, Theorem 0.1] shows that

$$\mathsf{cd}_{\mathcal{F}}(G) \leqslant \mathsf{gd}_{\mathcal{F}}(G) \leqslant \max\{3, \mathsf{cd}_{\mathcal{F}}(G)\}.$$

Together with Theorem 1.1, this yields the following. Here as before $\mathcal{F}$ is the collection of all the conjugates of $\{P_\lambda\}$, all their subgroups, and all finite subgroups of G.

Corollary 1.4. *Let G be relatively hyperbolic. Then $\mathsf{cd}_{\mathcal{F}}(G)$ is finite.*

The *homological Dehn function* $\mathsf{FV}_X(k)$ of a simply connected cell complex X measures the difficulty of filling cellular 1-cycles with 2-chains. For a finitely presented group G and X a model for EG with G-cocompact 2-skeleton, the growth rate of $\mathsf{FV}_G(k) := \mathsf{FV}_X(k)$ is a group invariant [10, Theorem 2.1]. The function $\mathsf{FV}_G(k)$ can also be defined from algebraic considerations under the weaker assumption that G is FP_2, see [12, Section 3]. Analogously, for a group G and a family of subgroups $\mathcal{F}$ with a cocompact model for $E_{\mathcal{F}}G$, there is relative homological Dehn function $\mathsf{FV}_{G,\mathcal{F}}(k)$ whose growth rate is an invariant of the pair $(G, \mathcal{F})$, see [18, Theorem 4.5].

Gersten proved that a group G is hyperbolic if and only if it is FP_2 and the growth rate of $\mathsf{FV}_G(k)$ is linear [11, Theorem 5.2]. An analogous characterisation for relatively hyperbolic groups is proved in [18, Theorem 1.11] relying on the following corollary. We remark that a converse of Corollary 1.5 requires an additional condition that $\{P_\lambda\}$ is an almost malnormal collection, see [18, Theorem 1.11(1) and Remark 1.13].

Corollary 1.5. *Let G be relatively hyperbolic. Then G is $FP_{2,\mathcal{F}}$ and $\mathsf{FV}_{G,\mathcal{F}}(k)$ has linear growth.*

Proof. The existence of a cocompact model $X = \Gamma_n^{\blacktriangle}$ for $E_{\mathcal{F}}(G)$ implies that G is $FP_{2,\mathcal{F}}$. Since X has fine and hyperbolic 1-skeleton and has finite edge G-stabilisers, it follows that $\mathsf{FV}_{G,\mathcal{F}}(k) := \mathsf{FV}_X(k)$ has linear growth by [19, Theorem 1.7]. □

Organisation. In § 2 we discuss the basic properties of the n-Rips complex $\Gamma_n^{\blacktriangle}$, and state our main results on the fixed-point sets, Theorems 2.5 and 2.6. We prove them in § 3 using a graph-theoretic notion called dismantlability. We also rely on a thin triangle Theorem 3.4 for relatively hyperbolic groups, which we prove in § 4.

2. Rips complex

2.1. Rips graph

We introduce relatively hyperbolic groups following Bowditch's approach [3]. A *circuit* in a graph is an embedded closed edge path. A graph is *fine* if for every edge e and every integer n, there are finitely many circuits of length at most n containing e.

Let G be a group, and let $\mathcal{P} = \{P_\lambda\}_{\lambda\in\Lambda}$ be a finite collection of subgroups of G. A $(G, \mathcal{P})$*-graph* is a fine δ-hyperbolic connected graph with a G-action with finite edge stabilisers, finitely many orbits of edges, and such that $\mathcal{P}$ is a set of representatives of distinct conjugacy classes of vertex stabilisers such that each infinite stabiliser is represented.

Suppose G is finitely generated, and let S be a finite generating set. Let Γ denote the Cayley graph of G with respect to S. Let V denote the set of vertices of Γ, which is in correspondence with G. A *peripheral left coset* is a subset of G of the form gP_λ. Let W denote the set of peripheral left cosets, also called *cone vertices*. The *coned-off Cayley graph* $\hat{\Gamma}$ is the connected graph obtained from Γ by enlarging the vertex set by W and the edge set by the pairs $(v, w) \in V \times W$, where the group element v lies in the peripheral left coset w.

We say that G is *hyperbolic relative to* $\mathcal{P}$ if $\hat{\Gamma}$ is fine and δ-hyperbolic, which means that it is a $(G, \mathcal{P})$*-graph*. This is equivalent to the existence of a $(G, \mathcal{P})$-graph. Indeed, if there is a $(G, \mathcal{P})$-graph, a construction of Dahmani [7, P. 82, proof of Lemma 4] (relying on an argument of Bowditch [3, Lemma 4.5]) shows that $\hat{\Gamma}$ is a G-equivariant subgraph of a $(G, \mathcal{P})$-graph Δ, and therefore $\hat{\Gamma}$ is fine and quasi-isometric to Δ. In particular, the definition of relative hyperbolicity is independent of S. From here on, we assume that G is hyperbolic relative to $\mathcal{P}$.

Extend the word metric (which we also call S*-distance*) $|\cdot,\cdot|_S$ from V to $V \cup W$ as follows: the distance between cone vertices is the distance in Γ between the corresponding peripheral left cosets, and the distance between a cone vertex and an element of G is the distance between the corresponding peripheral left coset and the element. Note that $|\cdot,\cdot|_S$ is not a metric on $V \cup W$. It is only for $v \in V$ that we have the triangle inequality $|a, b|_S \leqslant |a, v|_S + |v, b|_S$ for any $a, b \in V \cup W$.

Definition 2.1. Let $n \geqslant 1$. The n*-Rips graph* Γ_n is the graph with vertex set $V \cup W$ and edges between $u, u' \in V \cup W$ whenever $|u, u'|_S \leqslant n$.

Lemma 2.2. *The* n*-Rips graph* Γ_n *is a* $(G, \mathcal{P})$*-graph.*

Proof. Note that the graphs $\hat{\Gamma}$ and Γ_n have the same vertex set. In particular, since $\hat{\Gamma}$ is connected and contained in Γ_n, it follows that Γ_n is connected.

Since Γ is locally finite and there are finitely many G-orbits of edges in Γ, it follows that there are finitely many G-orbits of edge paths of length n in Γ. Since $\mathcal{P}$ is finite, there are finitely many G-orbits of edges in Γ_n.

Since G acts on $\hat{\Gamma}$ with finite edge stabilisers and $\hat{\Gamma}$ is fine, it follows that for distinct vertices in $V \cup W$, the intersection of their G-stabilisers is finite [20, Lemma 2.2]. Thus the pointwise G-stabilisers of edges in Γ_n are finite, and hence the same holds for the setwise G-stabilisers of edges.

It remains to show that Γ_n is fine and δ-hyperbolic. Since there are finitely many G-orbits of edges in Γ_n, the graph Γ_n is obtained from $\hat{\Gamma}$ by attaching a finite number of G-orbits of edges. This process preserves fineness by a result of Bowditch [3, Lemma 2.3]; see also [20, Lemma 2.9]. This process also preserves the quasi-isometry type [20, Lemma 2.7], thus Γ_n is δ-hyperbolic. □

For a graph Σ, let $\Sigma^{\blacktriangle}$ be the simplicial complex obtained from Σ by spanning simplices on all cliques. We call $\Gamma_n^{\blacktriangle}$ the *n-Rips complex*.

Corollary 2.3. *The G-stabiliser of a barycentre of a simplex Δ in $\Gamma_n^{\blacktriangle}$ that is not a vertex is finite.*

Proof. Let F be the stabiliser of the barycentre of Δ. Then F contains the pointwise stabiliser of Δ as a finite index subgroup. The latter is contained in the pointwise stabiliser of an edge of Δ, which is finite by Lemma 2.2. Therefore, F is finite. □

Corollary 2.4. *The G-action on $\Gamma_n^{\blacktriangle}$ is cocompact.*

Proof. By Lemma 2.2, the n-Rips graph Γ_n is fine. Hence every edge e in Γ_n is contained in finitely many circuits of length 3. Thus e is contained in finitely many simplices of $\Gamma_n^{\blacktriangle}$. By Lemma 2.2, there are finitely many G-orbits of edges in Γ_n. It follows that there are finitely many G-orbits of simplices in $\Gamma_n^{\blacktriangle}$. □

2.2. Fixed-point sets

The first step of the proof of Theorem 1.1 is the following fixed-point theorem.

Theorem 2.5. *For sufficiently large n, each finite subgroup $F \leqslant G$ fixes a clique of Γ_n.*

The proof will be given in § 3.2. As a consequence we obtain the following.

Proof of Corollary 1.3. By Corollary 2.4, there are finitely many G-orbits of simplices in $\Gamma_n^{\blacktriangle}$. From each orbit of simplices that are not vertices pick a simplex Δ_i, and let F_i be the stabiliser of its barycentre. By Corollary 2.3, the group F_i is finite.

Choose n satisfying Theorem 2.5. Then any finite subgroup F of G fixes the barycentre of a simplex Δ in $\Gamma_n^{\blacktriangle}$. If Δ is a vertex, then F is contained in a conjugate of some P_λ. Otherwise, F is contained in a conjugate of some F_i. □

It was observed by the referee that if one proved in advance Corollary 1.3, one could deduce from it Theorem 2.5 (without control on n).

The second step of the proof of Theorem 1.1 is the following, whose proof we also postpone, to § 3.4.

Theorem 2.6. *For sufficiently large n, for any subgroup $H \leqslant G$, its fixed-point set in $\Gamma_n^{\blacktriangle}$ is either empty or contractible.*

We conclude with the following.

Proof of Theorem 1.1. The point stabilisers of $\Gamma_n^{\blacktriangle}$ belong to $\mathcal{F}$ by Corollary 2.3. For every $H \in \mathcal{F}$ its fixed-point set $(\Gamma_n^{\blacktriangle})^H$ is nonempty by Theorem 2.5. Consequently, $(\Gamma_n^{\blacktriangle})^H$ is contractible by Theorem 2.6. □

3. Dismantlability

The goal of this section is to prove Theorems 2.5 and 2.6, relying on the following.

3.1. Thin triangle theorem

We state an essential technical result of the article, a thin triangles result for relatively hyperbolic groups. We keep the notation from the Introduction, where Γ is the Cayley graph of G with respect to S, etc. By *geodesics* we mean geodesic edge paths.

Definition 3.1. [14, Definition 8.9] Let $p = (p_j)_{j=0}^{\ell}$ be a geodesic in Γ, and let ϵ, R be positive integers. A vertex p_i of p is (ϵ, R)-*deep* in the peripheral left coset $w \in W$ if $R \leqslant i \leqslant \ell - R$ and $|p_j, w|_S \leqslant \epsilon$ for all $|j - i| \leqslant R$. If there is no such $w \in W$, then p_i is an (ϵ, R)-*transition vertex* of p.

Lemma 3.2. *[14, Lemma 8.10] For each ϵ there is a constant R such that for any geodesic p in Γ, a vertex of p cannot be (ϵ, R)-deep in two distinct peripheral left cosets.*

Definition 3.3. For $a, b \in V \cup W$, a *geodesic from a to b in Γ* is a geodesic in Γ of length $|a, b|_S$ such that its initial vertex equals a if $a \in V$, or is an element of a if $a \in W$, and its terminal vertex equals b if $b \in V$, or is an element of b if $b \in W$.

Throughout the article we adopt the following convention. For an edge path $(p_j)_{j=0}^{\ell}$, if $i > \ell$, then p_i denotes p_ℓ.

Theorem 3.4 (Thin triangle theorem). *There are positive integers ϵ, R and D, satisfying Lemma 3.2, such that the following holds. Let $a, b, c \in V \cup W$ with $a \neq b$, and let p^{ab}, p^{bc}, p^{ac} be geodesics in Γ from a to b, from b to c, and from a to c, respectively. Let $\ell = |a, b|_S$ and let $0 \leqslant i \leqslant \ell$.*

If p_i^{ab} is an (ϵ, R)-deep vertex of p^{ab} in the peripheral left coset w, then let $z = w$, otherwise let $z = p_i^{ab}$.

Then $|z, p_i^{ac}|_S \leqslant D$ or $|z, p_{\ell-i}^{bc}|_S \leqslant D$.

Note that the condition $a \neq b$ is necessary, since for $a = b \in W$ we could take for p^{ab} any element of a, leading to counterexamples.

While Theorem 3.4 seems similar to various other triangle theorems in relatively hyperbolic groups, its proof is surprisingly involved, given that we rely on these previous results. We postpone the proof till § 4. In the remaining part of the section, ϵ, R, D are the integers guaranteed by Theorem 3.4. We can and will assume that $D \geqslant \epsilon$.

3.2. Quasi-centres

In this subsection we show how to deduce Theorem 2.5 from thin triangle Theorem 3.4. This is done analogously as for hyperbolic groups, using quasi-centres (see [5, Lemma III.Γ.3.3]).

Definition 3.5. Let U be a finite subset of $V \cup W$. The *radius* $\rho(U)$ of U is the smallest ρ such that there exists $z \in V \cup W$ with $|z, u|_S \leqslant \rho$ for all $u \in U$. The *quasi-centre* of U consists of $z \in V \cup W$ satisfying $|z, u|_S \leqslant \rho(U)$ for all $u \in U$.

Lemma 3.6. *Let U be a finite subset of $V \cup W$ that is not a single vertex of W. Then for any two elements a, b of the quasi-centre of U, we have $|a, b|_S \leqslant 4D$.*

Proof. Assume first $\rho(U) \leqslant D$. If U is a single vertex $v \in V$, then $|a, b|_S \leqslant |a, v|_S + |v, b|_S \leqslant 2D$ and we are done. If there are $u \neq u' \in U$, then let v be the first vertex on a geodesic from u to u' in Γ. Since the vertex v is not (ϵ, R)-deep, by Theorem 3.4 applied to u, u', a (respectively u, u', b), we obtain a vertex v_a (respectively v_b) on a geodesic in Γ from a (respectively b) to u or u' satisfying $|v_a, v|_S \leqslant D$ (respectively $|v_b, v|_S \leqslant D$). Consequently $|a, b|_S \leqslant |a, v_a|_S + |v_a, v|_S + |v, v_b|_S + |v_b, b|_S \leqslant 2\rho(U) + 2D \leqslant 4D$, as desired.

Henceforth, we assume $\rho(U) \geqslant D + 1$. Let $\ell = |a, b|_S$. If $\ell > 4D$, then choose any $2D \leqslant i \leqslant \ell - 2D$, and any geodesic p^{ab} from a to b in Γ. If p_i^{ab} is an (ϵ, R)-deep vertex of p^{ab} in the peripheral left coset w, then let $z = w$, otherwise let $z = p_i^{ab}$. We claim that for any $c \in U$ we have $|z, c|_S \leqslant \rho(U) - 1$, contradicting the definition of $\rho(U)$, and implying $\ell \leqslant 4D$.

Indeed, we apply Theorem 3.4 to a, b, c, and any geodesics p^{bc}, p^{ac}. Without loss of generality assume that we have $|z, p_i^{ac}|_S \leqslant D$. Note that if $i > |a, c|_S$, then p_i^{ac} lies in (or is equal to) c, and consequently $|z, c|_S \leqslant |z, p_i^{ac}|_S \leqslant D \leqslant \rho(U) - 1$, as desired. If $i \leqslant |a, c|_S$, then $|p_i^{ac}, a|_S = i \geqslant 2D$, and hence

$$|z, c|_S \leqslant |z, p_i^{ac}|_S + |p_i^{ac}, c|_S \leqslant D + (|c, a|_S - |p_i^{ac}, a|_S) \leqslant D + (\rho(U) - 2D),$$

as desired. □

Proof of Theorem 2.5. Let $n \geqslant 4D$. Consider a finite orbit $U \subset V \cup W$ of F. If U is a single vertex, then there is nothing to prove. Otherwise, by Lemma 3.6 the quasi-centre of U forms a fixed clique in Γ_n. □

3.3. Convexity

Definition 3.7. Let μ be a positive integer. A subset $U \subset V \cup W$ is *μ-convex* with respect to $u \in U$ if for any geodesic $(p_j)_{j=0}^{\ell}$ in Γ from u to $u' \in U$, for any $j \leqslant \ell - \mu$, we have

(i) $p_j \in U$; and

(ii) for each $w \in W$ with $|w, p_j|_S \leqslant \epsilon$ we have $w \in U$.

Definition 3.8. Let r be a positive integer and let $U \subset V \cup W$ be a finite subset. The *r-hull* U_r of U is the union of

(i) all the vertices $v \in V$ with $|v, u|_S \leqslant r$ for each $u \in U$; and

(ii) all the cone vertices $w \in W$ with $|w, u|_S \leqslant r + \epsilon$ for each $u \in U$.

Lemma 3.9. *If $|U| \geqslant 2$, then each U_r is finite.*

Proof. Choose $u \neq u' \in U$. Assume without loss of generality $r \geqslant |u, u'|_S$. Each vertex of U_r distinct from u and u' forms with u and u' a circuit in $\Gamma_{r+\epsilon}$ of length 3. There are only finitely many such circuits, since $\Gamma_{r+\epsilon}$ is fine by Lemma 2.2. □

Lemma 3.10. *Let $U \subset V \cup W$ be a finite subset with $|u, u'|_S \leqslant \mu$ for all $u, u' \in U$. Then each U_r, with $r \geqslant \mu$, is $(\mu + 2D)$-convex with respect to all $b \in U$.*

Proof. Let $b \in U$, and $a \in U_r$. Let $(p_j^{ab})_{j=0}^{\ell}$ be a geodesic from a to b in Γ, and let $\mu + 2D \leqslant j \leqslant \ell$. By Definition 3.8, we have $\ell \leqslant r + \epsilon$. To prove the lemma, it suffices to show that $p_j^{ab} \in U_r$.

Consider any $c \in U$ and apply Theorem 3.4 with $i = \ell$. In that case p_i^{ab} is not (ϵ, R)-deep and thus $z = p_i^{ab}$. Consequently, we have $|p_\ell^{ab}, p_\ell^{ac}|_S \leqslant D$ or $|p_\ell^{ab}, p_0^{bc}|_S \leqslant D$.

In the second case, using $\epsilon \leqslant D$, we have

$$|p_j^{ab}, c|_S \leqslant |p_j^{ab}, p_\ell^{ab}|_S + |p_\ell^{ab}, p_0^{bc}|_S + |p_0^{bc}, c|_S \leqslant \big((r+\epsilon) - (\mu + 2D)\big) + D + \mu \leqslant r,$$

as desired.

In the first case, if $\ell > |a, c|_S$, then p_ℓ^{ac} lies in (or is equal to) c and hence

$$|p_j^{ab}, c|_S \leqslant |p_j^{ab}, p_\ell^{ab}|_S + |p_\ell^{ab}, p_\ell^{ac}|_S \leqslant \big((r+\epsilon) - (\mu + 2D)\big) + D \leqslant r.$$

If in the first case $\ell \leqslant |a, c|_S$, then

$$|p_j^{ab}, c|_S \leqslant |p_j^{ab}, p_\ell^{ab}|_S + |p_\ell^{ab}, p_\ell^{ac}|_S + |p_\ell^{ac}, c|_S \leqslant \big(\ell - (\mu + 2D)\big) + D + (r + \epsilon - \ell) \leqslant r.$$

$\square$

3.4. Contractibility

In this subsection we prove Theorem 2.6. To do that, we use dismantlability.

Definition 3.11. We say that a vertex a of a graph is *dominated* by an adjacent vertex $z \neq a$, if all the vertices adjacent to a are also adjacent to z.

A finite graph is *dismantlable* if its vertices can be ordered into a sequence $a_1, \ldots, a_k$ so that for each $i < k$ the vertex a_i is dominated in the subgraph induced on $\{a_i, \ldots, a_k\}$.

Polat proved that the automorphism group of a dismantlable graph fixes a clique [25, Theorem A] (for the proof see also [13, Theorem 2.4]). We use the following strengthening of that result.

Theorem 3.12 ([2, Theorem 6.5], [13, Theorem 1.2]). *Let Γ be a finite dismantlable graph. Then for any subgroup $H \leqslant \mathrm{Aut}(\Gamma)$, the fixed-point set $(\Gamma^\blacktriangle)^H$ is contractible.*

Our key result is the dismantlability in the n-Rips graph.

Lemma 3.13. *Let $U \subset V \cup W$ be a finite subset that is $6D$-convex with respect to some $b \in U$. Then for $n \geqslant 7D$, the subgraph of Γ_n induced on U is dismantlable.*

Proof. We order the vertices of U according to $|\cdot, b|_S$, starting from $a \in U$ with maximal $|a, b|_S$, and ending with b.

We first claim that unless $U = \{b\}$, the set $U - \{a\}$ is still $6D$-convex with respect to b. Indeed, let $u \in U - \{a\}$ and let $(p_j)_{j=0}^{\ell}$ be a geodesic from b to u in Γ. Let $j \leqslant \ell - 6D$. Then $|p_j, b|_S \leqslant \ell - 6D < |a, b|_S$, so $p_j \neq a$ and hence $p_j \in U - \{a\}$ since U was

$6D$-convex. Similarly, if $w \in W$ and $|w, p_j|_S \leqslant \epsilon$, then $|w, b| \leqslant \epsilon + (\ell - 6D) < |a, b|_S$, so $w \neq a$ and hence $w \in U - \{a\}$. This justifies the claim.

It remains to prove that a is dominated in the subgraph of Γ_n induced on U by some vertex z. Let $(p_j^{ab})_{j=0}^{\ell}$ be a geodesic from a to b in Γ. If $\ell \leqslant 7D$, then we can take $z = b$ and the proof is finished. We henceforth suppose $\ell > 7D$. If p_{6D}^{ab} is an (ϵ, R)-deep vertex of p^{ab} in the peripheral left coset w, then let $z = w$, otherwise let $z = p_{6D}^{ab}$. Note that by definition of convexity, we have $z \in U$. We show that z dominates a.

Let $c \in U$ be adjacent to a in Γ_n, which means $|a, c|_S \leqslant n$. We apply Theorem 3.4 to $a, b, c, i = 6D$ and any geodesics (p_j^{bc}), (p_j^{ac}). Consider first the case where $|z, p_{6D}^{ac}|_S \leqslant D$. If $6D \leqslant |a, c|_S$, then

$$|c, z|_S \leqslant |c, p_{6D}^{ac}|_S + |p_{6D}^{ac}, z|_S \leqslant (n - 6D) + D < n,$$

so c is adjacent to z in Γ_n, as desired. If $|a, c|_S < 6D$, then p_{6D}^{ac} lies in (or is equal to) c and hence $|c, z|_S \leqslant D < n$ as well.

Now consider the case where $|z, p_{\ell-6D}^{bc}|_S \leqslant D$. Since a was chosen to have maximal $|a, b|_S$, we have $|a, b|_S \geqslant |c, b|_S$, and hence $|c, p_{\ell-6D}^{bc}|_S \leqslant 6D$. Consequently,

$$|c, z|_S \leqslant |c, p_{\ell-6D}^{bc}| + |p_{\ell-6D}^{bc}, z|_S + \leqslant 6D + D \leqslant n. \qquad \square$$

We are now ready to prove the contractibility of the fixed-point sets.

Proof of Theorem 2.6. Let $n \geqslant 7D$. Suppose that the fixed-point set $\mathrm{Fix} = (\Gamma_n^{\blacktriangle})^H$ is nonempty.

Step 1. *The fixed-point set* $\mathrm{Fix}' = (\Gamma_{4D}^{\blacktriangle})^H$ *is nonempty.*

Let U be the vertex set of a simplex in $\Gamma_n^{\blacktriangle}$ containing a point of Fix in its interior. Note that U is H-invariant. If U is a single vertex u, then $u \in \mathrm{Fix}'$ and we are done. Otherwise, by Lemma 3.6, the quasi-centre of U spans a simplex in $\Gamma_{4D}^{\blacktriangle}$. Consequently, its barycentre lies in Fix'.

Step 2. *If* Fix' *contains at least* 2 *points, then it contains a point outside* W.

Otherwise, choose $w \neq w' \in \mathrm{Fix}'$ with minimal $|w, w'|_S$. If $|w, w'|_S \leqslant 4D$, then the barycentre of the edge ww' lies in Fix', which is a contradiction. If $|w, w'|_S > 4D$, then $\rho(\{w, w'\}) \leqslant \lceil \frac{|w, w'|_S}{2} \rceil < |w, w'|_S$. Let U' be the quasi-centre of $\{w, w'\}$. By Lemma 3.6, we have that U' spans a simplex in $\Gamma_{4D}^{\blacktriangle}$, with barycentre in Fix'. If U' is not a single vertex, this is a contradiction. Otherwise, if U' is a single vertex $w'' \in W$, then $|w, w''|_S \leqslant \rho(\{w, w'\}) < |w, w'|_S$, which contradicts our choice of w, w'.

Step 3. Fix *is contractible.*

By Step 1, the set Fix' is nonempty. If Fix' consists of only one point, then so does Fix, and there is nothing to prove. Otherwise, let Δ be the simplex in $\Gamma_{4D}^{\blacktriangle}$ containing in its interior the point of Fix' guaranteed by Step 2. Note that Δ is also a simplex in $\Gamma_n^{\blacktriangle}$ with barycentre in Fix. Since Δ is not a vertex of W, by Lemma 3.9 all its r-hulls Δ_r are finite. By Lemma 3.10, each Δ_r with $r \geqslant 4D$ is $6D$-convex. Thus by Lemma 3.13, the

1-skeleton of the span $\Delta_r^{\blacktriangle}$ of Δ_r in $\Gamma_n^{\blacktriangle}$ is dismantlable. Hence by Theorem 3.12, the set $\mathrm{Fix} \cap \Delta_r^{\blacktriangle}$ is contractible. Note that Δ_r exhaust entire $V \cup W$. Consequently, entire Fix is contractible, as desired. □

3.5. Edge dismantlability

Mineyev and Yaman introduced for a relatively hyperbolic group a complex $X(\mathcal{G}, \mu)$, which they proved to be contractible [22, Theorem 19]. However, analysing their proof, they exhaust the 1-skeleton of $X(\mathcal{G}, \mu)$ by finite graphs that are not dismantlable but satisfy a slightly weaker relation, which we can call *edge-dismantlability.*

An edge (a, b) of a graph is *dominated* by a vertex z adjacent to both a and b, if all the other vertices adjacent to both a and b are also adjacent to z. A finite graph Γ is *edge-dismantlable* if there is a sequence of subgraphs $\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_k$, where for each $i < k$ the graph Γ_{i+1} is obtained from Γ_i by removing a dominated edge or a dominated vertex with all its adjacent edges, and where Γ_k is a single vertex.

In dimension 2 the notion of edge dismantlability coincides with collapsibility, so by [26] the automorphism group of Γ fixes a clique, similarly as for dismantlable graphs.

Question 3.14. Does the automorphism group of an arbitrary edge-dismantlable graph Γ fix a clique? For arbitrary $H \leqslant \mathrm{Aut}(\Gamma)$, is the fixed-point set $(\Gamma^{\blacktriangle})^H$ contractible?

4. Proof of the thin triangle theorem

4.1. Preliminaries

Given an edge path $p = (p_i)_{i=0}^{\ell}$, we use the following notation. The length ℓ of p is denoted by $l(p)$, the initial vertex p_0 of p is denoted by p_-, and its terminal vertex p_ℓ is denoted by p_+. For integers $0 \leqslant j \leqslant k \leqslant l(p)$, we denote by $p[j, k]$ the subpath $(p_i)_{i=j}^{k}$, and by $p[k, j]$ we denote the inverse path.

The group G is hyperbolic relative to $\mathcal{P}$ in the sense of Osin [23, Definition 1.6, Theorem 1.5]. We first recall two results from [23, 24]. Consider the alphabet $\mathscr{P} = S \sqcup \bigsqcup_\lambda P_\lambda$. Every word in this alphabet represents an element of G, and note that distinct letters might represent the same element. Let $\bar{\Gamma}$ denote the Cayley graph of G with respect to $\mathscr{P}$.

Theorem 4.1 [23, Theorem 3.26]. *There is a constant $K > 0$ with the following property. Consider a triangle whose sides p, q, r are geodesics in $\bar{\Gamma}$. For any vertex v of p, there exists a vertex u of $q \cup r$ such that $|u, v|_S \leqslant K$.*

Definition 4.2 [23, p. 17]. Let q be an edge path in $\bar{\Gamma}$. Subpaths of q with at least one edge are called *nontrivial.* A gP_λ*-component* of q is a maximal nontrivial subpath r such that the label of r is a word in $P_\lambda - \{1\}$ and a vertex of r (and hence all its vertices) belong to gP_λ. We refer to gP_λ-components as $\mathcal{P}$*-components* if there is no need to specify gP_λ. A gP_λ-component of q is *isolated* if q has no other gP_λ-components. Note that gP_λ-components of geodesics in $\bar{\Gamma}$ are single edges and we call them gP_λ*-edges.*

Theorem 4.3 [24, Proposition 3.2]. *There is a constant $K > 0$ satisfying the following condition. Let Δ be an n-gon in $\bar{\Gamma}$, which means that Δ is a closed path that is a concatenation of n edge paths $\Delta = q^1 q^2 \dots q^n$. Suppose that $I \subset \{1, \dots, n\}$ is such that*

(1) *for $i \in I$ the side q^i is an isolated $\mathcal{P}$-component of Δ; and*
(2) *for $i \notin I$ the side q^i is a geodesic in $\bar{\Gamma}$.*

Then $\sum_{i \in I} |q_-^i, q_+^i|_S \leqslant Kn$.

We now recall a result of Hruska [14] and another of Druţu–Sapir [9] on the relation between the geometry of $\bar{\Gamma}$ and the Cayley graph Γ of G with respect to S.

Definition 4.4. Let p be a geodesic in Γ, and let ϵ, R be positive integers. Let $w \in W$ be a peripheral left coset. An (ϵ, R)*-segment in w* of p is a maximal subpath such that all its vertices are (ϵ, R)-deep in w. Note that an (ϵ, R)-segment could consist of a single vertex.

Definition 4.5. Edge paths p and q in $\bar{\Gamma}$ are K*-similar* if $|p_-, q_-|_S \leqslant K$ and $|p_+, q_+|_S \leqslant K$.

Proposition 4.6 (see [14, Proposition 8.13]). *There are constants ϵ, R satisfying Lemma 3.2, and a constant K such that the following holds. Let p be a geodesic in Γ and let $\bar{p}$ be a geodesic in $\bar{\Gamma}$ with the same endpoints as p.*

(i) *The set of vertices of $\bar{p}$ is at Hausdorff distance at most K from the set of (ϵ, R)-transition vertices of p, in the metric $|\cdot, \cdot|_S$.*
(ii) *If $p[j, k]$ is an (ϵ, R)-segment in w of p, then there are vertices $\bar{p}_m$ and $\bar{p}_n$ of $\bar{p}$ such that $|p_j, \bar{p}_m|_S \leqslant K$ and $|p_k, \bar{p}_n|_S \leqslant K$.*
(iii) *For any subpath $\bar{p}[m, n]$ of $\bar{p}$ with $m \leqslant n$ there is a K-similar subpath $p[j, k]$ of p with $j \leqslant k$.*

Proof. The existence of ϵ, R, and K satisfying Lemma 3.2 and (i) is [14, Proposition 8.13], except that Hruska considers the set of transition points instead of vertices. However, after increasing his R by 1, we obtain the current statement, and moreover each pair of distinct (ϵ, R)-segments is separated by a transition vertex. Consequently, by increasing K by 1, we obtain (ii).

For the proof of (iii), increase K so that it satisfies Theorem 4.1. We show that $3K$ satisfies statement (iii). By (i), there is a vertex p_k such that $|\bar{p}_n, p_k|_S \leqslant K$. Let $q, \bar{q}$ be geodesics in $\Gamma, \bar{\Gamma}$ from $\bar{p}_n$ to p_k. Let $\bar{r}$ be a geodesic in $\bar{\Gamma}$ from p_0 to p_k. Consider the geodesic triangle in $\bar{\Gamma}$ with sides $\bar{p}[0, n]$, $\bar{q}$, and $\bar{r}$. By Theorem 4.1, there is a vertex v of $\bar{q} \cup \bar{r}$ such that $|\bar{p}_m, v|_S \leqslant K$.

Suppose first that v lies in $\bar{r}$. By (i) there is a vertex p_j of $p[0, k]$ such that $|p_j, v|_S \leqslant K$. It follows that $|p_j, \bar{p}_m|_S \leqslant 2K$. Now suppose that v lies in $\bar{q}$. By (i) the vertex v is at S-distance $\leqslant K$ from a vertex of q. Since $l(q) \leqslant K$, it follows that $|p_k, \bar{p}_m|_S \leqslant 3K$, and we can assign $j = k$. □

Lemma 4.7 (Quasiconvexity, [9, Lemma 4.15]). *There is $K > 0$ such that the following holds. Let $w \in W$ be a peripheral left coset, let A be a positive integer, and let p be a* *geodesic in* Γ *with* $|p_-, w|_S \leqslant A$ *and* $|p_+, w|_S \leqslant A$*. Then any vertex* p_i *of* p *satisfies* $|p_i, w|_S \leqslant KA$*.*

Assumption 4.8. From here on, the constants (ϵ, R, K) are assumed to satisfy the statement of Proposition 4.6. By increasing K, we also assume that K satisfies the conclusions of Theorems 4.1 and 4.3, the quasiconvexity Lemma 4.7, and $K \geqslant \max\{\epsilon, R\}$.

We conclude with the following application of Theorem 4.3.

Lemma 4.9. *Let* p *be a geodesic in* Γ*, and let* $\bar{p}$ *be a geodesic in* $\bar{\Gamma}$ *with the same endpoints as* p*. Let* $p[j,k]$ *be an* (ϵ, R)*-segment of a peripheral left coset* $w \in W$*. If* $k - j > 8K^2$*, then* $\bar{p}$ *contains a* w*-edge which is* $5K^2$*-similar to* $p[j,k]$*.*

Proof. There are vertices r_- and r_+ of w such that $|p_j, r_-|_S \leqslant \epsilon \leqslant K$ and $|p_k, r_+|_S \leqslant \epsilon \leqslant K$. Let r be a w-edge from r_- to r_+. By Proposition 4.6(ii), there are vertices $\bar{p}_m, \bar{p}_n$ at S-distance at most K from p_j, p_k, respectively. Let $[r_-, \bar{p}_m]$ and $[\bar{p}_n, r_-]$ be geodesics in Γ between the corresponding vertices; note that the labels of these paths are words in the alphabet S, and they both have length at most $2K$.

Suppose for contradiction that $\bar{p}$ does not contain a w-edge. Consider the closed path $[r_-, \bar{p}_m]\bar{p}[m,n][\bar{p}_n, r_+]r$, viewed as a polygon Δ obtained by subdividing $[r_-, \bar{p}_m]$ and $[\bar{p}_n, r_+]$ into edges. Since $\bar{p}[m,n]$ is a geodesic in $\bar{\Gamma}$, the number of sides of Δ is at most $2 + |r_-, \bar{p}_m|_S + |r_+, \bar{p}_n|_S \leqslant 6K$. We have that r is an isolated w-component of Δ. Then Theorem 4.3 implies that $|r_-, r_+|_S \leqslant 6K^2$. It follows that $|p_j, p_k|_S \leqslant 8K^2$. This is a contradiction, hence $\bar{p}[m,n]$ contains a w-edge t.

Now we prove that $|t_-, p_j|_S \leqslant 5K^2$. Let $[\bar{p}_m, t_-]$ be the subpath of $\bar{p}[m,n]$ from $\bar{p}_m$ to t_-, and note that this is a geodesic in $\bar{\Gamma}$ without w-components. Let $[t_-, r_-]$ be a w-edge from t_- to r_-. Consider the polygon $[\bar{p}_m, t_-][t_-, r_-][r_-, \bar{p}_m]$, where the path $[r_-, \bar{p}_m]$ is subdivided into at most $2K$ edges. Observe that $[t_-, r_-]$ is an isolated w-component. Theorem 4.3 implies that $|t_-, p_j|_S \leqslant |t_-, r_-|_S + |r_-, p_j|_S \leqslant K(2K+2) + K \leqslant 5K^2$.

Analogously one proves that $|t_+, p_k|_S \leqslant 5K^2$. □

4.2. Proof

We are now ready to start the proof of Theorem 3.4. Let $D = 53K^2$.

Let $a, b, c \in V \cup W$ with $a \neq b$, and let p^{ab}, p^{bc}, p^{ac} be geodesics in Γ from a to b, from b to c, and from a to c, respectively. Let $\ell = |a, b|_S$ and let $0 \leqslant i \leqslant \ell$. If p^{ab}_i is an (ϵ, R)-deep vertex of p^{ab} in the peripheral left coset w then let $z = w$, otherwise let $z = p^{ab}_i$.

We define the following paths illustrated in Figure 1. Let $\bar{p}^{ab}, \bar{p}^{bc}, \bar{p}^{ac}$ be geodesics in $\bar{\Gamma}$ with the same endpoints as p^{ab}, p^{bc}, p^{ac}, respectively. If $a \in W$ and $\bar{p}^{ab}$ starts with an a-edge, then we call this edge s^a; otherwise let s^a be the trivial path. We define s^b analogously. Then $\bar{p}^{ab}$ is a concatenation

$$\bar{p}^{ab} = s^a q^{ab} s^b.$$

Similarly, the paths $\bar{p}^{ac}, \bar{p}^{bc}$ can be expressed as concatenations

$$\bar{p}^{ac} = u^a q^{ac} u^c, \qquad \bar{p}^{bc} = t^b q^{bc} t^c.$$

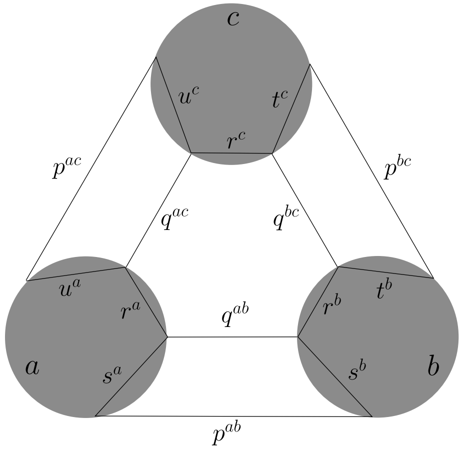

Figure 1. Paths in the proof of the thin triangle theorem.

Let r^a be a path in $\bar{\Gamma}$ from u^a_+ to s^a_+ that is a single a-edge if $u^a_+ \neq s^a_+$, or the trivial path otherwise. We define r^b, r^c analogously. Let Π be the geodesic hexagon in $\bar{\Gamma}$ given by

$$\Pi = r^a q^{ab} r^b q^{bc} r^c q^{ca}.$$

Step 1. *Paths p^{ab} and q^{ab} are $2K$-similar. The same is true for the pair p^{ac} and q^{ac}, and the pair p^{bc} and q^{bc}.*

Proof. By Proposition 4.6(i), there is a vertex p^{ab}_j such that $|s^a_+, p^{ab}_j|_S \leqslant K$. Since p^{ab} is a geodesic from a to b in Γ, it follows that $j \leqslant K$, and consequently $|s^a_-, s^a_+|_S \leqslant 2K$. The remaining assertions are proved analogously. □

In view of Proposition 4.6(i), there is a vertex of $\bar{p}^{ab}$ at S-distance $\leqslant K$ from p^{ab}_i. While this vertex might be s^a_- or s^b_-, Step 1 guarantees that there is a vertex q^{ab}_h at S-distance $\leqslant 3K$ from p^{ab}_i.

The following step should be considered as the bigon case of Theorem 3.4.

Step 2. *If q^{ac} contains a vertex in b, in the case where $b \in W$, or equal to b, in the case where $b \in V$, then $|z, p^{ac}_i|_S < D$.*

Similarly, if q^{bc} contains a vertex in a, in the case where $a \in W$, or equal to a, in the case where $a \in V$, then $|z, p^{bc}_{\ell-i}|_S < D$.

Note that here we keep the convention that for $i > l(p^{ac})$ the vertex p^{ac}_i denotes p^{ac}_+, and similarly if $\ell - i > l(p^{bc})$, then the vertex $p^{bc}_{\ell-i}$ denotes p^{bc}_+.

Proof. By symmetry, it suffices to prove the first assertion. We focus on the case $b \in W$, the case $b \in V$ follows by considering x below as the trivial path.

Let q^{ac}_n be the first vertex of q^{ac} in b. Let x be a b-edge joining q^{ab}_+ to q^{ac}_n. The edges r^a and x are isolated $\mathcal{P}$-components of the 4-gon $r^a q^{ab} x q^{ac}[n, 0]$. By Theorem 4.3, we have $|r^a_-, r^a_+|_S, |x_-, x_+|_S \leqslant 4K$. Consequently by Step 1 we have $|p^{ab}_-, p^{ac}_-|_S \leqslant 8K$.

First consider the case, where p_i^{ab} is an (ϵ, R)-transition vertex of p^{ab}. Let q_h^{ab} be the vertex defined after Step 1. Subdividing the 4-gon $r^a q^{ab} x q^{ac}[n, 0]$ into two geodesic triangles in $\bar{\Gamma}$, and applying twice Theorem 4.1, gives h^* such that $|q_h^{ab}, q_{h^*}^{ac}|_S \leqslant 6K$. By Proposition 4.6(i), there is i^* such that $|q_{h^*}^{ac}, p_{i^*}^{ac}|_S \leqslant K$. It follows that $|p_i^{ab}, p_{i^*}^{ac}|_S \leqslant 3K + 6K + K = 10K$ and hence $|i - i^*| \leqslant 8K + 10K = 18K$. Therefore, $|p_i^{ab}, p_i^{ac}|_S \leqslant 10K + 18K$.

Now consider the case, where p_i^{ab} is an (ϵ, R)-deep vertex of p^{ab} in the peripheral left coset w. Then p_i^{ab} lies in an (ϵ, R)-segment $p^{ab}[j, k]$ of p^{ab} in w. Thus $\max\{|p_j^{ab}, w|_S, |p_k^{ab}, w|_S\} \leqslant \epsilon \leqslant K$. By Proposition 4.6(ii) and Step 1, there is a vertex q_m^{ab} such that $|p_j^{ab}, q_m^{ab}|_S \leqslant 3K$. As in the previous case, we obtain j^* such that $|p_j^{ab}, p_{j^*}^{ac}|_S \leqslant 10K$. Analogously, there is k^* such that $|p_k^{ab}, p_{k^*}^{ac}|_S \leqslant 10K$. In particular, we have $\max\{|p_{j^*}^{ac}, w|_S, |p_{k^*}^{ac}, w|_S\} \leqslant 11K$. By quasiconvexity of w, Lemma 4.7, every vertex of $p^{ac}[j^*, k^*]$ is at S-distance $\leqslant 11K^2$ from w. Moreover, we have $|j - j^*| \leqslant 8K + 10K$, and analogously, $|k - k^*| \leqslant 18K$. Hence i is at distance $\leqslant 18K$ from the interval $[j^*, k^*]$. (Note that we might have $j^* > k^*$, but that does not change the reasoning.) It follows that $|w, p_i^{ac}|_S \leqslant 11K^2 + 18K$. □

By Step 2, we can assume that $a \neq c$ and $b \neq c$. Moreover, we can assume that there is no b-component in q^{ac}, nor an a-component in q^{bc}. Consequently, we can apply Theorem 4.3 to Π, viewing r^a and r^b as isolated components and the remaining four sides as geodesic sides. It follows that

$$\max\{|r_-^a, r_+^a|_S, |r_-^b, r_+^b|_S\} \leqslant 6K. \tag{1}$$

Together with Step 1, this implies

$$\max\{|p_-^{ab}, p_-^{ac}|_S, |p_+^{ab}, p_-^{bc}|_S\} \leqslant 10K. \tag{2}$$

Step 3. *If p_i^{ab} is an (ϵ, R)-transition vertex of p^{ab} or an endpoint of an (ϵ, R)-segment of p^{ab}, then $|p_i^{ab}, p_i^{ac}|_S \leqslant 34K$ or $|p_i^{ab}, p_{\ell-i}^{bc}|_S \leqslant 34K$.*

Proof. By Step 1, the vertex p_i^{ab} is at S-distance $\leqslant 3K$ from some q_h^{ab}. We split the hexagon Π into four geodesic triangles in $\bar{\Gamma}$ using diagonals. Repeated application of Theorem 4.1 and inequality (1) yields a vertex of $q^{ac} \cup q^{bc}$ at S-distance $\leqslant 8K$ from q_h^{ab}. Without loss of generality, suppose that this vertex is $q_{h^*}^{ac}$. By Proposition 4.6(i), the vertex $q_{h^*}^{ac}$ is at S-distance $\leqslant K$ from some $p_{i^*}^{ac}$. Consequently, $|p_i^{ab}, p_{i^*}^{ac}|_S \leqslant 12K$. By inequality (2), it follows that $|i - i^*| \leqslant 22K$. Therefore, $|p_i^{ab}, p_i^{ac}|_S \leqslant 34K$. □

To prove Theorem 3.4, it remains to consider the case where p_i^{ab} is an (ϵ, R)-deep vertex of p in a peripheral left coset w. Let $p^{ab}[j, k]$ be the (ϵ, R)-segment of p^{ab} in w containing p_i^{ab}.

Note that if $w = c$, then $|a, c|_S \leqslant j + \epsilon$. Consequently p_i^{ac} is at S-distance $\leqslant \epsilon$ from $p_+^{ac} \in w$, and the theorem follows. Henceforth, we assume $w \neq c$.

Also note that if $k - j \leqslant 18K^2$, then assuming without loss of generality that Step 3 yields $|p_j^{ab}, p_j^{ac}|_S \leqslant 34K$, we have:

$$|w, p_i^{ac}|_S \leqslant |w, p_j^{ab}|_S + |p_j^{ab}, p_j^{ac}|_S + |p_j^{ac}, p_i^{ac}|_S \leqslant \epsilon + 34K + 18K^2 \leqslant D,$$

and the theorem is proved. Henceforth, we assume $k - j > 18K^2$.

Step 4. *The path q^{ab} has a w-component $q^{ab}[m, m+1]$ which is $5K^2$-similar to $p^{ab}[j, k]$. Moreover, q^{bc} or q^{ac} has a w-component.*

Proof. The first assertion follows from Lemma 4.9. In particular, w is distinct from a and b. For the second assertion, suppose for contradiction that q^{bc} and q^{ac} have no w-components. Consequently, since $w \neq c$, the w-edge $q^{ab}[m, m+1]$ is an isolated w-component of Π. We apply Theorem 4.3 with Π interpreted as an 8-gon with a side $q^{ab}[m, m+1]$ as the only isolated component. This contradicts $|q^{ab}_m, q^{ab}_{m+1}|_S > 18K^2 - 2 \cdot 5K^2 \geqslant 8K$. $\square$

Step 5. *Suppose that q^{ac} has a w-component $q^{ac}[n, n+1]$ and q^{bc} does not have a w-component. Then $|w, p^{ac}_i|_S < D$. Similarly, if q^{bc} has a w-component and q^{ac} does not have a w-component, then $|w, p^{bc}_{\ell-i}|_S < D$.*

Proof. By symmetry, it suffices to prove the first assertion. Let x be a w-edge in $\bar{\Gamma}$ from q^{ab}_m to q^{ac}_n. Consider the geodesic 4-gon $r^a q^{ab}[0, m] x q^{ac}[n, 0]$ in $\bar{\Gamma}$. Observe that x is an isolated w-component and hence Theorem 4.3 implies that $|q^{ab}_m, q^{ac}_n|_S \leqslant 4K$. Analogously, by considering a geodesic 6-gon, we obtain $|q^{ab}_{m+1}, q^{ac}_{n+1}|_S \leqslant 6K$.

Proposition 4.6(iii) implies that $q^{ac}[n, n+1]$ is K-similar to a subpath $p^{ac}[j^*, k^*]$ of p^{ac}. By Step 4, the paths $p^{ab}[j, k]$ and $p^{ac}[j^*, k^*]$ are $(5K^2 + 6K + K)$-similar, hence $12K^2$-similar. By inequality (2), it follows that $|j - j^*| \leqslant 10K + 12K^2 \leqslant 22K^2$ and similarly $|k - k^*| \leqslant 22K^2$. Hence $i \in [j^* - 22K^2, k^* + 22K^2]$. By Lemma 4.7, we have $|p^{ac}_i, w|_S \leqslant K \cdot K + 22K^2$. $\square$

It remains to consider the case where both q^{ac} and q^{bc} have w-components, which we denote by $q^{ac}[n, n+1]$ and $q^{bc}[\tilde{n}, \tilde{n}+1]$.

The argument in the proof of Step 5 shows that

$$\max\{|q^{ab}_m, q^{ac}_n|_S, |q^{ac}_{n+1}, q^{bc}_{\tilde{n}+1}|_S, |q^{ab}_{m+1}, q^{bc}_{\tilde{n}}|_S\} \leqslant 4K. \tag{3}$$

By Proposition 4.6(iii) there are integers $0 \leqslant \alpha \leqslant \beta \leqslant l(p^{ac})$ and $0 \leqslant \gamma \leqslant \delta \leqslant l(p^{bc})$ such that $q^{ac}[n, n+1]$ and $p^{ac}[\alpha, \beta]$ are K-similar, and $q^{bc}[\tilde{n}, \tilde{n}+1]$ and $p^{bc}[\gamma, \delta]$ are K-similar.

Step 6. *We have*

$$\alpha - 20K^2 \leqslant i \leqslant \beta + 33K^2 \quad \textit{or} \quad \gamma - 20K^2 \leqslant \ell - i \leqslant \delta + 33K^2.$$

Proof. By inequality (3) and Step 4, we have the following estimates:

$$\begin{aligned}
&|p^{ac}_\beta, p^{bc}_\delta|_S \leqslant |p^{ac}_\beta, q^{ac}_{n+1}|_S + |q^{ac}_{n+1}, q^{bc}_{\tilde{n}+1}|_S + |q^{bc}_{\tilde{n}+1}, p^{bc}_\delta|_S \leqslant 6K,\\
&|p^{ab}_j, p^{ac}_\alpha|_S \leqslant |p^{ab}_j, q^{ab}_m|_S + |q^{ab}_m, q^{ac}_n|_S + |q^{ac}_n, p^{ac}_\alpha|_S \leqslant 5K^2 + 4K + K \leqslant 10K^2,\\
&|p^{ab}_k, p^{bc}_\gamma|_S \leqslant |p^{ab}_k, q^{ab}_{m+1}|_S + |q^{ab}_{m+1}, q^{bc}_{\tilde{n}}|_S + |q^{bc}_{\tilde{n}}, p^{bc}_\gamma|_S \leqslant 10K^2.
\end{aligned}$$

These estimates and the triangle inequality imply

$$k - j \leqslant (\beta - \alpha) + (\delta - \gamma) + 26K^2. \tag{4}$$

From inequality (2) it follows that $|j-\alpha| \leqslant 10K + 10K^2 \leqslant 20K^2$, and analogously, $|(\ell-k)-\gamma| \leqslant 20K^2$. In particular, $\alpha - 20K^2 \leqslant j \leqslant i$ and $\gamma - 20K^2 \leqslant \ell - k \leqslant \ell - i$, as desired. To conclude the proof we argue by contradiction. Suppose that $i > \beta + 33K^2$ and $\ell - i > \delta + 33K^2$. It follows that

$$i - j \geqslant i - \alpha - 20K^2 > \beta - \alpha + 13K^2$$

and

$$k - i = (\ell - i) - (\ell - k) \geqslant (\ell - i) - \gamma - 20K^2 > \delta - \gamma + 13K^2.$$

Adding these two inequalities yields $k - j > (\beta - \alpha) + (\delta - \gamma) + 26K^2$, which contradicts inequality (4). $\square$

We now conclude the proof of Theorem 3.4. Since the endpoints of $q^{ac}[n, n+1]$ are in w, it follows that the endpoints of $p^{ac}[\alpha, \beta]$ are at S-distance $\leqslant K$ from w. By quasiconvexity of w, Lemma 4.7, all the vertices of $p^{ac}[\alpha, \beta]$ are at S-distance $\leqslant K^2$ from w. Analogously, all the vertices of $p^{bc}[\gamma, \delta]$ are at S-distance $\leqslant K^2$ from w. Then Step 6 yields $|w, p_i^{ac}|_S \leqslant K^2 + 33K^2 < D$ or $|w, p_{\ell-i}^{bc}|_S \leqslant 34K^2 < D$.

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